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THEORETICAL MECHANICS

A Vectorial Treatment

BY

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PREFACE

The study of Theoretical Mechanics has undergone profound changes in recent years both in the formulation of the fundamental principles on which it rests and in the analytic procedure by which its conclusions are derived. For centuries the basic postulates concerning time, space and matter which formed the foundations of geometry and mechanics were supposed to enjoy a favored position among the physical laws in that they possessed an absolute rather than an approximate validity in the physical universe. But starting in 1904 Lorentz, Einstein, Minkowski and others pointed out that the conclusions of mechanics were not in theoretical accord with the equally well substantiated theories of electricity and magnetism as expounded by Maxwell and others. They therefore abandoned the simple postulates of the older *classical* mechanics and starting afresh began to build up on the basis of different postulates a more elaborate structure of theory to include not only mechanics, but electricity, magnetism, light and gravitation. This new *relativity* theory is free from the grosser inconsistencies of the older theories and its conclusions are in very close accord with the physical facts.

It thus appears that classical mechanics can no longer be regarded as giving a rigorously true picture of those aspects of the material world with which it deals. The present point of view may perhaps be expressed by saying that we regard geometry, kinematics, classical mechanics, relativity mechanics as different theories, each resting on its own postulates, each consistent in itself and each possessing certain advantages in its degree of simplicity and in the degree of accuracy with which it interprets the physical world.

Theoretical Mechanics is also changing in the method by which its results are derived and expressed. The concept of vector is of course inherent in mechanics and must appear in any exposition of it, but the special methods and notation of vector analysis have only recently come into general use in the study of mechanics. The virtues of this analysis are evident. As Burgatti says in his *Lezioni di Meccanica Razionale*, "The generality of this analysis,

which speaks and writes a language common to all the branches of mathematical physics; its clear and often eloquent conciseness; its rapidity in translating an idea into a formula and a formula into an idea; its peculiar property of holding together intuition and logic, synthesis and analysis, make of it a scientific and didactic instrument of the very first order."

In view of the above facts it has been the author's aim in preparing this book to present an introduction to classical mechanics, and through it to mathematical physics, which shall make clear the simple postulates on which they rest and at the same time to explain and apply immediately the basic principles of vector analysis. After adopting as a set of postulates for mechanics the postulates of Euclidean geometry and Newton's Laws of Motion, the rectilinear motion of a particle is discussed, using only scalar quantities. Vector Algebra is then introduced and briefly applied to Euclidean and Analytic Geometry. The vector function of a scalar is then treated and employed to discuss the curvilinear motion of a particle and the displacement and motion of a rigid body. A discussion of systems of sliding vectors forms the basis for the treatment of the statics of particles, of rigid bodies and the flexible cord. The Principle of Virtual Work then appears, either as proven from Newton's laws or as an independent postulate. Chapters on the kinetics of particles and of the rigid body follow and embody a simple introduction to and use of the linear vector function. A chapter on the variational principles of mechanics introduces alternative sets of postulates on which mechanics may be founded, and also a development of Lagrange's and Hamilton's equations. Finally a discussion of vector calculus leads naturally to a brief exposition of potential theory. This concludes the book but leaves the way open for a continuation of the method by studies in tensor analysis with application to relativity mechanics.

In teaching this subject the author has found that a double advantage accrues to the student by this approach. The student learns the vector analysis almost incidentally in the study of the mechanics, and learns it well because he not only studies it directly but continues to use it thereafter. And the vector analysis clears away many of the technical difficulties in the discussion of the mechanics and permits its basic ideas to be proven readily and to stand forth clearly.

In his courses at the University of Michigan the author has found that this book presents sufficient material for an introductory and an advanced course, each running through the year. If desired there should be no difficulty in omitting certain topics and putting more emphasis on the others. A year's work in calculus should form an adequate mathematical preparation for the study of the book, although the last three chapters will be more readily mastered by the student who has had a first course in differential equations.

A considerable selection of exercises is offered, both in application of the theory and in simple extensions of it. A goodly percentage of these are new and all have been carefully worked through and chosen for their value in clarifying the subject and arousing interest in it. As a whole the subjects treated and the exercises on them tend to become more difficult as the student progresses through the book.

CARL JENNESS COE

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September, 1938

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THEORETICAL MECHANICS

CHAPTER I

INTRODUCTION

1. Divisions of Mechanics.

As an introduction to the study of mechanics perhaps we can not do better than to quote the words of Appell, a famous authority on the subject and one time rector of the University of Paris. He says:

"Among the mathematical sciences the first is the science of calculation which rests on the concept of number and upon which one is obliged to base all the others. Next comes geometry which introduces a new concept, that of space. In geometry we consider points which describe curves, curves which describe surfaces, etc., but we are not concerned in any way with the time in which these movements are carried out. If this concept of time be introduced we obtain a more complex science called kinematics which studies the geometric properties of movements in their relations with time without questioning as to the physical cause of these movements. This last question is studied in dynamics, but we observe that it is impossible to discover the true cause of physical phenomena and hence we content ourselves with substituting for them fictitious causes called forces capable of producing the same effects.

"Dynamics has for its object the solution of the two following problems:

1. To determine the motion of a system of bodies under the action of given forces.
2. To find a system of forces capable of producing a given motion in a given system of bodies."

The divisions of mechanics are,

Geometry	treating of space,
Kinematics	treating of space and time,
Dynamics	treating of space, time and mass.

We may of course study geometry or either of the other branches of mechanics by various methods. We might for instance employ a graphical method and avoid the use of numbers entirely, the resulting subjects being called graphical kinematics, graphical dynamics, etc. If, however, we apply mathematical analysis, i.e. the study of quantity or number, to our study of space we have analytic geometry and similarly we may have analytic kinematics and analytic dynamics.

Dynamics is for convenience divided into two branches. If the problems studied are ones in which the bodies involved do not move we call the study *statics,* while the study of the general case in which the bodies may move is called *kinetics.*

2. Postulates of Mechanics.

Mechanics, like other branches of mathematics, proceeds by proving successive theorems each resting in general on some of the preceding. Also, whenever convenient, new terms may be introduced, but they must of course be carefully defined in terms of concepts already familiar. It is clearly evident that this procedure by theorem and definition can not apply at the very beginning of the study since there we have no preceding theorems and we have no previously defined terms. To get a start we must then leave some terms undefined and some propositions unproven.

The necessity of this will be further appreciated if we think of it from another point of view. A thing may be too simple, too familiar to be properly defined. For to define something we should explain it in terms more likely to be familiar to the listener than is the term defined. Thus we tell the small child that the world is shaped like a big round ball instead of saying that it is an oblate spheroid. If then a thing is extremely simple and familiar we may find it difficult to find other even more simple and familiar things in terms of which to define it. And of course if the thing is thus very simple and familiar there is no real reason for defining it, but it may be desirable to enumerate certain of its properties upon which we may agree before studying about it. Such a statement of a property of an undefined object is called a *postulate.* Thus in geometry we find "straight line" a term well left undefined but we may agree upon certain postulates of the line such as:

Two points determine a straight line.

A straight line is the shortest distance between two points.

Similarly certain very simple, obvious propositions can hardly be given a proper proof. For to prove a given statement to someone we must start with its hypotheses and lead him by easy steps to its conclusion. These steps must be so simple and clear that he can not avoid admitting their validity. But if the original statement is already extremely simple it may be difficult to find still simpler steps into which to divide it. Now since we must leave certain propositions unproven at the outset of our study, it is well to choose them from these simple propositions which are difficult to prove. These are also called postulates. Such a postulate is for instance:

Through a given point not on a given line it is possible to draw just one line parallel to the given line.

Mechanics assumes all the postulates of analysis and geometry and in addition with the introduction of its two new undefined elements, time and mass, come certain new postulates. Thus we have the postulate of time:

Time is absolute.

This simply means that an event A may be said to occur ten seconds before an event B without the necessity of any statement as to the position or state of motion of the observer or the events. A complete formulation of exactly those postulates which are necessary to the development of mechanics is a difficult task and beyond the limitations of this discussion. However there are three postulates of mechanics which are of great interest both for their content and for their historical associations. These are the three *Laws of Motion* of Sir Isaac Newton published by him in his "Principia" in 1687. It is on the foundation of these three famous principles that all of the so-called classical mechanics may be made to rest. At the time when Newton published the "Principia" mechanics was in something of a jumble of conflicting assumptions often vaguely stated or not stated at all. The statement of the three laws of motion of Newton served remarkably to clarify the ideas of the mathematicians of the time upon the subject and to dispel many of the then existing misconceptions. A great deal has since been written concerning these laws in which their sufficiency for their purpose and their consistency with each other have been minutely examined. In studying them we must not forget the state of the science at the time they were enun-

ciated. If Newton were alive today he might not put them in exactly the same form because he would not have the same misconceptions to combat.

The first of these laws is,

I. *Every body continues in its state of rest or uniform motion in a straight line unless it is acted upon by some exterior force.*

Previous to the time of Galileo (1564–1642) it had been generally supposed that rest relative to the earth was the natural state of a body and that it returned to that state unless something kept it in motion. Galileo observed that the more we remove impediments to the motion of a body the longer and more steadily it continues that motion, and he concluded that if the body were left entirely alone the motion would continue uniformly and indefinitely. The second law is,

II. *The rate of change of momentum is proportional to the force acting and in the direction of its action.*

This second law of motion serves as a definition of force since it gives the means by which forces are recognized and measured. The muscular reaction which we experience when we hold a heavy body at arm's length may give us an independent conception of force and even a rough determination of its amount and direction, but this can scarcely serve any scientific purpose. The third law is,

III. *To every action there is an equal and opposite reaction.*

This third law states that in nature forces always occur in pairs, the two forces being equal in magnitude and acting along the same line but in opposite senses. We often concentrate our attention upon only one of the forces of a pair, as in firing a gun, the explosion of a charge of dynamite on a rock, a team of horses pulling on a post to which they are attached, but the opposing force will always be found to be present. One can not pull himself up by his boot straps. We shall go further into the interpretation of these new postulates when we use them.

3. Validity of Mechanics.

On the basis of the three laws of motion just discussed and with the assumption of the law of universal gravitation, likewise due to Newton (1666), the scientists of his time succeeded in forming a

satisfactory explanation of the solar system. In other words they found that with these additional postulates it could be demonstrated mathematically that the bodies of the solar system would, to a high degree of approximation, move as they actually do move. This, one of the greatest and most brilliant achievements of science, led at once to an effort to extend the application of these principles of mechanics to other physical phenomena. These efforts met with surprising success. The physical properties of gases and liquids were readily treated in this way and even ultimately the phenomena of sound, light, electricity and magnetism. The physicists of the nineteenth century were busy perfecting this mechanistic view of the universe, showing in each case that if the laws of mechanics be obeyed by certain particles or in certain media the actual observed phenomena would result to a high degree of approximation.

The general success of these efforts was so great and their failures so comparatively slight that many eminent scientists seem to have felt that the whole physical universe could be *exactly* explained on the basis of the laws of mechanics. In looking back at this situation now it seems somewhat as if these men, great scholars as they were, deliberately closed their eyes to the cases of the failure of these mechanical explanations. But shortly after the beginning of the present century many new phenomena began to be observed extremely difficult to fit into the old mechanistic theories and certain exceptionally honest and clear headed men began to point out insistently the discrepancies between fact and theory. It became apparent that the comparatively simple classical mechanics could not be expected to explain exactly the complex phenomena of nature and the attempt to formulate an explanation along these lines was abandoned. Instead different types of more complicated mechanical theories were set up resting on entirely new postulates. Thus we have relativity mechanics, quantum mechanics, wave mechanics, etc.

It would seem at first that since classical mechanics has been found to have no actual validity in the physical world there is little reason for studying it or, for that matter, for studying its simplest branch geometry. This is however far from being the case. We should doubtless adopt a different point of view toward mechanics from that of our predecessors of previous generations, since we can no longer feel as they did that the theorems of geom-

etry are statements of actual facts true in the physical world. The present point of view may be put somewhat as follows:

The physical world seems to be extremely complicated. We can not at present hope to explain completely all its varied phenomena by means of a few simple laws. But we know from experience that if we create in imagination a fictitious simplified universe with only a few kinds of things in it and only a few simple laws to control it we may very well be able to understand this simplified universe and this understanding may throw great light on the real more complicated world. Thus we regard geometry, kinematics, classical mechanics, relativity mechanics as studies of different fictitious worlds each approximating the real world to a certain degree and each having its own peculiar advantages in simplicity and in the degree to which information is given concerning the real world.

We have been here discussing these theories from the physical point of view. From the mathematical point of view the question as to whether a given body of mathematical theory has physical applications may not arise.

We have said that the fictitious universe of classical mechanics fits the real world only approximately. We shall give here a simple example of a contradiction between the experimental facts and the theoretical conclusions of the old mechanics. In 1881 Michelson and Morley carried out a famous determination of the velocity of light and later Comstock and De Sitter made observations on double stars both of which point absolutely to the conclusion that the velocity of light in vacuo is a physical constant c independent of the velocity of the source and of the velocity of the observer. This is readily seen to be in contradiction to the simple reasoning of classical mechanics. For example, suppose a flash of light to emanate from an electric lamp situated on a railway platform and imagine an observer able to watch this flash of light travel outward from the source. He sees it going outward in all directions and in particular traveling along the track with the velocity c. The observer next boards a train moving in the same direction as the flash with a velocity v and discovering that the apparent velocity of the flash as observed from the train is still c he concludes that the velocity of the flash as observed from the platform must now be $c + v$. He next boards a train going in the opposite direction with a velocity u and again finding that the

apparent velocity of the flash is c as observed from this train he now concludes that the velocity as observed from the platform must be $c - u$. But all this time an observer remaining on the platform would declare that the flash had continued with its original velocity c. Classical mechanics can not explain this discrepancy.

Although it is thus true that the structure of mechanics as built on the three laws of Newton does not give results which are exactly verified by experiment, still the discrepancies are few and extremely difficult to observe. We are in fact fully justified in studying this classical mechanics and we find the extent to which it explains the physical world much more surprising than the fact that there are cases in which it does not.

CHAPTER II

RECTILINEAR MOTION OF A PARTICLE

4. Scalar Quantities.

In the study of mathematics we employ various different kinds of numbers, positive and negative, integral and fractional, real, imaginary and complex. And in applying mathematics to various problems we find use for all of these. But whenever we use one of these numbers to characterize some property of an object it is necessary to have a previous agreement on certain points. Let us consider a few examples of this.

The simplest kind of number, the one of which we learned before starting to study arithmetic, is the positive integer or natural number as $1, 2, 3, 4, \cdots$. We use these of course to characterize groups of objects which can be counted, such as the apples in a bag, the people in a country. But even here we often make an agreement as to the unit in which we are to express the result. Thus we usually agree to count eggs or oranges by dozens, sheets of paper by quires, the people in a country by millions, etc. Likewise in employing the positive real numbers we must always have an understanding as to the size of the unit employed. It would mean nothing to say that the size of this bin is $2\frac{1}{2}$, the circumference of this circle is 4π unless the unit referred to in each case were clearly understood.

When we employ numbers which may be both positive and negative we find it necessary to specify not only the unit to be employed in the measurement but also the initial value or starting point from which we measure and the sense in which the measurement has been made. Thus when we say, "The temperature in this room is 23° Centigrade" we are in effect specifying that the starting point in the measurement which gives the number 23 is the freezing point of water, that the unit is 1/100 of the interval from the freezing to the boiling point and we further imply that the sense is warmer rather than colder than zero. If the temperature were in the neighborhood of zero it would be necessary to make this last point specific. A great many quantities are like

the temperature of the room, i.e. capable of being completely characterized by the giving of one real number with sign attached. We call these quantities *scalar quantities.* In giving the value of any scalar quantity it will be found that we must always state or clearly imply the three elements just mentioned, the origin, the unit and the sense.

Two examples of scalar quantities of great importance in mechanics are the position of a point on a line and the time of an event. Let us suppose that we have an indefinitely long straight line such as is considered in elementary geometry and let us imagine this line fixed before us in a horizontal position. Suppose there were two points A and B on the line and that we wished to describe their relative positions. We might apply a unit of length such as the inch to the segment AB and by giving the number of inches in it we would state something concerning the relative position of the two points. But this would still leave an important question unanswered, namely whether A lies to the right or left of B. Now let us agree not only that the inch shall be used in the measurement but that after measuring the segment AB we shall prefix a $+$ sign to the resulting number if the second point mentioned, B, lies to the right of the first, A, and a $-$ sign if the second lies to the left of the first. With this understanding the equation

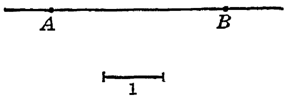

FIG. 1

$$AB = +5$$

may be interpreted so as to describe completely the relative position of the two points on the line. If we imagine one point O to remain fixed on the line and another P to move about on the line then the position of P at any instant will be completely determined by giving the value of the real number $OP = s$ with proper sign attached. The position of the point P on the line is thus a scalar quantity.

Similarly the time of an event is a scalar quantity. For suppose we have two events A and B and we wish to describe the relative times of their occurrence. Simply to measure the time interval AB and state that it is a certain number of seconds leaves open the important question as to which comes first. But if we adopt an agreement not only as to the unit of time employed in measuring

the interval AB but agree further that we shall prefix a $+$ sign to the resulting number if the first event, A, precedes the second, B, and a $-$ sign if the second mentioned precedes the first, then the equation

$$AB = +5$$

may be interpreted so as to describe completely the relative time of the two events. If we adopt the moment of some event O as a basis of reference and consider the possibility of another event P occurring at various moments then the time of its occurrence will be completely determined by giving the value of the real number $OP = t$ with proper sign attached. The moment of the event P is thus a scalar quantity.

We have seen that the simpler sorts of numbers may be employed to characterize certain simple properties of objects and the more complicated sorts of numbers may similarly characterize more complicated properties. The scalar quantity represents a certain stage in this development. In a later section we shall discuss the *vector quantity* which is a more advanced stage.

5. Velocity and Acceleration.

As in the previous section let us imagine a straight line before us in a horizontal position and two points O and P upon it. Let us imagine P moving relative to O on either or both sides. In thinking of this motion of P we often say that O is fixed and P moving but this means nothing more than that *we are considering* the motion of P relative to O. If we adopt the agreements of the preceding section then corresponding to every position of P will be a value of the quantity s and corresponding to the instant when P occupies this position will be a value of the quantity t. This value of s and this value of t are thus connected and we have a value of s associated with every value that t takes on during the motion. We express this fact by saying that s is a *function* of t and write it as,

FIG. 2

$$s = s(t). \tag{1}$$

This situation of one quantity being a function of another is familiar to us from the calculus and it naturally occurs to us that

possibly some of the associated concepts studied in the calculus may have interesting applications here. Consider some particular position of P. Call it P_0 and the corresponding values of t and s respectively t_0 and s_0. Now when P has moved to a new position t and s will take on new values which we shall simply designate by t and s. Let us employ Δt and Δs to indicate the variation in these quantities.

$$\Delta t = t - t_0, \qquad \Delta s = s - s_0.$$

We shall say that the quotient $\frac{\Delta s}{\Delta t}$ is the *average velocity* of the moving point while passing from P_0 to P. It is a number which evidently depends on the motion of this point all during the time interval Δt. If we now find this average velocity for several successively shorter intervals all having P_0 as their first point but having the successive values of Δt approaching zero, then we shall often discover that the successive average velocities approach some limiting value. This limiting value we call the *velocity* at the point P_0 and write it as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

The reader will at once recognize that we have here an entirely familiar situation. The velocity at P_0 is exactly the derivative of s with respect to t at that point.

$$v = \frac{ds}{dt} = \tag{2}$$

We now have in general a value of v corresponding to each position of P. There is also a value of t associated with the instant at which P occupies each position and if we now associate the values of v and t corresponding to the same position of P then there is a value of v associated with each value which t takes on during the motion. In other words, v is a function of t.

$$v = v(t). \tag{3}$$

We now proceed with v exactly as we did with s. Choosing any particular position of P we label it P_0 and the corresponding values of t and v we label t_0 and v_0. For any other position of P we have new values of t and v which we shall simply call t and v.

We designate the change in these two quantities by Δt and Δv,

$$\Delta t = t - t_0, \qquad \Delta v = v - v_0,$$

and call the quotient $\frac{\Delta v}{\Delta t}$ the *average acceleration* of the moving point in this interval. Now keeping P_0 fixed we choose a succession of positions of P in such a fashion that Δt approaches zero as a limit. This may cause the successive average accelerations to approach a limit also and if this limit exists we shall call it the *acceleration* at the point P_0.

$$j = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}.$$

Here again the situation is familiar. We recognize that the acceleration j is the value which the derivative of v with respect to t takes on at the point P_0 and is therefore also the second derivative of s with respect to t for that point.

$$j = \frac{dv}{dt} = \frac{d^2s}{dt^2} = s''. \tag{4}$$

Evidently we might continue in this fashion assigning names to the successive higher derivatives of s with respect to t but it has not been carried beyond this point, possibly because the higher derivatives are harder to visualize. In theory any of these derivatives may fail to exist but the study of such exceptional cases is beyond our scope and has few simple applications.

Equations (2) and (4) readily yield another useful relation. We have by a well known rule of calculus,

$$\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt},$$

and hence we have,

$$j = v\frac{dv}{ds}, \tag{5}$$

which is a relation which does not involve the time t explicitly.

It must not be forgotten that the definitions here given apply only to rectilinear motion. When we come to the study of curvilinear motion of a point we shall find the treatment analogous but not exactly the same.

As an example of the use of these ideas in the study of a given motion let us consider the following problem.

A ladder 20 feet long leans against a house. A man pulls the lower end of the ladder away from the house at a uniform rate of 2 feet per second. Determine the velocity and acceleration of the top of the ladder when the bottom is 12 feet from the house. Let us represent the height of the top of the ladder by s, the distance of the bottom of the ladder from the house by x and let us count time from the instant when the ladder stood upright against the house. Then evidently,

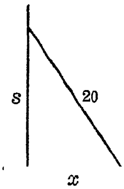

FIG. 3

$$x = 2t$$

and

$$s = \sqrt{400 - x^2} = \sqrt{400 - 4t^2}.$$

Hence we have by the usual rules for differentiation,

$$v = \frac{ds}{dt} = \frac{-4t}{\sqrt{400 - 4t^2}}, \qquad j = \frac{dv}{dt} = \frac{-1600}{(400 - 4t^2)^{3/2}}.$$

At the instant under consideration we have $t = 6$ and therefore,

$$v = -\frac{\ }{2}, \qquad j = -\frac{25}{64}.$$

The velocity is of course expressed in feet per second and the acceleration in feet per second per second. Since $v = \dfrac{ds}{dt}$ is here negative s must be diminishing and since j is negative v must also be diminishing. But since v is a negative quantity this means that v is numerically increasing. The top of the ladder is therefore sliding downward at an increasing rate.

EXERCISES

1. The height of a stone thrown vertically into the air is given by the formula,

$$s = 16t(3 - t).$$

What are the initial velocity and acceleration? How high will the stone go? *Ans.* 48, − 32, 36

2. A block of wood sliding down an inclined plane has the distance covered given by the formula,

$$s = 4t(t + 1).$$

What is the average velocity during the last half of the third second? At what time does the true velocity equal this value?

Ans. 26, 2.75

3. A barge whose deck is 12 feet below the level of a dock is pulled in by means of a cable attached to a ring on the floor of the dock, the cable being hauled by a windlass on the deck at the rate of 8 feet per minute. Compute the velocity and acceleration of the barge when the windlass is 16 feet from the dock. *Ans.* 10, 2.25

4. A man is walking over a bridge at the rate of 4 miles an hour and a boat passes under the bridge immediately below him at 8 miles an hour. The man is 20 feet above the boat. How fast are the man and boat separating 1 minute later? *Ans.* 13.11 ft./sec.

5. A steamboat sails north from a certain port at 20 miles per hour and one hour later a second boat sails from the same port N.60°E. at 15 miles per hour. How fast are the two boats separating one hour after the second boat leaves? *Ans.* 16.43

6. Two straight railroad tracks intersect at right angles. At noon there is a train on each track approaching the crossing at 40 miles per hour, one being 100 miles and the other 200 miles from the crossing. How fast are they approaching each other? When will they be closest together and what will this minimum distance be?

Ans. 53.67, 3.75, 70.71

7. A horse runs 10 miles per hour on a circular mile track in the center of which is a lamp post which casts the shadow of the horse on a straight fence tangent to the track at the starting point. What are the velocity and acceleration of the shadow when the horse has completed $\frac{1}{8}$ of a circuit? Would these answers be changed if it were a 2 mile track? *Ans.* 20, 2513

8. A locomotive runs 30 miles an hour over a high bridge and dislodges a stone lying near the track. The stone begins to fall just as the locomotive passes. How fast are the stone and locomotive separating 2 seconds later? (Assume that the distance fallen by the stone is given in feet by the formula $s = 16t^2$.)

Ans. 73.23 ft. per sec.

9. A flywheel of radius 2 feet is rotating in a vertical plane at the rate of 15 revolutions per minute. A lug on the rim casts a shadow on the floor, the rays of light being vertical. What are the velocity and acceleration of the shadow when the lug is at the top of the wheel; $\frac{1}{3}$ second later; 1 second later?

Ans. 3.142, 0, 2.721, $-$ 2.467, $-$ 4.935

10. A flywheel with center O and radius 4 feet rotates uniformly at one revolution a second. A rod of length 8 feet has one end A attached to the circumference of the wheel while the other end B slides along a produced diameter of the wheel. Find the velocity

and acceleration of the point B when the angle AOB is a right angle. *Ans.* -25.13, $+91.17$

11. If a point moves so as to constantly satisfy the relation, $v^2 = as + b$, where a and b are constants, show that the acceleration remains constant.

6. The Equation of Motion.

We are now in a position to make use of the basic postulates of mechanics known as Newton's laws. The proper interpretation of these laws is however a matter requiring some care. Firstly the moving body must be a *particle*, where by a particle we mean simply a point in space having a number associated with it called its *mass*. This amounts to saying that we assume that all the matter of the body has been concentrated at a single point which thus becomes a particle representing the body. This assumption will of course often introduce an error but in certain important cases the error can be proven to be extremely small and in other cases we find means of obviating its effects.

For our present purpose we shall assume that the particle moves continuously in a straight line so that the definitions of velocity and acceleration developed in the previous section are applicable. The product of the mass and velocity of the particle is called its *momentum* and following Newton's second law we assume that the derivative of the momentum with respect to the time is proportional to the force. Thus we have,

$$\frac{d(mv)}{dt} = \lambda f, \tag{1}$$

where λ is the constant proportionality factor. Since the law states that the change in momentum has the direction of the force acting it follows that λ is essentially positive. In the great majority of cases the mass of the particle is regarded as constant so that the equation may be written,

$$mj = \lambda f. \tag{2}$$

We have here an expression of Newton's first and second laws for rectilinear motion of a particle, but there is an important additional implication in the laws.

Forces are assumed to be additive. Suppose that under a given set of conditions we find our particle to have a force f_1 and under other conditions a force f_2, then we shall assume New-

ton's second law to imply that if these conditions are both satisfied the force will be the algebraic sum $f_1 + f_2$. Thus if one horse pulling alone on a wagon produces a force of 75 and another when pulling alone a force of 50, then pulling together in the positive sense they produce a force of $+ 125$. But if the second horse produces a force of $- 50$, i.e. pulls in the negative sense, then the effect of the two is a force of $+ 25$.

In saying that m is the mass, f the force, etc. the reader will of course understand that as usual in mathematics these letters really stand for numbers, namely the number of times the unit of measure will go into the quantity mentioned, suitable conventions having been adopted as to the origin, the unit of measure and the positive sense as explained in § 4. The units employed in measuring s are the familiar foot, yard, meter, etc.; for m the pound, the ton, the gram, etc. and for t the second, the hour, etc. In measuring f we may conveniently use the absolute unit of force which is so chosen that λ in the above equation has the value unity. That is, the unit of force will be the force which accompanies a unit acceleration of a particle of unit mass. On the other hand in measuring f it is often convenient to adopt the gravitational system in which we take as the unit of force that force with which the earth attracts a unit mass at its surface, i.e. the unit force is the weight of the unit mass. In this case λ must be determined experimentally for the place on the earth's surface where the motion takes place. In fact, however, this value of λ does not vary greatly from place to place over the earth's surface and it usually may be treated as a constant represented by the letter g. A tabulation of the units most frequently employed and the corresponding values of λ follows.

Absolute System

	λ	*length*	*mass*	*time*	*force*
English measure	1	foot	pound	second	poundal
Metric measure	1	cm.	gram	second	dyne

Gravitational System

	λ	*length*	*mass*	*time*	*force*
English measure	32	foot	pound	second	pound
Metric measure	980	cm.	gram	second	gram

The reader must carefully distinguish between a pound of mass and a pound of force which, although they unfortunately

have the same name, are entirely different sorts of things. Ten pounds of iron is a certain quantity of matter. If we hold it at arm's length we feel it pulled downward with a force of ten pounds. This remark also holds for the gram of mass and of force.

If the force f of a particle were known to be a certain function φ of the time, the position, and the velocity of the particle its value in terms of these quantities could be substituted in equation (2) and we would have,

$$(3) \qquad m \frac{\quad}{dt^2} \quad \lambda\, \varphi\left(t, s, \frac{ds}{dt} \right)$$

This is a differential equation of the second order known as the *equation of motion* of the particle. It may be possible to integrate this differential equation, that is to express the same relation between t and s by an equation free from derivatives but involving two arbitrary constants. If conditions are given by which these constants of integration may be determined we may finally be able to express s as an explicit function of t, thus completely determining the motion of the particle. Thus we would solve in this simple case the first problem of mechanics as formulated by Appell, § 1. In the remaining sections of this chapter we shall consider in detail several of the more important cases of this type.

EXERCISES

1. If the earth attracts different bodies at its surface with forces proportional to their masses, show that all bodies falling freely near the earth's surface will have the same acceleration.

2. Given that 1 meter = 39.37 inches and 1 kilogram = 2.2046 pounds, compute the number of dynes in a poundal.
Ans. 13,825.7

3. If at a certain point of the earth's surface a pound of force is exactly 32 poundals, compute the number of dynes in a gram of force.
Ans. 975.362

4. If a horse is trotting at 15 miles per hour, how many feet does it go in a second? (It will be found convenient to memorize this result.)
Ans. 22

5. The velocity of light in vacuo is about 300,000,000 meters per second. How much is this in miles per second? *Ans.* 186,000

6. A force of 30 grams acts upon a mass of 2 kilograms. What is the resulting acceleration? *Ans.* 14.7 cm. per sec. per sec.

7. Atwood's machine consists of two weights fastened to opposite ends of a light cord which passes over a light free pulley. If the two weights are 15 pounds and 17 pounds, what is their acceleration?

Ans. 2 ft. per sec. per sec.

7. Uniformly Accelerated Motion.

Up until the time of Galileo, 1564–1642, it was generally believed that the speed with which a body would fall when released near the surface of the earth would be proportional, among other things, to its weight. It is hard for us to understand how such an erroneous idea could be held by learned men when the simplest experiment would have shown it to be false. But it was the custom of that time to attempt to settle such questions by philosophical reasoning and direct experiment was not regarded as a proper philosophical argument. Galileo himself was one of the earliest of scholars to state clearly the principle that science must be founded on experimental facts. From 1588 to 1591 he carried out from the leaning tower of Pisa a series of experiments the results of which we may sum up by the statement,

All bodies falling freely near the surface of the earth have the same acceleration.

If we recall the equation, § 6,(2),

$$mj = \lambda f$$

for Newton's second law for rectilinear motion of a particle of constant mass we see that if j is thus a constant then f must be a constant for any given body. For this reason we take as the first case of rectilinear motion of a particle which we propose to study the simple and important case in which the force acting is a constant. The function φ of equation § 6, (3) being then a constant the equation of motion of the particle reduces simply to,

$$\frac{\quad}{dt^2} = \frac{dv}{dt} = j = \frac{\lambda f}{m}, \tag{1}$$

where f, and consequently j, are understood to be constants.

To integrate this differential equation we seek to express s as such a function of t that it and its derivatives expressed in terms of t will satisfy the equation identically. To do this we here employ the device known as *separation of the variables,* writing the equation as,

$$dv = j\,dt. \tag{2}$$

We may now form the definite integrals of the two members of the equation separately just as is done in the integral calculus, it being only necessary to take care that the limits of integration in the two cases are corresponding values of the two variables v and t. Thus if v_0 be the value taken on by v when t has some value t_0 and if v_1 be the value taken on by v when t has some other value t_1 then equation (2) must yield,

$$\int^{v_1} dv = \int^{t_1} j\, dt,$$

since essentially the same operation has been performed on the two members of equation (2). In practice, however, we shall find it convenient to let t_0 be zero and to write v_1 and t_1 as simply v and t since this simplifies the notation and causes no confusion. We therefore write,

$$\int_{v_0}^{v} dv = \int_{0}^{t} j\, dt,$$

and this gives on evaluation,

$$v = jt + v_0, \tag{3}$$

it being borne in mind that j is in this case a constant. This equation is called a *first integral* of our given differential equation (1). To get the *second integral* we replace v in equation (3) by its value $\frac{ds}{dt}$ and again separate the variables obtaining,

$$ds = (jt + v_0)\, dt.$$

We now again integrate between corresponding limits,

$$\int^{s} ds = \int^{t} (jt + v_0)\, dt,$$

it being understood that s and s_0 are the values assumed by s when t takes on the values t and 0, respectively. Evaluating these integrals gives us our second integral of the given equation (1),

$$s = \tfrac{1}{2}jt^2 + v_0 t + s_0. \tag{4}$$

Equations (3) and (4) would enable us to discuss completely any

case of uniformly accelerated motion, but an additional equation obtained from these two by elimination of t between them is also very useful,

$$v^2 - v_0^2 = 2j\,(s - s_0). \tag{5}$$

Galileo put the conclusions of this article into three easily visualized theorems given in Problem 7 of this article.

Let us apply the above results to the following problem.

A professional diver weighing 150 pounds dives into a tank four feet deep from a height of 36 feet above the water. Assuming that the water exerts a constant retarding force on the diver, determine how much this force is and how long it acts if the diver just fails to strike the bottom of the tank.

Let us take downward as the positive sense and as usual take 32 feet per second per second as the acceleration of gravity. Then for the motion in the air we have,

$$j = 32, \qquad v_0 = 0, \qquad s - s_0 = 36$$

and equation (5) yields, $v = 48$. For the motion under water we therefore have,

$$v_0 = 48, \qquad v = 0, \qquad s - s_0 = 4$$

and equation (5) gives us, $j = -288$. To determine the time in which the diver is brought to rest we substitute

$$v = 0, \qquad v_0 = 48, \qquad j = -288$$

into equation (3) obtaining $t = \frac{1}{6}$. To obtain the amount of the force exerted during this time we recall equation, § 6,(2),

$$mj = \lambda f.$$

For $m = 150$, $j = -288$, $\lambda = 32$ we find $f = -1350$ pounds. This force however is the sum of two forces, the pressure of the water upward and the 150 pound weight of the man acting downward. The total force exerted by the water is thus 1500 pounds, $\frac{3}{4}$ of a ton!

EXERCISES

In these problems assume that the force of gravity near the earth's surface will produce a downward acceleration represented by the letter g which has a value about 32 when the foot and second are the units of space and time.

1. A bullet is shot vertically upward with an initial velocity of 1200 feet per second.

(a) How high will it ascend?
(b) What is the velocity at a height of 16,000 feet?
(c) When will it reach the ground again?
(d) At what time is it at a height of 16,000 feet?

Ans. 22,500, 645.0, 75, 17.34 and 57.66

2. A stone is dropped from a balloon while ascending at the rate of 25 feet per second and reaches the ground in 6 seconds. What was the height of the balloon when the stone was dropped? *Ans.* 426

3. A bale is hoisted to a window 32 feet from the ground by a rope which can give it an acceleration of not over 8 ft./sec.^{2} when working against the force of gravity. What is the minimum time in which the bale can be hoisted from the ground to come to rest at the window? *Ans.* 3.162

4. A stone is dropped down a well and t seconds later the sound of the splash is heard. If the velocity of sound is c, how far down is the surface of the water? Express the result in terms of c, g and t and evaluate for $g = 32$, $c = 1100$, $t = 6$. *Ans.* 493.1

5. Two drops of water fall over Niagara Falls, 167 feet, the second starting when the first has gone 1 inch. How far apart are they when the first lands? *Ans.* 7.378 ft.

6. A certain locomotive can just keep a 3000 ton train moving up a 1% grade. How long would it take this locomotive to get this train up to a velocity of 15 miles per hour on the level? If the draw-bar on the locomotive weighs 50 pounds, how much harder does the locomotive pull on the draw-bar than the draw-bar pulls on the train? *Ans.* 68.75 sec., 0.5 lb.

7. Galileo first formulated his observations on falling bodies into three theorems:

I. The velocities acquired at the ends of the successive seconds increase as the natural numbers, 1, 2, 3, $\cdots$.
II. The distances covered during the successive seconds increase as the odd numbers, 1, 3, 5, $\cdots$.
III. The total distances covered from the beginning to the ends of the successive seconds increase as the squares of the natural numbers, 1, 4, 9, $\cdots$.

Prove that these theorems follow if the acceleration is constant.

8. Obtain equation (5) of this article by a single integration of the differential equation, $j = \text{const.}$

9. A train weighing 1000 tons and running at the rate of 60 miles an hour is brought to rest by its brakes after running $\frac{1}{4}$ mile. With what force does the train push forward on the rails during this time? *Ans.* 91.67 tons

10. A runner weighing 150 pounds starting from rest attains a speed of 30 feet per second in 3 seconds. With what average force does he push backward on the ground during this time? *Ans.* 46.88 lbs.

11. A body is projected vertically upward with a velocity which will carry it to a height $2g$; show that 3 seconds after being projected it is descending with a velocity g.

8. Acceleration Inversely Proportional to Square of the Distance.

In 1687 Sir Isaac Newton enunciated the law of universal gravitation which may be expressed as follows:

Every particle of matter in the universe attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of the distance between them.

Thus two particles of masses m_1 and m_2 and separated by a distance r would attract each other with a force of amount,

$$f = k^2 \frac{m_1 m_2}{r^2},$$

where k^2 is the constant proportionality factor. This constant of gravitation k^2 is so small that its experimental determination is very difficult. The value obtained * is,

Absolute c.g.s. system, $k^2 = 6.670 \times 10^{-8}$,
Absolute English system, $k^2 = 1.066 \times 10^{-9}$.

Newton proved in 1685 that if the above law of gravitation holds then a sphere whose density at any point depends only on the distance of that point from the center attracts any exterior particle exactly as if the entire mass of the sphere were concentrated at its center. The sun and planets are nearly spheres and we may accordingly apply the law of gravitation to them as if they were particles.

At present we shall apply the law of gravitation only to the following simple case. Let a particle of mass M be thought of as fixed at a point O. We may do this by assuming the mass M to be so large as to be practically unmoved by the forces to be considered, or else rigidly attached to such a large mass. Now let a particle P of mass m move along a straight line through O subject to the gravitational

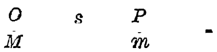

FIG. 4

* *A Redetermination of the Constant of Gravitation*, P. R. Heyl, *Bureau of Standards Journal of Research*, Vol. 5, No. 6 (1930).

attraction of the mass M. The second law of motion, $mj = \lambda f$, and the law of gravitation,

$$f = k^2 \frac{Mm}{s^2},$$

give us the equation of motion of P,

$$(1) \qquad j = -\frac{\mu}{s^2}, \qquad \mu = \lambda k^2 M,$$

the minus sign being due to the fact that since the force acting between the bodies is an attraction j is opposite in sense to s. If the force be a repulsion obeying the same law as to its amount we have only to regard μ as negative. If we now replace j in equation (1) by $\frac{dv}{dt}$ and multiply both members by $2v\,dt = 2\,ds$ the variables are separated and we may integrate the two members between corresponding limits obtaining,

$$\int^{v} 2v\,dv = -2\mu \int^{s} \frac{ds}{s^2}$$

or

$$(2) \qquad v^2 - v_0^2 = 2\mu \left(\frac{1}{s} - \frac{1}{s_0} \right)$$

as our first integral.

We may at once derive an interesting conclusion from this equation. If the particle P fall from rest toward O until it arrives at a certain distance s from O the velocity which it will then have can not exceed a certain limit no matter how far distant P may have been from O when it started to fall. For if we set $v_0 = 0$ then no matter how large s_0 be taken the value of v can not quite attain the value it would have for $1/s_0 = 0$. This limiting value for v is called the *velocity from infinity* at the distance s and is given by the formula,

$$(3) \qquad v_\infty = \sqrt{\frac{2\mu}{s}}.$$

To get the second integral we first separate the variables in

equation (2), writing it as,

$$dt = \pm \frac{ds}{\sqrt{2\mu/s - (2\mu/s_0 - v_0^2)}},$$

where the $+$ sign is used if P is moving to the right and the $-$ sign if P is moving to the left. If we then set,

$$2\mu/s_0 - v_0^2 = \mu/a, \qquad \text{i.e.} \qquad a = \frac{\mu s_0}{2\mu - v_0^2\, s_0};$$

and integrate the last equation between corresponding limits we obtain,

$$(4) \qquad \int^{t} dt = \pm \sqrt{\frac{a}{\mu}} \int \cdot \qquad s\, ds$$

When a/μ is positive we may evaluate the integral in the second member by making the change of variables,

$$s = a\,(1 + \sin\theta),$$

obtaining the result,

$$(5) \qquad t = \pm \sqrt{a/\mu} \left[a \arcsin \frac{s-a}{a} - \sqrt{2as - s^2} \right]_{s_0}^{s}$$

Evaluating the integral in the second member of equation (4) in the cases when a/μ is negative and when $2\mu - v_0^2\, s_0 = 0$ is left as an exercise for the reader.

As an application of the above discussion let us consider the following problem.

Two homogeneous spheres of diameter 1 foot and mean density that of the earth have their surfaces 1 foot apart. If they are acted upon only by their mutual attraction, how long will it take them to come together? (Assume the radius of the earth to be, $R = 3960$ miles.)

In the formula for the force of gravitation,

$$f = k^2 \frac{m_1\, m_2}{r^2},$$

if we let m_1 be the mass of the earth, m_2 the mass m of one of the above spheres and r the radius R of the earth, then since the masses of the earth and of the sphere are proportional to the cubes of

their diameters we have,

$$m_1 = 8R^3\, m,$$

while at this distance R from its center the earth of course exerts a force mg on the sphere. Substituting these values into the formula for f yields,

$$k^2 = \frac{g}{8Rm}.$$

Hence for the attraction of the two spheres upon each other we have,

$$f = k^2 \frac{m^2}{s^2} = \frac{gm}{8Rs^2},$$

where s is the distance between their centers. In the case previously discussed one of the bodies was fixed and their distance s was affected only by the motion of the other. Now, however, s is affected by the motion of both spheres and since these are equal we have,

$$m \frac{d^2s}{dt^2} = -\,2f \qquad (\lambda = 1),$$

and combining the last two equations gives,

$$\frac{d^2s}{dt^2} \quad \cdot \frac{\mu}{s^2} \qquad \text{where} \qquad \mu = \frac{}{4R}\cdot$$

With this value for μ and with $s_0 = 2$, $s = a = 1$, formula (5) gives us,

$$t = \sqrt{\frac{4R}{g}}\left(1 + \frac{\pi}{2}\right) = 4156 = 1 \text{ hr. } 9 \text{ min. } 16 \text{ sec.}$$

EXERCISES

1. Find the velocity with which a body would arrive at the surface of the earth if dropped from a height equal to the earth's radius. Determine also the time of falling. Take $g = 32$, $R = 3960$ miles.
Ans. 4.899 miles per sec., 34 min. 38 sec.

2. Show that at the surface of the earth the velocity from infinity is $v_\infty = \sqrt{2gR}$ and compute it in miles per second to three significant figures. *Ans.* 6.93

3. A meteorite is approaching the earth with the velocity it would have acquired in falling from infinity. Find how long it would take to fall the last 1000 miles. *Ans.* 2 min. 33.1 sec.

4. The constant of gravitation in the absolute c.g.s. system is $k^2 = 6.670 \times 10^{-8}$. Taking the radius of the earth as 6.37×10^8 cm. and g as 981, compute the mean density ρ of the earth to three significant figures. *Ans.* 5.51

5. Find how long it would take the earth to fall to the sun if its orbital velocity were arrested. Take the acceleration of gravity at the sun's surface as 905 feet per second per second, its diameter as 860,000 miles and its distance as 93,000,000 miles. Neglect the diameter of the earth and regard the sun as fixed.
Ans. 64.74 days

6. It is found that most comets travel in their orbits about the sun with the velocity at each point which they would have acquired in falling from infinity to that point. With what velocity would such a comet cross the earth's orbit? Use the data of Problem 5.
Ans. 26.11 miles per sec.

7. We have seen that under an acceleration inversely proportional to the square of the distance to a fixed point the time of the motion is given by the formula,

$$t = \pm \sqrt{\frac{a}{\mu}} \int^{s} \frac{s\,ds}{\sqrt{2as - s^2}} \qquad \text{where} \qquad a = \frac{\mu s_0}{2\mu - v_0^2 s_0}.$$

We have evaluated this for the case $a/\mu > 0$. Obtain corresponding formulas for the case $a/\mu < 0$ and for the case $2\mu - v_0^2 s_0 = 0$.

8. If a particle fall from rest to a fixed attracting particle, find the ratio of the time required for the first half of the fall to the time required for the second half. *Ans.* $\dfrac{\pi + 2}{\pi - 2}$

9. Continuing the discussion of the text show that if we call,

$$\frac{s_0 - s}{s} = \epsilon$$

we have,

$$v^2 - v_0^2 = (1 + \epsilon)\, 2j_0\,(s - s_0).$$

What approximate formula have we therefore for the case where $s_0 - s$ remains very small in comparison with s? Mention such a case.

10. The moon's mass is 1/81 that of the earth and the distance between their centers is 237,000 miles. If a body start from the position of equilibrium between the earth and the moon and be given a slight displacement towards the earth with what velocity will it strike the earth's surface? Take $g = 32$, $R = 4000$.
Ans. 6.898 miles per sec.

9. Acceleration Directly Proportional to the Distance.

If an elastic body be distorted from its state of equilibrium a certain force is required and if the distortion is not too great it has been found that the force is approximately proportional to the distortion. Briefly we may say, *The stress is proportional to the strain*, a statement known as *Hooke's Law*. Examples of this are the stretching of a rubber string, the compression of a steel spring, the bending of a steel rod, etc. In such cases we may write,

$$f = ks$$

where f is the force applied, s is the distance of some point of the body from its position of equilibrium and k is a constant proportionality factor.

Now let us consider the motion of a particle P along a straight line when the particle is acted upon by a force directly proportional to the distance s of P from some fixed point O of the line and where we shall assume the force to be always directed toward O. From what has been said above it is evident that such a force might be produced by the distortion of some elastic body attached to the particle P where O is the position of equilibrium of P. The above equation and Newton's second law,

$$mj = \lambda f,$$

give us as the equation of motion of the particle,

$$(1) \qquad j = -\mu^2 s \qquad \text{where} \qquad \mu^2 = \frac{\lambda k}{m},$$

the minus sign being due to the fact that f is always opposite in sense to s. To get a first integral of equation (1) we replace j by $\frac{dv}{dt}$ and multiply both members of the equation by $2v\,dt = 2\,ds$ thus separating the variables and giving us,

$$2v\,dv = -\,2\mu^2 s\,ds.$$

We then integrate the two members between corresponding limits obtaining as our first integral of equation (1),

$$\int_{v_0}^{v} 2v\,dv = -\,\mu^2 \int_{s_0}^{s} 2s\,ds$$

or

$$(2) \qquad v^2 - v_0^2 = -\mu^2 (s^2 - s_0^2).$$

To obtain the second integral we solve this for v,

where $\quad a^2 = \mu^2 s_0^2 + v_0^2$

and once more separate the variables obtaining after integration between corresponding limits,

$$\int_0^t dt = \pm \frac{1}{\mu} \int_{s_0}^{s} \frac{ds}{\sqrt{a^2 - s^2}}$$

or

$$(3) \qquad t = \pm \frac{1}{\mu} \left(\arcsin \frac{s}{a} - \arcsin \frac{s_0}{a} \right).$$

This is our second integral and completes the theoretical discussion of this problem, but it is of interest to note that if we count time from the instant that the body passes through O in the positive sense our equations (3) and (2) take on the simple forms,

$$(4) \qquad s = a \sin \mu t, \qquad v = a\mu \cos \mu t.$$

It is evident from these last equations that if we observe the values of s and v at any instant and then consider an instant $2\pi/\mu$ later, the angle μt will have increased by 2π and consequently s and v will have resumed their former values. We express this fact briefly by saying that the motion is *periodic*, the interval $2\pi/\mu$ being called the *period*. This particular type of periodic motion is termed *simple harmonic* and is of the greatest importance in physics and applied mechanics. The constant a is called the *amplitude* of the simple harmonic motion.

As an interesting application of the above theory we have the following problem. Newton first proved that if a body be below the surface of the earth it is in effect attracted only by that portion of the earth lying nearer the center than it does, and by that portion as if its entire mass were concentrated at the center. If we assume the earth to be of uniform density ρ it thus appears that the attraction on a unit mass at a distance s from the center

and below the surface would be,

$$f = k^2 \frac{4\pi s^3}{3} \rho \frac{1}{s^2} = \left(\frac{4\pi k^2 \rho}{3}\right) s,$$

and consequently directly proportional to the distance s from the center. If we then imagine the earth at rest and a hole bored through along a diameter, a body falling in a vacuum in this hole would obey equation (1) and the motion of the body could therefore be determined from equations (2) and (3).

As another application of the discussion in this section let us consider the following problem.

A buoy consisting of a cylindrical spar 12 feet long and having one leaded end floats with 4 feet out of water. If pushed under water until just submerged and then released, how high will the end jump out of water? How fast will it be going when the end is 2 feet out of water and when will this first occur?

Let us indicate by x the length of the spar protruding from the water at any instant and call,

$$4 - x = s.$$

We may consider that two forces act on the spar, the force of gravity acting downward and the buoyancy of the water acting upward. Since the first of these tends to increase s we shall consider it as positive and the second as negative. The buoyancy will be proportional to the length of the spar under water. Hence we have,

$$f = mg - k(12 - x).$$

For $x = 4$ we have $f = 0$ and consequently, $k = mg/8$. Thus,

$$mj = f = -\frac{mg}{8}(4 - x)$$

or,

$$j = -\mu^2 s \qquad \text{where,} \qquad \mu^2 = \frac{g}{8} = 4.$$

This being exactly equation (1) of this section we know that the motion of the spar is simple harmonic and we may employ equations (2) and (3). From,

$$v^2 - v_0^2 = -\mu^2 (s^2 - s_0^2)$$

we have,

$$s = -4, \qquad x = 8.$$

To get the time required for this motion we have from,

$$t = \pm \frac{1}{\mu}\left(\arcsin\frac{s}{a} - \arcsin\frac{s_0}{a}\right), \quad \mu = 2, \quad s_0 = a = 4, \quad s = -4,$$

the conclusion that,

$$t = \pi/2 = 1.571.$$

For $s = 2$ we have from the above equations and values,

$$v = 4\sqrt{3} = 6.928, \qquad t = \pi/6 = 0.5236.$$

EXERCISES

1. A hole is bored through the center of the earth and a stone is dropped in. Find how long it will take the stone to reach the center and how fast it will be going when it gets there. $g = 32$, $R = 3960$ miles. *Ans.* 21 min. 10 sec., 4.899 miles per sec.

2. A particle moves on a straight line attracted by several (three) centers of force on that line, the magnitude of each force being proportional to the distance from the corresponding center but with different proportionality factors. Show that the motion of the particle is simple harmonic.

3. A horizontal shelf is moving up and down with a simple harmonic motion of amplitude 2 feet. What is the greatest number of cycles per second that it can make without objects being dislodged from the shelf? $g = 32$. *Ans.* 0.6366

4. A rubber cord 3 feet long is fastened at one end and a weight attached that stretches the cord to a length of 5 feet. If now the weight be pulled down 2 more feet and released, how far from the point of support will it be in $\pi/12$ seconds? How fast will it be going and what is the period of vibration? $g = 32$.
Ans. 6, $-$ 6.928, 1.571

5. If the weight in Problem 4 be dropped from the point of support how far will it descend? *Ans.* 9

6. Discuss the motion of a particle repelled from a fixed point with a force proportional to its distance from that point. Show in particular that if we count time from the instant that the particle passes through the fixed point going in the positive sense, that the equation of motion, $j = \mu^2 s$ will yield,

$$v = a\mu \cosh \mu t, \qquad s = a \sinh \mu t,$$

where $a = v_0/\mu$.

7. A cylinder floating in water in a vertical position is observed to oscillate vertically with a period of 2 seconds. If the radius is 5 cm. how much does the cylinder weigh? $g = 980$.

Ans. 7799. gms.

10. Motion in a Resisting Medium.

When a body moves through the air or water these media oppose a resistance to its passage which depends upon the velocity, but the expression of this resistance as a function of the velocity has been found to be very difficult. Especially is this true for velocities near that of sound in the medium, the resistance having been found to vary in an extremely erratic fashion for such velocities. Certain simple formulas for the resistance however are satisfactory for velocities in limited ranges and in particular for low velocities up to about 10 miles an hour *the resistance can be taken as proportional to the velocity,*

$$f = av. \tag{1}$$

For higher velocities, but still well below the velocity of sound in the medium, the formula,

$$f = av^2, \tag{2}$$

gives a sufficient approximation for many cases that arise.

Let us discuss the motion in the first case, assuming that the resistance of the medium is the only force acting. The equation of motion is evidently,

$$mj = - av, \tag{3}$$

where a is a positive constant and the minus sign is due to the fact that the resistance always has the opposite sense from v. We may at once separate the variables obtaining,

$$dt = - \frac{m}{a}\frac{dv}{v},$$

and after integrating between corresponding limits find,

$$\int^{t} dt = \int^{v} - \frac{m}{a}\frac{dv}{v}$$

or,

$$t = \frac{m}{a} \log \frac{v_0}{v} \qquad \text{or} \qquad v = v_0\, e^{-at/m} \tag{4}$$

From this first integral of our equation of motion it would appear that the body would never come to rest, but would continue to move more and more slowly.

To get s in terms of t we write equation (4) as,

$$ds = v_0 \, e^{-at/m} \, dt,$$

thus separating the variables and upon once more integrating between corresponding limits we have,

$$\int^{s} ds = \int^{t} v_0 \, e^{-at/m} \, dt$$

or,

$$s = \frac{mv_0}{a}\left(1 - e^{-at/m}\right) + s_0. \tag{5}$$

Evidently the distance $s - s_0$ passed over by the body will gradually approach the value mv_0/a but never quite reach it. If we eliminate t between equations (4) and (5) we obtain the useful simple relation,

$$s - s_0 = \frac{m}{a}(v_0 - v). \tag{6}$$

In applying any of these formulas to physical problems we must not forget that since the law, $f = av$ holds only approximately, our conclusions are likewise only approximately true of the real motion.

If the resistance of the medium varies as in equation (2) and if no other force is acting the equation of motion is evidently,

$$mj = - av^2, \tag{7}$$

where a is a constant having the sign of the velocity. After separating the variables and integrating between corresponding limits as usual we find for the first integral,

$$t = \frac{m}{a}\left(\frac{1}{v} - \frac{1}{v_0}\right. \qquad \text{or} \qquad v = \frac{}{A + Bt} \tag{8}$$

where,

$$A = \frac{1}{v_0}, \qquad B = \frac{a}{m}.$$

It thus appears that as in the preceding case v will become zero

only after an infinite time, i.e. the body will move more and more slowly but never come to rest. The reader will find no difficulty in again separating the variables in equation (8) and will find after integrating between corresponding limits,

$$(9) \qquad s - s_0 = \frac{1}{B} \log \left(\frac{A + Bt}{A} \right) = \frac{m}{a} \log \left(1 + \frac{av_0}{m} t \right).$$

If we could be sure that the assumed law for the retardation were accurately obeyed we might here conclude that the body would cover any desired distance if sufficient time were allowed, although the time required would become enormous for large distances.

EXERCISES

1. Prove the relation, $s - s_0 = \dfrac{m}{a}(v_0 - v)$ directly from the differential equation of the motion, $mj = - av$.
2. A steady pull of 3 pounds maintains a speed of 3 miles per hour in a 300 pound boat. If the pull be stopped how far will the boat glide during the next 20 seconds? Obtain an answer on each of the two assumptions, $f = av$ and $f = av^2$. *Ans.* 46.37 ft., 54.33 ft.
3. A steady pull of 3 pounds maintains a speed of 3 miles per hour in a 300 pound boat. If the pull be stopped, what will be the velocity of the boat after drifting 20 seconds; after drifting 20 feet? *Ans.* 2.945 ft. per sec., 1.027 ft. per sec.
4. Discuss the motion of a particle acted upon only by the resistance of the medium if the resistance is proportional to the cube of the velocity.
5. Show that any motion in which,
$$t = as^2 + 2bs + c \qquad (a, b, c \text{ constants})$$
could be brought about by the action of a medium on a freely moving body if the resistance is proportional to the cube of the velocity.

11. Motion under an Attracting Force with Damping.

In the preceding article we discussed the rectilinear motion of a body in a resisting medium but we confined our attention to the case in which the retardation of the medium could be regarded as the only force acting. A much more important problem is that in which this force is combined with some other. The simplest case of this is that in which the additional force is constant, a case which we encounter for instance in the vertical motion of a body through the air near the surface of the earth,

or in the motion of a boat with constant engine speed. If we assume the retardation of the medium to be proportional to the square of the velocity the equation of motion of the body will be,

$$(1) \qquad mj = a - bv^2,$$

where a and b are constants. It will be convenient to choose as the positive sense along the line of motion the sense in which the constant force acts. Thus a will always be a positive quantity while b will be positive only when the motion is in this positive sense. Let us discuss the case when b is positive. As usual we may separate the variables in equation (1) obtaining,

$$\frac{a}{m}\,dt = \frac{dv}{1 - \mu^2 v^2} \qquad \text{where} \qquad \mu^2 = b/a$$

and upon integrating between corresponding limits we find,

$$\frac{a}{m}\int_0^t dt = \int_{v_0}^{v} \frac{dv}{1 - \mu^2 v^2}$$

or,

$$\frac{2a\mu}{m}\,t = \log\left(\frac{1 + \mu v}{1 - \mu v}\right) - \log\left(\frac{1 + \mu v_0}{1 - \mu v_0}\right).$$

If for convenience we call,

$$1/2 \log \frac{1 + \mu v_0}{1 - \mu v_0} = k, \qquad \frac{\sqrt{ab}}{m} = l$$

and solve the above equation for v we find as the first integral of equation (1),

$$(2) \qquad v = \frac{1}{\mu}\frac{e^{(k+lt)} - e^{-(k+lt)}}{e^{(k+lt)} + e^{-(k+lt)}} = \sqrt{\frac{a}{b}}\tanh(k + lt).$$

We may now read off from this equation an interesting property of the motion; namely that after a long interval of time the velocity of the body very nearly approaches the value,

$$v_\infty = \sqrt{a/b},$$

a quantity entirely independent of the initial velocity v_0. This follows at once from the fact that for large values of t the negative powers of e in the above expression for v become very small

at the same time that the positive powers are becoming very large so that v approaches the value $1/\mu$.

To complete the integration of equation (1) we write equation (2) in the form,

$$ds = \sqrt{a/b}\tanh(k + lt)\, dt$$

and integrate between corresponding limits obtaining,

$$\int_{s_0}^{s} ds = \sqrt{a/b}\int_{0}^{t}\tanh(k + lt)\, dt$$

or,

$$s = s_0 + \frac{m}{b}\log\left(\frac{\cosh(k + lt)}{\cosh k}\right) \tag{3}$$

as our second integral.

As an example of motion of this type let us consider the following.

A mouse weighing 1 ounce falls from a height of 36 feet. Compare his velocity on landing with that of a man falling from the same height. Assume that the mouse presents 2 square inches to the retardation of the air, which retardation is proportional to the square of the velocity and is 2 pounds per square foot of surface at a velocity of 30 feet per second. Assume that the retardation of the air will not materially affect the motion of the man in this distance.

The velocity of the man is at once given by the formula, § 7,(5),

$$v^2 - v_0^2 = 2j\,(s - s_0),$$

as 48 feet per second, while the equation of motion of the mouse is our equation,

$$mj = a - bv^2, \tag{1}$$

in which in our case $m = 1/16$, $a = mg = 2$, $b = 2/2025$. It now appears that neither formula (2) nor (3) is exactly what we need in the present problem for we require a relation between the distances and the velocities. We could of course obtain such a relation by eliminating t between equations (2) and (3) but it may be easily derived directly from equation (1). If we replace j by $v\dfrac{dv}{ds}$ we may at once separate the variables and on inte-

grating between corresponding limits obtain,

$$\int_{v_0}^{v} \frac{v\,dv}{a - bv^2} = \int_{s_0}^{s} \frac{1}{m}\,ds$$

or,

$$\log\left(\frac{a - bv^2}{a - bv_0^2}\right) = -\frac{2b}{m}(s - s_0).$$

If we count s downward from the point where the mouse started to fall we have $s_0 = v_0 = 0$ and our last equation takes the form,

$$v^2 = \frac{a}{b}(1 - e^{-2bs/m}).$$

Evaluating this for $s = 36$ we have,

$$v^2 = 2025\,(1 - e^{-256/225}), \qquad v = 37.09.$$

Thus in falling only 36 feet the mouse has already acquired 82% of his limiting velocity of 45 feet per second, while the man in falling 36 feet is already going faster than the mouse would go no matter how far he fell.

The discussion of the case in which the motion is opposed to the constant force and in which b is therefore negative follows very similar lines. The details are left to the reader.

We discussed in § 9 the motion of a particle under the action of a force proportional to the distance from a fixed point. Let us now consider in addition to that force the resistance of the medium, supposing it to be proportional to the distance. In amount this gives us

$$f = ks + lv,$$

where k and l are constants, and if we consider the first force an attraction toward the origin while the second is a resistance then the equation of motion of the particle is,

$$\text{(4)} \qquad mj = -ks - lv \qquad (\lambda = 1).$$

If we write this in the form,

$$\text{(5)} \qquad \frac{d^2s}{dt^2} + 2\rho\frac{ds}{dt} + \mu^2 s = 0 \qquad \rho = \frac{l}{2m}, \qquad \mu^2 = \frac{k}{m},$$

we recognize that we have here a homogeneous linear differential

equation of the second order with constant coefficients. This type of differential equation is readily integrated by any one of a variety of methods and in fact we may show by direct substitution that it is satisfied by,

$$s = Ae^{-\rho t} \sin \left(\sqrt{\mu^2 - \rho^2}\, t - \alpha \right),$$

no matter what values the two constants A and α may have. To visualize the motion in the case $\mu > \rho$ we consider separately the two factors $e^{-\rho t}$ and $\sin (\sqrt{\mu^2 - \rho^2}\, t - \alpha)$. If only the first factor were to vary s would diminish constantly toward zero as t increased but would never quite reach zero. If only the second factor were to vary we recognize by comparison with equation, § 9, (4) that the motion would be simple harmonic with a period $2\pi/\sqrt{\mu^2 - \rho^2}$ and amplitude $Ae^{-\rho t}$. Since both factors actually vary we may roughly describe the motion as being simple harmonic with a continuously diminishing amplitude. We agree to continue to call the quantity $2\pi/\sqrt{\mu^2 - \rho^2}$ the *period* although the motion is not strictly periodic. In the case of any individual motion of this type the constants A and α would have to be determined by the conditions of the problem.

As an example let us consider the following.

A rubber cord is supported at one end and a weight attached to the other end that stretches the cord 2 feet. If the weight is then pulled down 1 more foot and released it is found that at its tenth descent after release it comes down only 6 inches below the position of equilibrium. Determine the period of the motion and compare with what it would be in a vacuum.

Let us indicate by s the distance of the weight below the position of equilibrium. There are three forces acting on the weight, the force of gravity mg acting downward and therefore to be regarded as positive, the pull of the cord which will be proportional to the amount it is stretched and acting upward and the friction of the air which is proportional to the velocity and oppositely directed. We may therefore write as the equation of motion of the weight,

$$mj = - k\,(s + 2) + mg - lv.$$

If the weight were at rest at the position of equilibrium we would have $j = s = v = 0$ and consequently $k = mg/2$. We may there-

fore write the equation of motion as,

$$\frac{d^2s}{dt^2} + 2\rho\frac{ds}{dt} + \mu^2 s = 0 \qquad \rho = \frac{l}{2m} \qquad \mu^2 = \frac{g}{2}.$$

We have seen that the general solution of this equation is,

$$s = Ae^{-\rho t}\sin(\sqrt{\mu^2 - \rho^2}\,t - \alpha),$$

where A and α are constants. To determine the values of A and α we first observe that by differentiation we have,

$v = Ae^{-\rho t}\{\sqrt{\mu^2 - \rho^2}\cos(\sqrt{\mu^2 - \rho^2}\,t - \alpha) - \rho\sin(\sqrt{\mu^2 - \rho^2}\,t - \alpha)\}$.

At the start of the motion $t_0 = v_0 = 0$, $s_0 = 1$ from which the last two equations give us,

$$A = \frac{\mu}{\sqrt{\mu^2 - \rho^2}}, \qquad \sin\alpha = -\frac{\sqrt{\mu^2 - \rho^2}}{\mu}, \qquad \cos\alpha = \frac{\rho}{\mu}.$$

At the tenth descent we have $t = 20\pi/\sqrt{\mu^2 - \rho^2}$, $s = 1/2$, while the factor, $\sin(\sqrt{\mu^2 - \rho^2}\,t - \alpha)$, resumes its initial value. Hence,

$$e^{-20\pi/\sqrt{\mu^2-\rho^2}} = \frac{1}{2} \qquad \text{or,} \qquad \rho = \frac{\mu\log 2}{\sqrt{(20\pi)^2 + (\log 2)^2}},$$

which for $g = 32$ gives $\rho = 0.044125$. As we have seen the period is given by the formula, $2\pi/\sqrt{\mu^2 - \rho^2}$ while in a vacuum we would have $\rho = 0$ and the period would be $2\pi/\mu$. Performing the numerical calculation yields,

Present period	1.570892
Period in a vacuum	1.570796
Ratio	1.000061

It appears that any moderate amount of resistance in the medium has only an extremely slight effect on the period.

The solution of the equation of motion (5) and the interpretation of the results for the cases $\rho = \mu$ and $\rho > \mu$ is left as an exercise for the reader

EXERCISES

1. Discuss the motion of a body under the equation of motion,

$$mj = a - bv^2$$

for the case of positive a and negative b, i.e. for the case in which the motion is opposed to the constant force.

2. Show that under the conditions of Problem 1 the body will come to rest after traveling a distance,

$$s_0 - s = -\frac{m}{2b}\log\left(1 - \frac{b}{a}v_0^2\right)$$

and after an interval of time,

$$t = \frac{m}{\sqrt{-ab}}\arctan\left(-\sqrt{\frac{-b}{a}}\,v_0\right).$$

3. A bullet of mass 1/200 pound is fired vertically upward from an automatic pistol the muzzle velocity being 900 feet per second. If the resistance of the air is proportional to the square of the velocity and is initially 1 ounce, how high will the bullet rise; how long will it rise? *Ans.* 2635. ft., 10.30 sec.

4. If a body start from rest under the equation of motion,

$$mj = a - bv^2,$$

how long will it take to acquire a velocity r times its limiting velocity? $(r < 1)$.

Ans. $t = \dfrac{m}{2\sqrt{ab}}\log\left(\dfrac{1+r}{1-r}\right)$

5. If the bullet of Problem 3 were dropped from a height, what would its limiting velocity be? How long would it take to acquire a velocity 9/10 of its limiting velocity?

Ans. 254.6 ft. per sec., 11.71 sec.

6. A steamship of 10,000 tons has a limiting velocity of 15 miles per hour, the engines being capable of exerting a force of 25 tons. How long will it take the ship to come from rest to a velocity of 10 miles per hour, assuming the resistances proportional to the square of the velocity. *Ans.* 3 min. 41.3 sec.

7. A rubber cord is supported at one end and a weight attached that stretches the cord 2 feet. The weight is then pulled down 2 more feet and released and the period of the motion observed. Wings of negligible weight are then attached and the experiment repeated, it being found that the period has increased to 25/24 of its previous value. Assuming that the resistance of the air was negligible in the first experiment and proportional to the velocity in the second, compute how far below the position of equilibrium the weight will descend at its first descent after release; at its fourth descent.

Ans. 0.32000 ft., 0.0013106 ft.

8. Show that for the case $\rho > \mu$ the equation,

$$\frac{d^2s}{dt^2} + 2\rho\frac{ds}{dt} + \mu^2 s = 0,$$

is satisfied by,

$$s = Ae^{-mt} + Be^{-nt},$$

where A and B are any two constants and where,

$$m = \rho + \sqrt{\rho^2 - \mu^2}, \qquad n = \rho - \sqrt{\rho^2 - \mu^2}.$$

Describe the resulting motion.

9. A man and his parachute weigh 150 pounds. What must be the diameter of the parachute if the man may entrust himself to it at any height, 25 feet per second being a safe velocity with which to land. Assume that the resistance of the air is proportional to the square of the velocity and is 2 pounds per square foot of surface at 30 feet a second. *Ans.* 11.73 feet

12. Friction.

The resistance which a medium such as the air or water offers to the passage of a body through it are examples of a general class of forces which we ordinarily call *friction*. Another important force coming under this head is the resistance which two surfaces in contact offer to a sliding motion of the one over the other. If the surfaces are dry and hard and fairly smooth this friction is found by experiment to be almost independent of the velocity of the motion but to be directly proportional to the force with which the surfaces press on each other. Thus we have,

$$f = \mu p,$$

where f is the friction, p the normal force between the two surfaces and where the proportionality factor μ is known as the *coefficient of friction*. The coefficient of friction of course depends on the nature of the two surfaces in contact. If μ is so small as to render f negligible in the problem to be considered we say that the surfaces are *smooth*. If not, they are called *rough*.

When the two surfaces in contact are at rest relative to each other the frictional force f is in the nature of a *constraint* (cf. § 63) and has any value not greater than μp which will be just sufficient to maintain the surfaces at rest relative to each other. In theory we usually consider that the friction can never exceed the value μp but actual experiment shows that when the surfaces are at rest relative to each other this value may be slightly surpassed.

For example let us consider the following problem.

An ice boat weighing 600 pounds is driven straight before a 30 mile wind. The force exerted by the wind is proportional to the amount by which the velocity of the wind exceeds the velocity

of the boat and is 60 pounds when the boat is at rest. The coefficient of friction between the ice and the boat is 1/50. Show that as the boat continues to sail its velocity approaches a certain limiting velocity independent of the initial velocity. Compute its value.

The force exerted by the wind being of the form $k\,(30 - v)$ where k is a constant, and having the value 60 for $v = 0$, k must be 2. The force exerted by the friction is the weight of the boat times the coefficient of friction, i.e. $600 \times 1/50 = 12$. Taking account of the opposite senses of these two forces we may write the equation of motion of the boat as

$$600\ddot{\jmath} = g\,\{2\,(30 - v) - 12\}.$$

Separating the variables we have,

$$\frac{dv}{24 - v} = \frac{g}{300}\,dt$$

and on integrating the two members between corresponding limits we find,

$$\int_{v_0}^{v} \frac{dv}{24 - v} = \frac{g}{300}\int_{0}^{t} dt$$

or

$$\log\left(\frac{24 - v}{24 - v_0}\right) = -\frac{g}{300}\,t.$$

If we write this equation in the form,

$$v = 24 - (24 - v_0)\,e^{-gt/300},$$

it becomes evident that as t increases indefinitely the velocity approaches the value 24 miles per hour as a limit.

Many of the problems involving friction in conjunction with other forces may be solved along the above lines.

EXERCISES

1. A small boy saw a slide on the ice ahead and ran for it. He reached it with a speed of 8 miles per hour and slid 15 feet. What was the coefficient of friction between his shoes and the ice? *Ans.* 0.1434

2. An automobile in starting on an icy pavement takes 30 seconds to get up a speed of 5 miles per hour. What is the coefficient of friction between the tires and the pavement? *Ans.* 0.007639

3. A boy is bringing home a block of ice on a sled, the coefficient of friction between the block and the sled being 1/50. What is the shortest time in which he might bring the block to a velocity of 4 miles per hour? *Ans.* 9.167 sec.

4. A flexible chain 2.5 feet long lies in a straight line on a table, the coefficient of friction being 1/4. What is the greatest length of chain which might hang over the edge of the table without the chain starting to slide? What would be the velocity acquired by the chain in sliding off the table if started from rest from a position very slightly beyond this point? If the chain were started from rest with 5/8 of a foot hanging over the edge how long would it take to slide off? *Ans.* 0.5 ft., 11.314 ft. per sec., 0.6125 sec.

5. A block of wood lies on a horizontal table and is attached to a post by an elastic cord whose natural length is l. The coefficient of friction is μ and the cord would be stretched an amount e if the block were hung on it vertically. If the block be drawn out to a distance $s_0 > l$ from the post and projected directly away from it with an initial velocity v_0, where will the block first come to rest? What is the greatest value that v_0 might have without the block starting to return after once coming to rest?

6. A block of wood lies on a horizontal table and is fastened to a post by an elastic cord whose natural length is 1 foot. The coefficient of friction is 1/4 and the cord would be doubled in length if the block were hung on it vertically. The block is set at a distance of 1 foot from the post and projected directly away from it with a velocity of (*a*) 2 feet per second; (*b*) 3 feet per second. Find where the block will finally come to rest.
Ans. Distance from post is (*a*) 1.183 ft., (*b*) 1.164 ft.

CHAPTER III

INTRODUCTION TO VECTORS

13. Vector Quantities.

We have seen in the last chapter that many of the quantities studied in mechanics are scalar quantities, i.e. capable of being characterized by the giving of one real number. Some quantities of an even simpler sort have been encountered, as for instance the mass of a body which is characterized by the giving of a positive number. We shall often encounter however objects of a more complicated sort for which the giving of one real number would form an inadequate characterization. Suppose for instance that we wish to describe the motion of the air at a certain instant at a certain point of the room. It is true that by giving one real number, e.g. $+2$, we might indicate that the velocity of the air is 2 feet per second, but this leaves the direction of the motion and the sense along that direction still entirely undetermined. But by giving three real numbers in a definite order we may by any one of various schemes determine the amount, direction and sense of the motion of the air at the point and so completely characterize it. Any quantity which can thus be completely characterized by the giving of three real numbers in a definite order we shall call a *vector quantity.*

A set of three real numbers in a definite order is called a *triple of numbers.* It should be observed that the order of the numbers is here as important as the numbers themselves. That is, we regard the two triples, $(3, 2, 4)$ and $(4, 3, 2)$, as different although the numbers are the same in the two cases. It is clear from the definition of vector quantity in the previous paragraph that such a triple is itself a vector quantity and may indeed be regarded as a sort of ideal simple vector quantity capable of being employed to characterize other vector quantities just as the number 5 is an abstraction capable of characterizing such quantities as 5 potatoes, 5 dynes, 5 seconds. In this way these triples are like numbers and we therefore propose to extend our number system, already consisting of several kinds of numbers of

varying degrees of complexity, and to make it now include these triples as a new kind of number. These new numbers are called *vectors*.

If such triples of real numbers are to be called vectors and included in the number system the question next arises as to what symbols had best be employed to represent them. Some authors have found it convenient to reserve the Roman letters for scalars and to employ Greek letters or Gothic characters for vectors, but the scheme now coming into general use is to use the ordinary italics for scalars as usual and to represent vectors by the black-faced Clarendon letters. Thus we have,

$$a = 6, \qquad \mathbf{a} \equiv (3, 2, -4).$$

In writing on paper or blackboard this scheme is not practicable and we usually distinguish vectors from scalars by drawing a short line above each letter used to represent a vector.

In dealing with scalar quantities we naturally did not find it necessary to discuss the various operations which are possible with the real numbers representing them since these are already familiar to the reader, and we were able to confine our attention to the interpretation of these operations on the quantities represented. In dealing with these new numbers however we must devote some attention from time to time to the actual operations on vectors before interpreting these in terms of the quantities represented. In the ensuing sections of this chapter we shall therefore concern ourselves with certain vector operations and their interpretation in extremely simple applications.

It can not of course be expected that all the operations of vector analysis will have a significance for all vector quantities. The same thing holds true for scalars. The temperature at a given point and instant is clearly a scalar quantity but we can scarcely imagine an occasion on which we should be called upon to multiply one temperature by another for the product so obtained would seem to have no important significance.

14. Geometric Representation.

We observed in § 4 that the position of a point on a line is a scalar quantity. The geometric representation which the points of a line thus afford the real numbers is so simple, complete and convenient that it has become very firmly established in our

minds and we can scarcely be concerned for any length of time with the real numbers without referring in some way to their corresponding points on a line. In like manner vectors have an extremely simple and convenient geometric representation which we come to practically identify with the vector represented. This graphic representation of a vector is in the form of an *arrow* thought of as having a definite length, direction and sense, but no definite position so that it is free to move provided it remains parallel to its original position. The two ends of the arrow are called its initial and terminal points, the sense indicated by the arrow being from the initial to the terminal point.

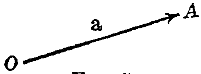

FIG. 5

To set up the correspondence between this arrow and the triple of real numbers constituting the vector we employ a rectangular coördinate system which we regard as fixed in space. We then move the arrow parallel to its original position and place its initial point O on the origin of coördinates. The coördinates of its terminus A then form the triple of real numbers constituting the vector represented. It is clear that conversely being given a triple of real numbers we may find the point A having them as coördinates and thus determine the corresponding arrow OA. The coördinates of A are also called the *coördinates of the vector OA.*

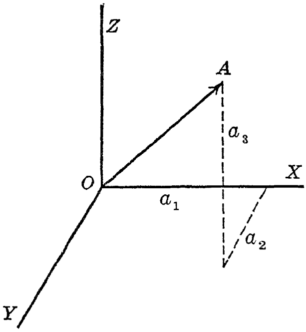

FIG. 6

$$OA = \mathbf{a} \equiv (a_1, a_2, a_3).$$

It will be observed that the geometric vector or arrow fits into a sequence of concepts already familiar. First we have the *line* of elementary geometry which we think of as possessing no definite length but as indicating a certain direction. If we place an arrow head on this line thus indicating a certain sense along it we have an *axis*, and if we cut out a segment of definite length from this axis we have a *vector* provided we do not associate with it any definite position in space.

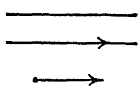
FIG. 7

The graphical representation of a vector gives rise to certain terms connected with it. Thus we have the *length* of the vector **a** represented by $|\mathbf{a}|$ or a and given by the formula,

$$a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

This is a real number either positive or zero and is evidently simply the length of the arrow representing the vector.

Being given a vector $OA = \mathbf{a}$ and a fixed line λ we define the *projection of* **a** *on* λ to be a new vector $O'A' = \mathbf{a}'$ where O' is the foot of the perpendicular let fall from O upon λ and A' is the foot of the perpendicular let fall from A upon λ. It will be observed that we may properly speak of $\mathbf{a}'$ as the projection $\mathbf{p}_\lambda(\mathbf{a})$ of **a** upon λ since if λ is given $\mathbf{a}'$ depends only upon the length, direction and sense of **a** and not upon its position. The projection of a vector upon a plane is defined in an analogous manner.

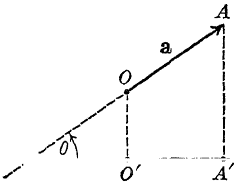

FIG. 8

Being given a vector $OA = \mathbf{a}$ and a fixed axis Δ we define the *measure of* **a** *on* Δ as the length of the projection of **a** on Δ if that projection has the same sense as the axis Δ and as the negative of the length of this projection if the projection has the opposite sense from the axis. The measure of **a** on Δ is thus given by the formula,

$$m_\Delta(\mathbf{a}) = |\mathbf{a}| \cos \theta,$$

where θ is the angle from the positive sense of Δ to the positive sense of **a**. It is evident that the measures of a vector on the three coördinate axes are simply the coördinates of the vector.

In determining the arrow which graphically represents a vector whose coördinates are known we employ a rectangular coördinate system. Such rectangular coördinate systems in space are of two kinds, right handed and left handed as illustrated in the accompanying figure. The right handed system is so called because the thumb and first two fingers of the right hand may be

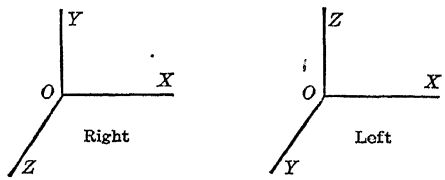

FIG. 9

held so as to point in the direction and sense of the X, Y and Z axes respectively, while the left hand similarly fits a left handed system. It would be convenient if a certain one of these two systems were universally adopted by writers on geometric subjects, but such is unfortunately not the case. In our use of the coördinate system we shall not find it necessary to specify which type has been adopted but we shall assume that one of them has been chosen and retained throughout.

By the use of our chosen coördinate system we may define what is known as the *sense of rotation* of a triple of non-coplanar vectors or axes. The coördinate system being chosen we pick a point on each of the positive coördinate axes, call them X, Y and Z respectively, pass a circle through them and view this circle from the side opposite to that on which the origin lies. We shall see the three points X, Y, Z arranged around the circle in a certain order, clockwise or counter-clockwise. Now being given any triple of non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we place their origins all at some point O and pass a circle through their termini A, B, C and view this circle from the side opposite to that on which their common origin lies. *If the order of the points A, B, C around their circle is the same as the order of the points X, Y, Z around their circle then we say that the triple of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ has a positive sense of rotation.* If not, the sense is said to be negative. This concept of the sense of rotation of a triple of vectors is of importance in certain definitions and theorems to be encountered later.

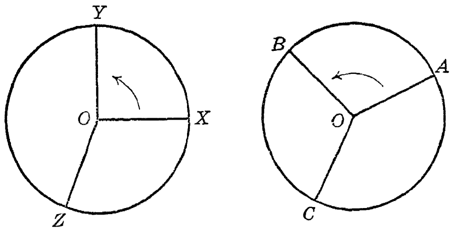

FIG. 10

Closely associated with the sense of rotation of a triple of non-coplanar vectors or axes is the concept of the sense of rotation of a figure relative to a given vector or axis. *The positive sense of rotation relative to the vector $\mathbf{a}$ of the above figure is the sense of the rotation of less than 180° which would carry the plane AOB into the plane AOC, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ being a triple of non-coplanar vectors with a positive sense of rotation.* Thus the positive sense of rotation

relative to the X-axis is the sense in which the XY-plane may be rotated through 90° to bring it into the position of the ZX-plane.

EXERCISES

1. If the triple of vectors **a**, **b**, **c** has a positive sense of rotation, show that the triple **a**′, **b**, **c** has also a positive sense of rotation, where **a**′ is the projection of **a** on a plane perpendicular to **b**; on a plane perpendicular to **c**.

2. If **a**′ is a vector having the direction of the vector **a** but opposite in sense show that if the triple **a**, **b**, **c** has a positive sense of rotation, then the triple **a**′, **b**, **c** has a negative sense of rotation.

15. Elementary Operations.

A vector is said to have the value *zero* if its length is zero. Clearly this is true if and only if every coördinate of the vector is zero. It will be convenient to think of a zero vector as having *any* direction and sense so that it may be regarded as perpendicular to or parallel to any other vector. We shall see that this convention enables us to state certain theorems without the exceptions that would otherwise have to be specified.

Two vectors are *equal* if and only if they have the same coördinates in the same order. Thus the vector equation,

$$\mathbf{a} = \mathbf{b}, \qquad \text{where} \qquad \mathbf{a} \equiv (a_1, a_2, a_3), \qquad \mathbf{b} \equiv (b_1, b_2, b_3),$$

is equivalent to the three scalar equations,

$$a_1 = b_1, \qquad a_2 = b_2, \qquad a_3 = b_3.$$

From the geometric point of view this amounts to the statement that two vectors are equal if and only if they have the same length, direction and sense. As a special case of this we have the fact that any two zero vectors are equal since their lengths are equal, being both zero, and their directions and senses are equal by the convention explained in the previous paragraph.

By the *product* $\mathbf{b} = k\,\mathbf{a} = \mathbf{a}\,k$ of a scalar k and a vector **a** we agree to mean a vector having a length $|k|$ times the length of **a**, a direction the same as that of **a** and a sense the same as that of **a** or the opposite sense according as k is positive or negative. If the vector **a** and the vector $\mathbf{b} = k\,\mathbf{a}$ be both projected on the same line λ then we have $\mathbf{b}' = k\,\mathbf{a}'$ where **a**′ is the projection of **a** and **b**′ is the projection of **b**. The truth of this statement be-

comes obvious on inspection of the accompanying figures, the first being for positive k and the second for negative k. For in the first figure it is evident that if $\mathbf{b}$ differs from $\mathbf{a}$ only in that its

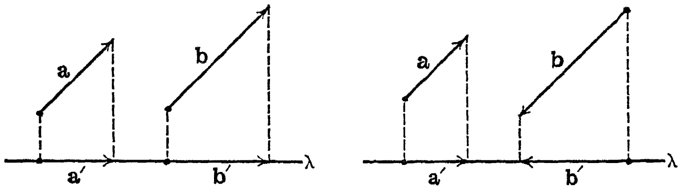

FIG. 11

length is k times that of $\mathbf{a}$ then $\mathbf{b}'$ will differ from $\mathbf{a}'$ only in that its length will be k times that of $\mathbf{a}'$; while from the second figure it appears that if $\mathbf{b}$ also differs in sense from $\mathbf{a}$ then $\mathbf{b}'$ will likewise differ in sense from $\mathbf{a}'$. This statement may be written,

$$(1) \qquad \mathrm{p}_\lambda(k\,\mathbf{a}) = k\,\mathrm{p}_\lambda(\mathbf{a}),$$

showing that the scalar factor k may be transferred across the sign of projection p_λ.

If the vector $\mathbf{a}$ and the vector $\mathbf{b} = k\,\mathbf{a}$ be both measured on the same axis Δ then the *measure of* $\mathbf{b}$ *is* k *times the measure of* $\mathbf{a}$,

$$(2) \qquad m_\Delta(k\,\mathbf{a}) = k\,m_\Delta(\mathbf{a}).$$

The truth of this statement is an immediate consequence of the theorem proven in the previous paragraph. For a complete proof we should discuss four cases depending on whether k is positive or negative and whether $\mathrm{p}_\Delta(\mathbf{a})$ has the same sense as the axis Δ or the opposite sense. Let us consider for example the case in which k is positive and $\mathrm{p}_\Delta(\mathbf{a})$ has the same sense as Δ. Here the equation,

$$\mathrm{p}_\Delta(k\,\mathbf{a}) = k\,\mathrm{p}_\Delta(\mathbf{a}),$$

states that $\mathrm{p}_\Delta(k\,\mathbf{a})$ differs from $\mathrm{p}_\Delta(\mathbf{a})$ only in that the former vector is k times as long as the latter. But the length of $\mathrm{p}_\Delta(k\,\mathbf{a})$ is by definition in this case exactly the measure of $k\,\mathbf{a}$ on Δ and the length of $\mathrm{p}_\Delta(\mathbf{a})$ is by definition the measure of $\mathbf{a}$ on Δ. Hence we have at once the proof of equation (2) for this case. The discussion of the remaining three cases follows similar lines. The details are left to the reader. As an immediate application of this theorem we have the formula for the coördinates of the

product of a given scalar and a given vector,

(3) $\quad k\,\mathbf{a} \equiv (ka_1, ka_2, ka_3) \qquad \text{where} \qquad \mathbf{a} \equiv (a_1, a_2, a_3).$

This follows at once from equation (2) and the fact that the coördinates of a vector are the measures of that vector on the coördinate axes.

Being given two vectors $\mathbf{a}$ and $\mathbf{b}$ we define their *sum* $\mathbf{a} + \mathbf{b}$ as a new vector obtained as follows. Place the initial point of $\mathbf{b}$ on the terminal point of $\mathbf{a}$. Then the vector extending from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$ is defined as the vector $\mathbf{a} + \mathbf{b}$. We may easily prove from this definition that addition of vectors obeys the commutative and associative laws which hold for addition of scalars. That is,

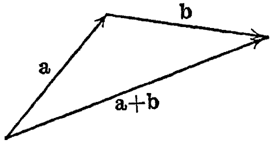

FIG. 12

(4) $\quad \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \qquad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$

For instance, to prove the commutative law we construct both $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{a}$ and show from a study of the figure that these vectors are the same in length, direction and sense.

We define the *difference* $\mathbf{a} - \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ as the sum of $\mathbf{a}$ and -1 times $\mathbf{b}$. A convenient construction for $\mathbf{a} - \mathbf{b}$ is to place the origins of $\mathbf{a}$ and $\mathbf{b}$ together and draw the vector extending from the terminus of $\mathbf{b}$ to the terminus of $\mathbf{a}$. It will be seen that we are here conforming to the definition for this is equivalent to reversing the sense of $\mathbf{b}$ and then adding $\mathbf{a}$ to the resulting vector. The subtraction of two vectors thus involves a combination of the operations of addition of vectors and multiplication of vectors by scalars. The laws of such combinations are readily shown to be the same as if all the quantities involved were scalars. These laws are expressed by the following identities:

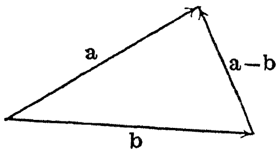

FIG. 13

(5) $\quad k\,(\mathbf{a} + \mathbf{b}) = k\,\mathbf{a} + k\,\mathbf{b}, \qquad (k + l)\,\mathbf{a} = k\,\mathbf{a} + l\,\mathbf{a}.$

If the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ are all projected on the same line λ then their projections are connected by the relation,

(6) $$p_\lambda(\mathbf{a} + \mathbf{b}) = p_\lambda(\mathbf{a}) + p_\lambda(\mathbf{b}).$$

In other words, *projection of vectors on a given line is distributive with respect to addition.* This is an immediate consequence of the *principle of projection* that the projection of a broken line path joining two given points depends only on those two end points and not on the particular path chosen. Thus in the figure the projection on λ of any succession of vectors running from A to C will be a succession of vectors running along λ from A' to C'. The two members of the above equation are therefore equal because they are both equal to the vector $A'C'$. Similarly if we measure the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ on the same axis Δ then their measures are connected by the relation,

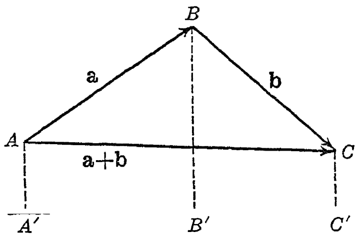

FIG. 14

$$m_\Delta(\mathbf{a} + \mathbf{b}) = m_\Delta(\mathbf{a}) + m_\Delta(\mathbf{b}). \tag{7}$$

That is, *the measure of vectors on an axis is distributive with respect to addition.* To prove this formula we first observe from the definition of measure that any two vectors will have the same measure on an axis if they have the same projection on that axis. Thus if we represent by $\mathbf{a}'$, $\mathbf{b}'$ and $(\mathbf{a} + \mathbf{b})'$ the projections of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} + \mathbf{b}$ on the given axis Δ our formula (7) will be established if we can show that,

$$m_\Delta\{(\mathbf{a} + \mathbf{b})'\} = m_\Delta(\mathbf{a}') + m_\Delta(\mathbf{b}').$$

Now by formula (6) we have $(\mathbf{a} + \mathbf{b})' = \mathbf{a}' + \mathbf{b}'$ so that the equation to be established may be written,

$$m_\Delta(\mathbf{a}' + \mathbf{b}') = m_\Delta(\mathbf{a}') + m_\Delta(\mathbf{b}').$$

But in this formula we have only to do with the measures of different directed line segments all lying on the same axis and the truth of the statement is well known for this case. It is known as *Chasles' Theorem* and lies at the foundations of analytic geometry. Formula (7) is thus established. Since the coördinates of a vector are simply its measures on the three coördinate axes we have from equation (7) the following simple formula for the sum of the two vectors $\mathbf{a} \equiv (a_1, a_2, a_3)$ and

$\mathbf{b} \equiv (b_1, b_2, b_3),$

$$\mathbf{a} + \mathbf{b} \equiv (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3).$$

If along a given axis Δ we draw a vector $\mathbf{u}$ one unit long having the sense of the axis, it follows from the definition of the measure of a vector $\mathbf{a}$ on an axis and from the definition of the product of a scalar and a vector that the projection of $\mathbf{a}$ on Δ is given by multiplying the vector $\mathbf{u}$ by the measure of $\mathbf{a}$ on Δ,

(9) $$\mathbf{p}_\Delta(\mathbf{a}) = \{m_\Delta(\mathbf{a})\}\ \mathbf{u}.$$

Likewise if we draw unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ having the direction and sense of the X, Y, Z-coördinate axes respectively, then since the coördinates of a vector are its measures on these axes we have the relations,

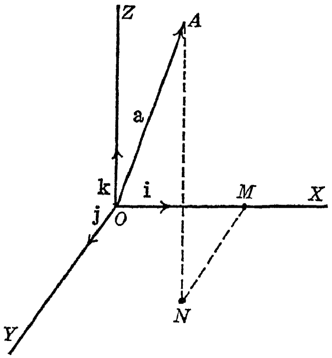

Fig. 15

$$\mathbf{p}_X(\mathbf{a}) = a_1\,\mathbf{i}, \qquad \mathbf{p}_Y(\mathbf{a}) = a_2\,\mathbf{j}, \qquad \mathbf{p}_Z(\mathbf{a}) = a_3\,\mathbf{k}.$$

Now it is evident from the figure that the vector $\mathbf{a} = OA$ is the sum of its projections OM, MN, NA on the three coördinate axes so that we have the following important formula expressing the vector $\mathbf{a}$ in terms of its coördinates and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

(10) $$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}.$$

The laws governing all the operations discussed in this article are summarized in the following formulas.

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a}, & (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \mathbf{a} + (\mathbf{b} + \mathbf{c}),\\ (k + l)\,\mathbf{a} &= k\,\mathbf{a} + l\,\mathbf{a}, & k\,(\mathbf{a} + \mathbf{b}) &= k\,\mathbf{a} + k\,\mathbf{b},\\ \mathbf{p}_\lambda(\mathbf{a} + \mathbf{b}) &= \mathbf{p}_\lambda(\mathbf{a}) + \mathbf{p}_\lambda(\mathbf{b}), & \mathbf{p}_\lambda(k\,\mathbf{a}) &= k\,\mathbf{p}_\lambda(\mathbf{a}),\\ m_\Delta(\mathbf{a} + \mathbf{b}) &= m_\Delta(\mathbf{a}) + m_\Delta(\mathbf{b}), & m_\Delta(k\,\mathbf{a}) &= k\,m_\Delta(\mathbf{a}). \end{aligned}$$

Formal proofs of these have been given in some of the more difficult cases. Others are considered in the following problems.

EXERCISES

1. If $\mathbf{a}$ and $\mathbf{b}$ be any two vectors, show that,

$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}|+|\mathbf{b}|, \qquad |\mathbf{a}-\mathbf{b}| \geq |\mathbf{a}|-|\mathbf{b}|.$$

Present both a geometric and an analytic argument.

2. Prove geometrically that,

$$m_\Delta(k\,\mathbf{a}) = k\,m_\Delta(\mathbf{a})$$

for all the variations in sign of k and $m_\Delta(\mathbf{a})$.

3. Prove geometrically the associative law for vector addition,

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}).$$

4. Prove geometrically that,

$$k\,(\mathbf{a}+\mathbf{b}) = k\,\mathbf{a}+k\,\mathbf{b}, \qquad (k+l)\,\mathbf{a} = k\,\mathbf{a}+l\,\mathbf{a}.$$

5. Show that the principle of projection applies to projection upon a plane as well as to projection upon a line. That is, show that the projection upon a given plane of the sum of any number of vectors is the sum of the projections of these vectors on that plane.

6. The projection of a vector $\mathbf{a}$ on a given line is $\mathbf{a}'$. The vector $\mathbf{a}$ is not determined by $\mathbf{a}'$ but may vary in length and direction. What figure is traced by its terminus when its initial point is fixed? Answer the same question when the projections of the vector on two non-parallel lines are given; on three non-coplanar lines.

16. Euclidean Geometry; Linear Dependence.

The elementary operations on vectors discussed in the previous article may be employed in the proof of many of the theorems of elementary geometry. Let us for instance prove that,

The line joining the mid-points of two sides of a triangle is parallel to the third side and has a length one half that of the third side.

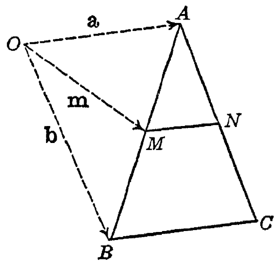

FIG. 16

Being given any triangle A, B, C, let M be the mid-point of AB and N the mid-point of AC and let O be any point in space thought of as fixed relative to the triangle. Let vectors run from O to the other points mentioned and call, $OA = \mathbf{a}$, $OB = \mathbf{b}$, $OC = \mathbf{c}$, etc. Then we have,

$$\mathbf{m} = OM = OA + \tfrac{1}{2}AB = \mathbf{a} + \tfrac{1}{2}(\mathbf{b}-\mathbf{a}) = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}).$$

Similarly we find,

$$\mathbf{n} = \tfrac{1}{2}(\mathbf{c} + \mathbf{a}),$$

and consequently,

$$MN = \mathbf{n} - \mathbf{m} = \tfrac{1}{2}(\mathbf{c} + \mathbf{a}) - \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{b}) = \tfrac{1}{2}BC.$$

If we recall the meaning of the product of a vector by a scalar, we see that the equation $MN = \frac{1}{2}BC$ states exactly the conclusions we wished to prove, namely that MN is parallel to BC and one half as long.

The application of vectors to the study of geometry is facilitated by the concept of *linear dependence* of vectors and its geometric interpretation. If we have a system of n vectors, $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ which are such that there exist n scalars, not all zero, $k_1, k_2, \cdots, k_n$ such that

$$k_1 \mathbf{a}_1 + k_2 \mathbf{a}_2 + \cdots + k_n \mathbf{a}_n = 0,$$

then we express this fact by saying that the given vectors are *linearly dependent.* Clearly a system consisting of a single vector $\mathbf{a}$ is linearly dependent when and only when the vector is of value zero, for the equation, $k\,\mathbf{a} = 0$ must hold without k being zero. If two vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent then we have by definition,

$$k\,\mathbf{a} + l\,\mathbf{b} = 0,$$

in which k and l are not both zero. If for example k were not zero we could write,

$$\mathbf{a} = -\frac{l}{k}\mathbf{b},$$

and $\mathbf{a}$ being the product of the vector $\mathbf{b}$ by a scalar would be parallel to $\mathbf{b}$. Thus $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, or as it is often expressed, *collinear.* Even if k were zero our conclusion would still hold for in that case $\mathbf{b}$ would be zero and therefore parallel to $\mathbf{a}$ by the convention previously adopted. Conversely if $\mathbf{a}$ and $\mathbf{b}$ are collinear then if we multiply each by the length of the other the resulting vectors can differ only in sense and we have,

$$|\mathbf{b}|\,\mathbf{a} \pm |\mathbf{a}|\,\mathbf{b} = 0.$$

If $\mathbf{a}$ and $\mathbf{b}$ are not both zero the scalars $|\mathbf{a}|$ and $|\mathbf{b}|$ are not both

zero and the above equation states that **a** and **b** are linearly dependent, while if **a** and **b** are both zero they are clearly linearly dependent for in that case the scalars k and l could be chosen at random.

We have now proven the two theorems:

The necessary and sufficient condition that a single vector be linearly dependent is that it be confined to a point.

The necessary and sufficient condition that two vectors be linearly dependent is that they be collinear.

Similar arguments may be employed to prove the analogous theorems:

The necessary and sufficient condition that three vectors be linearly dependent is that they be coplanar.

The necessary and sufficient condition that four vectors be linearly dependent is that they be cospatial.

Since all vectors are cospatial the last theorem may be stated as,

Any four or more vectors are linearly dependent.

The details of the proofs of these theorems are left to the reader.

If several vectors be so placed that they have the same initial point or origin we call them a system of *radius vectors.* On the basis of the above four theorems we may now easily establish the following three theorems.

The necessary and sufficient condition that two radius vectors **a** *and* **b** *have their termini at the same point is that there exist two scalars k and l, not both zero, such that,*

$$k\,\mathbf{a} + l\,\mathbf{b} = 0, \qquad k + l = 0.$$

The necessary and sufficient condition that three radius vectors **a**, **b**, **c** *have their termini on the same line is that there exist three scalars k, l, m, not all zero such that,*

$$k\,\mathbf{a} + l\,\mathbf{b} + m\,\mathbf{c} = 0, \qquad k + l + m = 0.$$

The necessary and sufficient condition that four radius vectors **a**, **b**, **c**, **d** *have their termini on the same plane is that there exist four scalars k, l, m, n, not all zero such that,*

$$k\,\mathbf{a} + l\,\mathbf{b} + m\,\mathbf{c} + n\,\mathbf{d} = 0, \qquad k + l + m + n = 0.$$

We shall establish the second of these three theorems, the argu-

ments for the others being entirely analogous. If the three radius vectors **a**, **b**, **c** have their termini on the same straight line it is evident from the figure that the vectors **b** − **a** and **c** − **a** lie along this line and consequently by the second theorem of the first group there must exist two scalars p and q, not both zero such that,

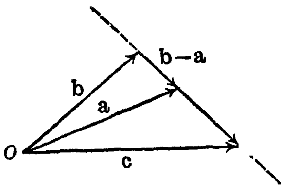

FIG. 17

$$p\,(\mathbf{b} - \mathbf{a}) + q\,(\mathbf{c} - \mathbf{a}) = 0.$$

Hence we may write,

$$-\,(p + q)\,\mathbf{a} + p\,\mathbf{b} + q\,\mathbf{c} = 0, \qquad -\,(p + q) + p + q = 0,$$

and the quantities $-\,(p + q)$, p, q constitute the k, l, m of the theorem. Conversely if we have the relations,

$$k\,\mathbf{a} + l\,\mathbf{b} + m\,\mathbf{c} = 0, \qquad k + l + m = 0,$$

in which **a**, **b**, **c** are three radius vectors and k, l, m are not all zero then if we eliminate k we obtain,

$$-\,(l + m)\,\mathbf{a} + l\,\mathbf{b} + m\,\mathbf{c} = 0 \quad \text{or} \quad l\,(\mathbf{b} - \mathbf{a}) + m\,(\mathbf{c} - \mathbf{a}) = 0,$$

where l and m can not both be zero for then k, l, m would all be zero, contrary to hypothesis. Hence **b** − **a** and **c** − **a** must be collinear and the termini of **a**, **b**, **c** must lie on the same straight line.

EXERCISES

Employ vector methods to prove the following propositions from Euclidean geometry.

1. The opposite sides of a parallelogram are equal.
2. The diagonals of a parallelogram bisect each other.
3. A quadrilateral having two sides which are both equal and parallel is a parallelogram.
4. A line parallel to one side of a triangle and passing through the mid-point of another side also passes through the mid-point of the third side.
5. The mid-points of the four sides of any quadrilateral form the vertices of a parallelogram.
6. The line joining one vertex of a parallelogram with the mid-point of an opposite side passes through a trisection point of a diagonal at one of its own trisection points.

7. A triangle may be constructed whose sides are equal and parallel to the medians of any triangle.

8. The medians of a triangle are concurrent at a trisection point of each.

9. The bisectors of the interior angles of a triangle are concurrent.

10. The mid-points of the diagonals of a complete quadrilateral are collinear.

11. A line is drawn connecting the mid-points of each pair of opposite edges of a tetrahedron. These three lines bisect each other.

12. The lines joining the mid-points of the opposite sides of a skew quadrilateral bisect each other.

13. Prove that the necessary and sufficient condition that three vectors be linearly dependent is that they be coplanar.

14. Prove that any four vectors are linearly dependent.

15. Prove that the necessary and sufficient condition that four radius vectors **a**, **b**, **c**, **d** have their termini on the same plane is that there exist four scalars k, l, m, n, not all zero, such that,

$$k\,\mathbf{a} + l\,\mathbf{b} + m\,\mathbf{c} + n\,\mathbf{d} = 0, \qquad k + l + m + n = 0.$$

16. Are the following theorems true? If not, what additional proviso will make them true?

(*a*) If $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$ are radius vectors and if,

$$\mathbf{p} = k_1\,\mathbf{p}_1 + k_2\,\mathbf{p}_2,$$

then the necessary and sufficient condition that their termini P, P_1, P_2 lie on the same straight line is that,

$$k_1 + k_2 = 1.$$

(*b*) If $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are radius vectors and if,

$$\mathbf{p} = k_1\,\mathbf{p}_1 + k_2\,\mathbf{p}_2 + k_3\,\mathbf{p}_3,$$

then the necessary and sufficient condition that their termini P, P_1, P_2, P_3 lie on the same plane is that,

$$k_1 + k_2 + \qquad 1.$$

17. Prove that the necessary and sufficient condition that the three vectors **a**, **b**, **c** be linearly dependent is that,

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} = 0.$$

Generalize this theorem to apply to n vectors.

17. Scalar Product of Two Vectors.

From two given vectors we may obtain by a certain operation a single scalar number. The operation by which this scalar is obtained resembles the operation of multiplication of two scalars in certain of its properties and for that reason we call the operation *multiplication* of the two given vectors. Since the result is a scalar we call it the *scalar product* of the two vectors. Unfortunately at present several different notations are employed to indicate the scalar product of **a** and **b**. The one we shall adopt is $\mathbf{a}\cdot\mathbf{b}$ formed by placing a dot between the symbols of the two vectors. To form this dot-product of two given vectors **a** and **b** we place them with their origins coinciding at some point O and call their termini A and B respectively. Then by definition the scalar product $\mathbf{a}\cdot\mathbf{b}$ is the product of their lengths and the cosine of the angle AOB.

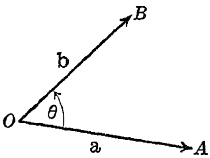

FIG. 18

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos(\angle AOB).$$

It will be observed that there is here no necessity of specifying which of the various determinations of the angle AOB is chosen since only its cosine is involved and this will have the same value for all determinations.

For the concept of the scalar product to be useful to us we must know its important properties and a formula for obtaining it when the coördinates of the two vectors are known. They are as follows.

I. *A necessary and sufficient condition that the vectors* **a** *and* **b** *be perpendicular is that,*

$$\mathbf{a}\cdot\mathbf{b} = 0.$$

Firstly the condition is necessary for if **a** and **b** are perpendicular then at least one of them must be zero or they must form a right angle and in either case the quantity $|\mathbf{a}|\ |\mathbf{b}| \cos(\angle AOB)$ vanishes. The condition is also sufficient for if $\mathbf{a}\cdot\mathbf{b}$ is to vanish it follows that $|\mathbf{a}|$ or $|\mathbf{b}|$ or $\cos(\angle AOB)$ must be zero. In the first two cases one or both of the given vectors is zero and they are perpendicular by convention while if $\cos(\angle AOB)$ is zero one determination of $\angle AOB$ must be a right angle.

II. *Scalar multiplication of two vectors is commutative, i.e.,*

$$(1) \qquad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}.$$

If either of the vectors is zero both members of this equation are zero and hence equal, while if not then no matter what determination of the angles $\angle AOB$ and $\angle BOA$ we choose we are still able to write,

$$\cos(\angle AOB) = \cos(\angle BOA).$$

Consequently we always have,

$$|\mathbf{a}|\ |\mathbf{b}|\cos(\angle AOB) = |\mathbf{b}|\ |\mathbf{a}|\cos(\angle BOA),$$

as we wished to prove.

III. *If k is any scalar and* $\mathbf{a}$ *and* $\mathbf{b}$ *are any two vectors then we have,*

$$(2) \qquad k\,(\mathbf{a}\cdot\mathbf{b}) = (k\,\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(k\,\mathbf{b}).$$

If k or $\mathbf{a}$ or $\mathbf{b}$ is zero the three members all vanish and the equations hold. If $\mathbf{a}$ and $\mathbf{b}$ are not zero and k is positive the three angles $\mathbf{a}$, $\mathbf{b}$ and $k\,\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}$, $k\,\mathbf{b}$ are equal and we may call them all θ. The equations to be proven then become,

$$k\,|\mathbf{a}|\ |\mathbf{b}|\cos\theta = |k\,\mathbf{a}|\ |\mathbf{b}|\cos\theta = |\mathbf{a}|\ |k\,\mathbf{b}|\cos\theta,$$

which hold as a consequence of the definition of the product of a vector by a scalar. If $\mathbf{a}$ and $\mathbf{b}$ are not zero and k is negative then the angles $k\,\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}$, $k\,\mathbf{b}$ are the supplement of the angle $\mathbf{a}, \mathbf{b} = \theta$ and the equations to be proven become,

$$k\,|\mathbf{a}|\ |\mathbf{b}|\cos\theta = -\ |k\,\mathbf{a}|\ |\mathbf{b}|\cos\theta = -\ |\mathbf{a}|\ |k\,\mathbf{b}|\cos\theta,$$

which hold as a consequence of the definition of the product of a vector by a scalar. The equations thus hold in every case.

IV. *The scalar product of two vectors is unaltered if either is replaced by its projection upon the other.*

This important fact is known as the *fundamental theorem of the scalar product* and we shall have frequent occasion to refer to it. Let us indicate by $\mathbf{a}'$ the projection of $\mathbf{a}$ on $\mathbf{b}$ so that the theorem to be proven may be written,

$$(3) \qquad \mathbf{a}\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{b}.$$

If as before we represent the angle **a**, **b** by θ we have from the figure,

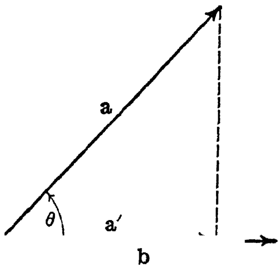

Fig. 19

$$|\mathbf{a}'| = |\mathbf{a}|\ |\cos\theta|,$$

and multiplying through by $|\mathbf{b}|$ gives us,

$$|\mathbf{a}|\ |\mathbf{b}|\ |\cos\theta| = |\mathbf{a}'|\ |\mathbf{b}|,$$

which is a statement that equation (3) holds in numerical value. It only remains to show that it holds in sign. If, as in the figure, θ is an acute angle the left member of equation (3) is positive while since $\cos(\mathbf{a}', \mathbf{b}) = +1$ the right member is also positive. While if θ is obtuse the left member of equation (3) is negative and since $\cos(\mathbf{a}', \mathbf{b}) = -1$ the right member is negative also. In the case where θ is a right angle the theorem is clearly true since both members vanish.

V. *Scalar multiplication of vectors is distributive with respect to addition.*

That is,

$$\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}. \tag{4}$$

Excluding the trivial case $\mathbf{a} = 0$ we represent by $\mathbf{b}'$, $\mathbf{c}'$, $(\mathbf{b} + \mathbf{c})'$ the projections of **b**, **c**, $(\mathbf{b} + \mathbf{c})$ on the vector **a** so that by virtue of Theorem IV just proven, equation (4) to be proven may be written,

$$\mathbf{a}\cdot(\mathbf{b} + \mathbf{c})' = \mathbf{a}\cdot\mathbf{b}' + \mathbf{a}\cdot\mathbf{c}'.$$

But by the principle of projection, § 15, (6), we have,

$$(\mathbf{b} + \mathbf{c})' = \mathbf{b}' + \mathbf{c}',$$

so that the equation to be proven becomes,

$$\mathbf{a}\cdot(\mathbf{b}' + \mathbf{c}') = \mathbf{a}\cdot\mathbf{b}' + \mathbf{a}\cdot\mathbf{c}'.$$

Now all the vectors involved in this equation are collinear and consequently the cosines involved in the scalar products are all $+1$ or -1 according as the vector multiplied by **a** has the same or the opposite sense to **a**. Thus except for the factor $|\mathbf{a}|$ running through it this equation is simply a statement that the measure of the sum of two directed segments *of a given line* is

the sum of their measures. But this statement, known as the *Theorem of Chasles*, lies at the foundation of analytic geometry and is well known to be true. Equation (4) thus holds.

By the *square of a vector* we mean its scalar product with itself. Since the cosine involved in this scalar product has the value $+1$ it follows that the square of a vector is simply the square of the length of the vector,

$$\mathbf{a}^2 = |\mathbf{a}|^2.$$

In addition to the above properties of the scalar product some formulas will be necessary for numerical computation. The first of these is in effect a formula for the measure of a vector $\mathbf{b}$ on a non-zero vector $\mathbf{a}$. We may show that,

$$|\mathbf{a}|\, m_{\mathbf{a}}(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}. \tag{5}$$

Leaving out the trivial case, $\mathbf{a} = 0$, we represent by $\mathbf{b}'$ the projection of $\mathbf{b}$ on $\mathbf{a}$ and write the equation (5) which we are to prove as,

$$|\mathbf{a}|\, m_{\mathbf{a}}(\mathbf{b}') = \mathbf{a}\cdot\mathbf{b}'.$$

This we may do because if two vectors have the same projection on an axis they have the same measure on that axis (§ 15) and because the scalar product of two vectors is unaltered if either is replaced by its projection upon the other. But now the truth of this last equation is obvious because both members are equal to $|\mathbf{a}|\,|\mathbf{b}'|$ with a plus or minus sign according as $\mathbf{a}$ and $\mathbf{b}'$ have like or unlike senses. In particular if $\mathbf{u}$ is any unit vector, formula (5) gives us,

$$m_{\mathbf{u}}(\mathbf{a}) = \mathbf{a}\cdot\mathbf{u}. \tag{5'}$$

If then we represent as usual by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the three unit vectors drawn along the positive sense of the X, Y, Z axes, then since the coördinates of a vector are its measures on the coördinate axes we have for the coördinates of the vector $\mathbf{a}$,

$$\mathbf{a} \equiv (\mathbf{a}\cdot\mathbf{i},\ \mathbf{a}\cdot\mathbf{j},\ \mathbf{a}\cdot\mathbf{k}). \tag{6}$$

Since the *coördinate vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are each one unit long we have,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,$$

while due to their mutual perpendicularity,

$$\mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{j} = 0.$$

If we now resolve any two vectors **a** and **b**,

$$\mathbf{a} \equiv (a_1, a_2, a_3), \qquad \mathbf{b} \equiv (b_1, b_2, b^0),$$

into their components parallel to the coördinate axes,

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}, \qquad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k},$$

and form the scalar products of these two equations, member for member, we find on applying the distributive law and the formulas just obtained relative to **i**, **j**, **k**, the following important formula,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{7}$$

From our fundamental Theorem IV comes the name, "inner product," often applied to the scalar product. This is due to the fact that this product is concerned with the component of either vector lying *in* the other. The name was first used by Grassmann in *Die Ausdehnungslehre von 1844*. He was the first to regard scalar multiplication of two vectors as a single operation upon them.

EXERCISES

1. Show that the angle θ between the two vectors **a** and **b** is given by the formula,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a}^2\, \mathbf{b}^2}}$$

2. Do any two of the following vectors form an angle of 90°, 60°, 45°? Are any two of equal length?

$$\mathbf{a} \equiv (2, 2, -1), \qquad \mathbf{b} \equiv (0, -3, 3), \qquad \mathbf{c} \equiv (2, -1, 2).$$

3. Show that the vectors $\mathbf{a} \equiv (1, 4, -1)$, $\mathbf{b} \equiv (-1, -1, 4)$, $\mathbf{c} \equiv (0, -3, -3)$ may form the sides of an equilateral triangle.

4. If $\mathbf{a} \equiv (a_1, a_2, a_3)$, $\mathbf{b} \equiv (b_1, b_2, b_3)$, show that the vector

$$\mathbf{c} \equiv (a_2 b_3 - a_3 b_2, \quad a_3 b_1 - a_1 b_3, \quad a_1 b_2 - a_2 b_1)$$

is perpendicular to both **a** and **b**.

5. Prove by vector methods that the sum of the squares of the four sides of a parallelogram equals the sum of the squares of the two diagonals.

6. Under what conditions is $(\mathbf{a} \cdot \mathbf{b})^2 = \mathbf{a}^2\, \mathbf{b}^2$? Under what conditions upon **a** and **b** will the vector $k\,\mathbf{a} + l\,\mathbf{b}$ be perpendicular to $l\,\mathbf{a} - k\,\mathbf{b}$ for every choice of k and l?

7. What is the geometric significance of the identities,

$$
\begin{aligned}
&(a) \quad (\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\,\mathbf{a}\cdot\mathbf{b},\\
&(b) \quad (\mathbf{a}+\mathbf{b})^2 + (\mathbf{a}-\mathbf{b})^2 = 2\,(\mathbf{a}^2 + \mathbf{b}^2),\\
&(c) \quad (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2?
\end{aligned}
$$

Ans. (*a*) The cosine law for plane triangles

8. If the length of the sum of two vectors equals the length of their difference, show that the vectors are perpendicular.

9. Show that the projection of a vector **a** upon a non-zero vector **b** is given by the formula,

$$\mathbf{p}_{\mathbf{b}}(\mathbf{a}) \quad \frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{\mathbf{b}^2}$$

18. Vector Product of Two Vectors.

We have seen that from two given vectors we may obtain by a certain operation a single scalar number which we call the scalar product of the two given vectors. Similarly from a pair of vectors, i.e. two vectors in a given order, we may by a certain rule obtain a third vector. As in the previous case this operation has properties similar to those of the ordinary multiplication of two scalars and for that reason we call the operation *vector multiplication* and the vector obtained is called the *vector product* of the given pair of vectors. Several different notations are still employed to indicate this vector product. The one we shall adopt is $\mathbf{a} \times \mathbf{b}$ formed by writing down the pair of vectors in the given order with a slanting cross between them.

In defining the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ we shall find it convenient to think of **a** and **b** as having their initial points coinciding at some point O and their termini falling at A and B respectively. We call the angle AOB, θ and the plane AOB, π. Then by definition,

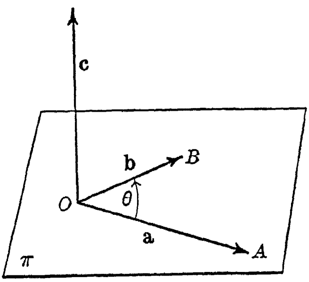

FIG. 20

The direction of **c** *is perpendicular to the plane* π.

The sense of **c** *is such that the triple of vectors* **a**, **b**, **c** *has a positive sense of rotation.* (§ 14.)

The length of **c** *is given by the formula,* $|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\sin\theta|$.

It will be observed that there is here no necessity of specifying which of the various different determinations of θ is chosen since only $|\sin \theta|$ is involved and this will be the same for all determinations. If **a** and **b** should be parallel the direction and sense of **c** become indeterminate but at the same time its length is zero so that **c** is here as completely determined as any zero vector. We might also note that the length of **c** equals the area of the parallelogram having **a** and **b** for two concurrent edges.

Although the vector and the scalar products of **a** and **b** are different kinds of numbers and quite different in definition, still there is a considerable formal similarity running through their properties, as will be seen by comparing those of the same number in this and the preceding article.

I. *A necessary and sufficient condition that the vectors* **a** *and* **b** *be parallel is that,* $\mathbf{a} \times \mathbf{b} = 0$.

The condition is sufficient, for if $\mathbf{a} \times \mathbf{b}$ is to vanish either **a** or **b** or $\sin \theta$ must be zero. In the first two cases at least one of the given vectors is zero and they are parallel by convention, (§ 15) while if $\sin \theta$ is zero one determination of θ must be zero or π. Conversely if **a** and **b** are parallel then at least one of them must be zero or they must form an angle of 0 or π and in each case the quantity $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\sin \theta|$ vanishes.

II. *Vector multiplication of two vectors is commutative with change of sign, i.e.,*

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}). \tag{1}$$

The statement obviously holds if **a** and **b** are parallel since both members of the equation vanish. If they are not parallel it is clear from the definition that the direction and length of the two cross products will be the same. But since the triple **a**, **b**, $\mathbf{a} \times \mathbf{b}$ has by definition a positive sense of rotation the triple **b**, **a**, $\mathbf{a} \times \mathbf{b}$ must have a negative sense of rotation. But **b**, **a**, $\mathbf{b} \times \mathbf{a}$ has a positive sense of rotation and therefore $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ must have opposite senses. Equation (1) is thus established.

III. *If k is any scalar and* **a** *and* **b** *are any two vectors then we have,*

$$k\,(\mathbf{a} \times \mathbf{b}) = (k\,\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\,\mathbf{b}). \tag{2}$$

If k or **a** or **b** is zero the three members all vanish and the equations hold. If k, **a** and **b** are none of them zero the plane π and

$|\sin\theta|$ of the definition are the same for all three members and the equations hold in direction and length. Furthermore since the sense of rotation of a triple of vectors is not altered by multiplying one of them by a positive scalar and is altered by multiplying one by a negative scalar, (§ 14, Prob. 2) it follows that the three members of our equations have the same sense as $\mathbf{a} \times \mathbf{b}$ or the opposite sense according as k is positive or negative. Thus the three members are the same in direction, length and sense in every case.

IV. *The vector product of two vectors is unaltered if either vector is replaced by its projection upon a plane perpendicular to the other.*

This important property is known as *the fundamental theorem of the vector product.* We shall find frequent use for it in the following sections. If $\mathbf{a}$, $\mathbf{b}$ is the given pair of vectors let us for convenience place their initial points both at some point O and call their termini A and B respectively. Draw a plane π perpendicular to $\mathbf{b}$ at O and let A' be the projection of A on this plane. Then $\mathbf{a}' = OA'$ is the projection of $\mathbf{a}$ on π and the property which we wish to prove is expressed by the equation,

$$\mathbf{a}' \times \mathbf{b} = \mathbf{a} \times \mathbf{b}. \tag{3}$$

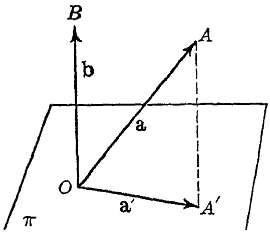

FIG. 21

Since the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}'$ all lie in the plane determined by the two parallel lines OB and AA' the two vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a}' \times \mathbf{b}$ must have the same direction being both perpendicular to that plane. We have seen (§ 14, Prob. 1) that the sense of rotation of the triple $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{c}$ is the same as that of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ where $\mathbf{c}$ is any vector not coplanar with $\mathbf{a}$ and $\mathbf{b}$. The sense of rotation of the triple $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ is thus positive because that of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ is positive. But since the sense of rotation of $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{a}' \times \mathbf{b}$ is likewise positive it follows that the two vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a}' \times \mathbf{b}$ have the same sense (§ 14, Prob. 2). Furthermore from the figure we have at once,

$$= |\mathbf{a} \quad |\sin\theta| \qquad \text{where} \qquad \theta = \angle \mathbf{a},\mathbf{b},$$

and consequently,

$$|\mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\ |\sin\theta|,$$

which is a statement that the two members of equation (3) are of the same length. Equation (3) is thus verified in every respect.

V. *Vector multiplication of vectors is distributive with respect to addition, i.e.,*

$$\text{(4)} \qquad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Excluding the trivial case, $\mathbf{a} = 0$, we represent by $\mathbf{b}'$, $\mathbf{c}'$, $(\mathbf{b} + \mathbf{c})'$ the projections of $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b} + \mathbf{c}$ on a plane perpendicular to $\mathbf{a}$. By Property IV, just proven, it follows that equation (4) will be proven if we can establish that,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c})' = \mathbf{a} \times \mathbf{b}' + \mathbf{a} \times \mathbf{c}'.$$

Now the principle of projection states that the projection of the sum of two vectors is the sum of their projections and consequently we have,

$$(\mathbf{b} + \mathbf{c})' = \mathbf{b}' + \mathbf{c}',$$

and the equation to be proven may therefore be written,

$$\mathbf{a} \times (\mathbf{b}' + \mathbf{c}') = \mathbf{a} \times \mathbf{b}' + \mathbf{a} \times \mathbf{c}'.$$

Let us now write the vector $\mathbf{a}$ in the form $|\mathbf{a}|\ \mathbf{u}$ where $\mathbf{u}$ is a unit vector having the direction and sense of $\mathbf{a}$. Since $|\mathbf{a}|$ is not zero it is evident from Property III that the above equation will be established if we can show that,

$$\mathbf{u} \times (\mathbf{b}' + \mathbf{c}') = \mathbf{u} \times \mathbf{b}' + \mathbf{u} \times \mathbf{c}'.$$

Let us draw the vectors $\mathbf{b}'$, $\mathbf{c}'$, $\mathbf{b}' + \mathbf{c}'$ with a common origin O. Evidently they will lie in a plane perpendicular to $\mathbf{u}$ and the vector $\mathbf{b}' + \mathbf{c}'$ will be the diagonal of the parallelogram having $\mathbf{b}'$ and $\mathbf{c}'$ for two concurrent edges. But now it appears from the definition of the vector product that the effect of cross multiplying a given vector by a unit vector perpendicular to it is to rotate the given vector through a right angle in a plane perpendicular to the unit vector. Thus when we cross multiply each of the vectors $\mathbf{b}'$, $\mathbf{c}'$, $\mathbf{b}' + \mathbf{c}'$ by $\mathbf{u}$ the resulting vectors form a figure exactly congruent with that formed by the original vec-

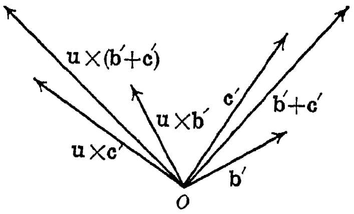

Fig. 22

tors but rotated through a right angle in their plane. Thus $\mathbf{u} \times (\mathbf{b}' + \mathbf{c}')$ will form the diagonal of the parallelogram having $\mathbf{u} \times \mathbf{b}'$ and $\mathbf{u} \times \mathbf{c}'$ for two concurrent edges. This establishes our last equation and equation (4) follows as we have shown.

It is now easy to derive a formula for the computation of the coördinates of the vector product $\mathbf{a} \times \mathbf{b}$ when the coördinates of $\mathbf{a}$ and $\mathbf{b}$ are given. As before let us take $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as the coördinate vectors, i.e. unit vectors with the direction and sense of the coördinate axes. As a direct consequence of the definition of the vector product we have the relations,

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \qquad \mathbf{i} \times \mathbf{j} = \mathbf{k},$$

while of course the vector product of each vector with itself is zero. If we now resolve the vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a} \equiv (a_1, a_2, a_3), \qquad \mathbf{b} \equiv (b_1, b_2, b_3),$$

into their components parallel to the coördinate axes,

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}, \qquad \mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k},$$

and form the vector product of these two equations member for member, we find on applying Properties II, III and IV and the above formulas relative to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ that,

$$(5) \quad \mathbf{a} \times \mathbf{b} = (a_2\,b_3 - a_3\,b_2)\,\mathbf{i} + (a_3\,b_1 - a_1\,b_3)\,\mathbf{j} + (a_1\,b_2 - a_2\,b_1)\,\mathbf{k}.$$

These coördinates of $\mathbf{a} \times \mathbf{b}$ can be readily remembered by observing that they are obtained from the matrix,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

by evaluating the second order determinants left when the columns are omitted in succession, the remaining two columns being arranged in the cyclic order, 1, 2, 3, 1. Thus we have,

$$(5) \qquad \mathbf{a} \times \mathbf{b} \equiv \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \quad \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

EXERCISES

1. Prove *Lagrange's Identity*, $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = \mathbf{a}^2\,\mathbf{b}^2$.
2. Evaluate, $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$ and $(\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{c})$ and interpret the results regarding $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as the radius vectors of three points A, B, C from a common origin O.

3. Let each face of a polyhedron be represented by an outward normal vector whose length measures the area of the face. Show that the sum of these vectors is zero. (Hint: Discuss the tetrahedron first.)
4. If $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ show that $\mathbf{a} - \mathbf{d}$ is parallel to $\mathbf{b} - \mathbf{c}$.

19. Products of Three Vectors.

If we form the vector product of the pair of vectors $\mathbf{b}$, $\mathbf{c}$ the result is itself a vector and we may therefore form its scalar or vector product with a third vector $\mathbf{a}$. Let us consider first the scalar product so obtained, which we may without ambiguity write as $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. From the notation employed to represent it this is often called the *dot-cross product* of the triple of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. It is a scalar having an interesting geometric significance. Let us place our three vectors with their origins at a common point O and call their termini A, B, C respectively. We then recall that by the fundamental theorem of the scalar product we may replace $\mathbf{a}$ by its projection $\mathbf{a}'$ upon $\mathbf{b} \times \mathbf{c}$ without altering the value of the expression, i.e.,

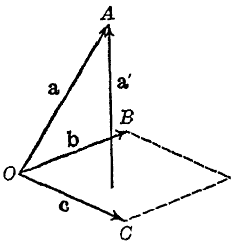

FIG. 23

$$\mathbf{a}' \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}.$$

Since $\mathbf{a}'$ and $\mathbf{b} \times \mathbf{c}$ are parallel the numerical value of their scalar product is the product of their lengths. But since the length of $\mathbf{b} \times \mathbf{c}$ is the area of the parallelogram formed upon the vectors $\mathbf{b}$ and $\mathbf{c}$ as two concurrent edges and since the length of $\mathbf{a}'$ is the perpendicular distance from A to the plane of this parallelogram it follows that the scalar product considered equals numerically the volume of the parallelopiped having $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as three concurrent edges. Furthermore if $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is positive the cosine of the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$ must be positive and $\mathbf{a}$ must extend on the same side of the plane of $\mathbf{b}$ and $\mathbf{c}$ that $\mathbf{b} \times \mathbf{c}$ does, while if $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is negative the opposite conclusion holds. Since the triple $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b} \times \mathbf{c}$ has by definition of $\mathbf{b} \times \mathbf{c}$ a positive sense of rotation it follows that the sense of rotation of the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is positive or negative according as the dot-cross product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is positive or negative. Thus both the numerical value and the sign of the scalar triple product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$

yield simple geometric facts concerning the triple of vectors **a**, **b**, **c** and their relation to the coördinate system.

The above geometric interpretation of $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ would clearly serve to completely determine it, both as to numerical value and as to sign, and still this interpretation depends only on the three separate vectors and on their cyclic order in the triple. It follows that,

The value of the scalar triple product is unaltered if the dot and cross be interchanged or if the vectors be rearranged in the triple in any way which maintains their cyclic order,

$$(1) \qquad \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}.$$

The property just proven geometrically becomes evident analytically immediately upon deriving the formula for $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ in terms of the coördinates of the vectors involved. Recalling that,

$$\mathbf{a} \equiv (a_1, a_2, a_3), \ \mathbf{b} \times \mathbf{c} \equiv (b_2 c_3 - b_3 c_2, \quad b_3 c_1 - b_1 c_3, \quad b_1 c_2 - b_2 c_1),$$

we find on forming the scalar product of these two vectors an expression which we recognize as the expansion of the determinant,

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Employing the above notation for the determinant we may therefore write.

$$(2) \qquad \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}],$$

and the facts expressed in equation (1) now appear as familiar properties of determinants. In particular we might note that if any two of the vectors **a**, **b**, **c** are parallel their dot-cross product vanishes.

If we form the vector product of the pair of vectors **b**, **c** and then cross-multiply this vector from the left by a third vector **a** the resulting vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is often known as the *cross-cross product* of the triple **a**, **b**, **c**. It is here essential to indicate or bear in mind the placing of the parentheses in the expression, for $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are by no means the same vector. We shall show by a geometric argument that we may always

write,

$$(3) \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = k\,\mathbf{b} + l\,\mathbf{c},$$

where k and l are properly chosen scalars. If $\mathbf{b} \times \mathbf{c} = 0$ we have our equation at once by choosing $k = l = 0$. If $\mathbf{b} \times \mathbf{c}$ is not zero we observe that since $\mathbf{b} \times \mathbf{c}$ is by definition perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$, and since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to this perpendicular, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be coplanar with $\mathbf{b}$ and $\mathbf{c}$. The three vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b}$, $\mathbf{c}$ being coplanar must be linearly dependent, i.e. there must exist three scalars p, q, r not all zero such that,

$$(4) \qquad p\,\mathbf{b} + q\,\mathbf{c} + r\,\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0.$$

We have now merely to show that r is not zero to establish equation (3), for if r is not zero equation (3) follows from equation (4) by calling $-p/r = k$ and $-q/r = l$. Suppose that r were zero and form the cross product of both members of equation (4) with $\mathbf{b}$. We have then,

$$q\,\mathbf{b} \times \mathbf{c} = 0,$$

and since $\mathbf{b} \times \mathbf{c}$ is not zero q must be zero. A similar argument shows that p is also zero and we would thus have p, q, r all zero, contrary to hypothesis. Hence r can not be zero and equation (3) is established.

The actual determination of the scalars k and l of equation (3) is most readily carried through analytically, although it may be done by geometric methods without serious difficulty. Let us find the coördinates of the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. If we employ the usual device we set up the matrix having for its top row the coördinates of $\mathbf{a}$ and for its second row the coördinates of $\mathbf{b} \times \mathbf{c}$,

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{array}$$

The first coördinate of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is then obtained by evaluating the determinant left when we omit the first column of this matrix. This yields,

$$a_2\,b_1\,c_2 - a_2\,b_2\,c_1 + a_3\,b_1\,c_3 - a_3\,b_3\,c_1,$$

which we may write as,

$$(c_1 a_1 + c_2 a_2 + c_3 a_3)\, b_1 - (a_1 b_1 + a_2 b_2 + a_3 b_3)\, c_1.$$

By advancing the subscripts cyclically in the order 1, 2, 3, 1 we may obtain the other two coördinates of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and in the form written they are all clearly the coördinates of the vector,

$$(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}.$$

We thus obtain what is known as *the fundamental identity of vector analysis*,

$$\text{(5)} \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}.$$

This important formula because of its frequent use should be memorized. The reader will have no difficulty in deriving the alternative form,

$$\text{(6)} \qquad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}.$$

EXERCISES

1. Prove that if $\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} = 0$ then $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$, $\mathbf{a}\times\mathbf{b}$ are parallel and if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be drawn with a common origin their termini will be collinear.
2. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are mutually perpendicular show that $[\mathbf{a\,b\,c}]^2 = \mathrm{a}^2\,\mathrm{b}^2\,\mathrm{c}^2$.

20. Vector Division.

In dealing with scalar quantities we may introduce the operation of division by regarding multiplication as a previously known operation and then defining division as the inverse operation to multiplication. Thus the equation in x,

$$\text{(1)} \qquad ax = b,$$

may be regarded as asking the question, "What number when multiplied by a will yield b?," and the answer gives us the definition of b/a. Similarly with vectors we may ask the question, "What vector $\mathbf{x}$ when multiplied by the vector $\mathbf{a}$ will yield the vector $\mathbf{b}$?," i.e. we seek the solution of the equation in $\mathbf{x}$,

$$\text{(2)} \qquad \mathbf{a} \times \mathbf{x} = \mathbf{b}.$$

In answering this question we shall be introducing an operation

upon the two vectors **a** and **b** analagous to the division of the scalar b by the scalar a.

It is a familiar fact that we shall find no solution to equation (1) if a is zero and b is not, and the corresponding fact is true of equation (2), but here in order to have a solution we must impose the additional condition, $\mathbf{a} \cdot \mathbf{b} = 0$. We have in fact the theorem,

A necessary and sufficient condition that the equation,

$$\mathbf{a} \times \mathbf{x} = \mathbf{b} \qquad (\mathbf{a} \neq 0)$$

possess a solution is that $\mathbf{a} \cdot \mathbf{b} = 0$.

The condition is necessary, for forming the scalar product of both members of the equation with **a** yields,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} \times \mathbf{x} = [\mathbf{a}\,\mathbf{a}\,\mathbf{x}] = 0.$$

The condition is also sufficient for if $\mathbf{a} \cdot \mathbf{b} = 0$ we may write,

$$\frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{a})}{\mathbf{a}^2} = \frac{\mathbf{a}^2\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{a}}{\mathbf{a}^2} = \mathbf{b},$$

which is a statement that,

$$\mathbf{x} = \mathbf{b} \times \mathbf{a}$$

is one possible solution of our equation. There are, in fact, infinitely many solutions. (Problem 1)

Corresponding to the two types of vector multiplication we have two types of vector division. Corresponding to the scalar product we have the solution of the equation in **x**,

$$\mathbf{a} \cdot \mathbf{x} = c \qquad (\mathbf{a} \neq 0)$$

in which the vector **a** and the scalar **c** are given. Here it is unnecessary to impose any further restriction on **a** and **c** because the identity,

$$\mathbf{a} \cdot \left(\frac{c\,\mathbf{a}}{\mathbf{a}^2} \right) = c$$

shows us at once that,

$$\mathbf{x} = \frac{c\,\mathbf{a}}{\mathbf{a}^2}$$

is one possible solution of our equation. There are, in fact, infinitely many solutions. (Problem 2)

The solution of the equation,

$$\mathbf{a} \times \mathbf{x} = \mathbf{b}$$

is a special case of the more general problem of determining an operation φ which when performed on a given vector **a** will yield the given vector **b**. We might symbolize this by the equation,

$$\varphi\,(\mathbf{a}) = \mathbf{b}.$$

The solution of this problem leads to the branch of mathematics known as *quaternions* invented by Sir William Hamilton about 1843.

EXERCISES

1. Prove that every solution of the equation,

$$\mathbf{a} \times \mathbf{x} = \mathbf{b} \qquad (\mathbf{a} \neq 0, \qquad \mathbf{a}\cdot\mathbf{b} = 0)$$

is given by

$$\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{\mathbf{a}^2} + k\,\mathbf{a},$$

in which k is an arbitrary scalar.

2. Prove that every solution of the equation,

$$\mathbf{a} \cdot \mathbf{x} = b \qquad (\mathbf{a} \neq 0)$$

is given by,

$$\mathbf{x} = \frac{b\,\mathbf{a}}{\mathbf{a}^2} + \mathbf{k} \times \mathbf{a},$$

in which **k** is an arbitrary vector.

3. Determine the vector **x** satisfying the simultaneous equations,

$$\mathbf{a} \times \mathbf{x} = \mathbf{b} \qquad \mathbf{c} \cdot \mathbf{x} = d$$

where $\mathbf{a} \neq 0$, $\mathbf{c} \neq 0$, $\mathbf{a}\cdot\mathbf{b} = 0$.
Evaluate **x** for,

(*a*) $\mathbf{a} = (3, 1, -1)$, $\mathbf{b} = (1, -11, -8)$, $\mathbf{c} = (1, 2, -1)$, $d = 1$
(*b*) $\mathbf{a} = (3, 1, -1)$, $\mathbf{b} = (1, -11, -8)$, $\mathbf{c} = (1, 2, 5)$, $d = 13$

Ans. (*a*) $(5, -1, 2)$

21. Vector Identities.

The three basic identities of § 19,

(1) $$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}],$$

(2) $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a},$$

being established, we may use them to prove a large number of

more complicated relations. This is very similar to the situation which we have encountered in trigonometry where on the basis of a few fundamental identities such as,

$$\cos^2 \alpha + \sin^2 \alpha = 1, \qquad \sin \alpha \csc \alpha = 1,$$

we are enabled to prove a great number of others. We shall deduce but one of these vector identities, leaving the proofs of others as exercises for the reader.

Let us show that if **a**, **b**, **c**, **d** be any four vectors then we have,

$$(3) \qquad [\mathbf{b\,c\,d}]\,\mathbf{a} - [\mathbf{c\,d\,a}]\,\mathbf{b} + [\mathbf{d\,a\,b}]\,\mathbf{c} - [\mathbf{a\,b\,c}]\,\mathbf{d} = 0.$$

Starting with the expression, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ we apply the fundamental identity (2) to it twice, once treating $\mathbf{a} \times \mathbf{b}$ as a single vector and once treating $\mathbf{c} \times \mathbf{d}$ in this way. We have,

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{d} \cdot \mathbf{a} \times \mathbf{b})\,\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\,\mathbf{d},$$
$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{c} \times \mathbf{d} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d})\,\mathbf{a},$$

or,

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{d\,a\,b}]\,\mathbf{c} - [\mathbf{a\,b\,c}]\,\mathbf{d},$$
$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{c\,d\,a}]\,\mathbf{b} - [\mathbf{b\,c\,d}]\,\mathbf{a}.$$

Equating these last two expressions gives us at once the desired identity (3).

EXERCISES

1. By the use of equations (1), (2), (3) prove the following identities.

(*a*) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0,$

(*b*) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \quad \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$

(*c*) $\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\} = (\mathbf{b}\cdot\mathbf{d})\,\mathbf{a} \times \mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\,\mathbf{d} \times \mathbf{a},$

(*d*) $[\mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a} \quad \mathbf{a} \times \mathbf{b}] = [\mathbf{a\,b\,c}]^2,$

(*e*) $[\mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a} \quad \mathbf{a} \times \mathbf{b}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$

(*f*) $[\mathbf{a\,b\,c}]\,\mathbf{d} = (\mathbf{d}\cdot\mathbf{a})\,\mathbf{b} \times \mathbf{c} + (\mathbf{d}\cdot\mathbf{b})\,\mathbf{c} \times \mathbf{a} + (\mathbf{d}\cdot\mathbf{c})\,\mathbf{a} \times \mathbf{b}.$

2. Prove the identity,

$$[\mathbf{b\,c\,d}]\,\mathbf{a} - [\mathbf{c\,d\,a}]\,\mathbf{b} + [\mathbf{d\,a\,b}]\,\mathbf{c} - [\mathbf{a\,b\,c}]\,\mathbf{d} = 0$$

by consideration of the following determinant,

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

3. Put the following scalar identities in vector form and prove them. The first is due to Lagrange.

$$(a)\quad (a^2 + b^2 + c^2)(k^2 + l^2 + m^2) = (ak + bl + cm)^2 + (bm - cl)^2 + (ck - am)^2 + (al - bk)^2,$$

$$(b)\quad (bm - cl)(qz - ry) + (ck - am)(rx - pz) + (al - bk)(py - qx) = (ap + bq + cr)(kx + ly + mz) - (ax + by + cz)(kp + lq + mr).$$

4. Prove analytically that if $\mathbf{p} \times \mathbf{a} = 0$ and $\mathbf{a} \neq 0$ then there exists a scalar k such that $\mathbf{p} = k\,\mathbf{a}$. Obtain an analytic expression for k.

Ans. $\mathbf{p} = \mathbf{a}\cdot\mathbf{p}$.

5. Prove analytically that if $[\mathbf{p\,a\,b}] = 0$ and $\mathbf{a} \times \mathbf{b} \neq 0$ then there exist two scalars k and l such that $\mathbf{p} = k\,\mathbf{a} + l\,\mathbf{b}$. Obtain analytic expressions for k and l.

$$\textit{Ans.}\quad \mathbf{p} = \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{p}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})^2}\,\mathbf{a} + \frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{p})}{(\mathbf{a}\times\mathbf{b})^2}\,\mathbf{b}$$

6. Prove analytically that if $[\mathbf{a\,b\,c}] \neq 0$ then there exist three scalars k, l, and m such that $\mathbf{p} = k\,\mathbf{a} + l\,\mathbf{b} + m\,\mathbf{c}$. Obtain analytic expressions for k, l, and m.

$$\textit{Ans.}\quad \mathbf{p} = \frac{[\mathbf{p\,b\,c}]}{[\mathbf{a\,b\,c}]}\,\mathbf{a} + \frac{[\mathbf{a\,p\,c}]}{[\mathbf{a\,b\,c}]}\,\mathbf{b} + \frac{[\mathbf{a\,b\,p}]}{[\mathbf{a\,b\,c}]}\,\mathbf{c}$$

7. Prove analytically that if $\mathbf{p}\cdot\mathbf{a} = 0$, $\mathbf{p}\cdot\mathbf{b} = 0$, $\mathbf{a}\times\mathbf{b} \neq 0$ then there exists a scalar k such that $\mathbf{p} = k\,\mathbf{a}\times\mathbf{b}$. Obtain an analytic expression for k.

$$\textit{Ans.}\quad \mathbf{p} = \frac{[\mathbf{a\,b\,p}]}{(\mathbf{a}\times\mathbf{b})^2}\,\mathbf{a}\times\mathbf{b}$$

8. Prove analytically that if $\mathbf{a}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ then $[\mathbf{a\,b\,c}]^2 = \mathbf{a}^2(\mathbf{b}\times\mathbf{c})^2$.

9. Prove that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three mutually perpendicular vectors then $[\mathbf{a\,b\,c}]^2 = \mathbf{a}^2\,\mathbf{b}^2\,\mathbf{c}^2$.

10. If a, b, c are the three sides and α, β, γ the opposite angles of a spherical triangle then the fundamental laws of spherical trigonometry are the following.

Cosine Law $\quad \cos a = \cos b \cos c + \sin b \sin c \cos \alpha,$

Sine Law $\quad \dfrac{\sin a}{\sin \alpha} = \dfrac{\sin b}{\sin \beta} = \dfrac{\sin c}{\sin \gamma}.$

Prove these laws by vector methods. (Hint: Employ three vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ running from the center of the sphere to the vertices of the triangle.)

11. Show that there exists a unique value of $\mathbf{p}$ satisfying the equation,

$$\mathbf{a}\times\mathbf{p} + (\mathbf{b}\cdot\mathbf{p})\,\mathbf{c} + k\,\mathbf{p} = \mathbf{d},$$

if and only if k is not a root of the cubic equation,

$$k^3 + (\mathbf{b}\cdot\mathbf{c})\,k^2 + (\mathbf{a}^2 + [\mathbf{a}\,\mathbf{b}\,\mathbf{c}])\,k + (\mathbf{a}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{c}) = 0.$$

In particular show that there exists a unique value of $\mathbf{p}$ satisfying the equation,

$$\mathbf{a}\times\mathbf{p} + \mathbf{b}\times(\mathbf{b}\times\mathbf{p}) = \mathbf{d},$$

if and only if $\mathbf{a}\times\mathbf{b} \neq 0$ and that this value of $\mathbf{p}$ is given by,

$$\mathbf{p} = -\frac{}{(\mathbf{a}\times\mathbf{b})^2}\,\mathbf{a} + \frac{(\mathbf{a}\times\mathbf{b} - \mathbf{b}^2\,\mathbf{b})\cdot\mathbf{d}}{(\mathbf{a}\times\mathbf{b})^2}\,\mathbf{b} - \frac{\mathbf{b}\cdot\mathbf{d}}{(\mathbf{a}\times\mathbf{b})^2}\,\mathbf{a}\times\mathbf{b}.$$

22. Analytic Geometry.

Many of the formulas and equations of solid analytic geometry may be expressed in vector notation in a very simple and convenient form. In this section we shall present briefly those which we shall find most useful in our later work.

Distance between Two Points. Let A and B be any two points and $\mathbf{a}$ and $\mathbf{b}$ the radius vectors running from the origin O to these points. Thus the coördinates of the points are also the coördinates of the corresponding radius vectors. The distance r between the points being the length of the vector $\mathbf{b} - \mathbf{a}$ is given by the formula,

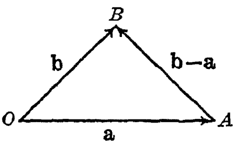

Fig. 24

(1) $$r = \sqrt{(\mathbf{b} - \mathbf{a})^2}.$$

Area of a Triangle. Let A, B, C be any three points and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the radius vectors running to them from the origin O. Then $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ are vectors forming two sides of the triangle ABC. The area T of this triangle will thus be half the length of the vector product of $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$. That is,

(2) $$T = \tfrac{1}{2}\sqrt{\{(\mathbf{b} - \mathbf{a})\times(\mathbf{c} - \mathbf{a})\}^2}.$$

This may be readily transformed to the symmetric form,

(2′) $$T = \tfrac{1}{2}\sqrt{(\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b})^2}.$$

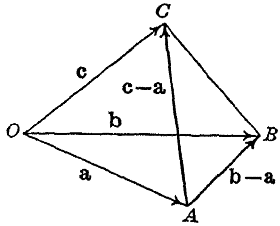

Fig. 25

Volume of a Tetrahedron. Continuing with the above notation we may find the volume V of the tetrahedron having vertices at the four points A, B, C, D. This volume will clearly be one sixth of that of the parallelopiped

having the vectors $\mathbf{b} - \mathbf{a}$, $\mathbf{c} - \mathbf{a}$, $\mathbf{d} - \mathbf{a}$ as three concurrent edges and it will therefore be given by the formula,

$$(3)\qquad V = \tfrac{1}{6}\left|[(\mathbf{b} - \mathbf{a})\ (\mathbf{c} - \mathbf{a})\ (\mathbf{d} - \mathbf{a})]\right|.$$

This may be readily transformed to the symmetric form,

$$(3')\qquad V = \tfrac{1}{6}\left|[\mathbf{b\ c\ d}] - [\mathbf{c\ d\ a}] + [\mathbf{d\ a\ b}] - [\mathbf{a\ b\ c}]\right|.$$

Equation of a Line. If we represent by $\mathbf{p}$ the radius vector running from the origin O to any point P, then an equation involving $\mathbf{p}$ may place a limitation on the values which $\mathbf{p}$ can assume and thus place a limitation on the positions which P can occupy. If we write down an equation in $\mathbf{p}$ which limits P to the points of a line we call this equation the equation of the line. Let us find the equation of a line passing through a given point A and parallel to a given vector $\mathbf{m}$. The vector $\mathbf{p} - \mathbf{a}$ running from A to any other point P on the line will then be parallel to $\mathbf{m}$ and we shall have,

$$(4)\qquad (\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0.$$

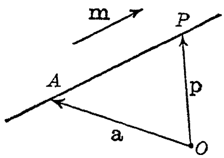

FIG. 26

Not only is this equation satisfied by the radius vectors of all the points on the line, but if it is satisfied then the point P will lie on the line, so that this is the desired equation in the *point, direction form.* If we wish to find the equation of a line passing through two given points A and B then we may write $\mathbf{b} - \mathbf{a}$ for $\mathbf{m}$ in the above equation and have as the *two point form* of the equation,

$$(5)\qquad (\mathbf{p} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a}) = 0.$$

This may be readily reduced to the symmetric form,

$$(5')\qquad \mathbf{p} \times (\mathbf{b} - \mathbf{a}) = \mathbf{a} \times \mathbf{b}.$$

If for the direction vector $\mathbf{m}$ of equation (4) we use a unit vector $\mathbf{u}$ the equation becomes,

$$(6)\qquad (\mathbf{p} - \mathbf{a}) \times \mathbf{u} = 0,$$

and is called the *normal form* of the equation of the line. It possesses an important property which we shall develop shortly.

Equations (4), (5), (6) are of the type,

$$\mathbf{k} \times \mathbf{l} = 0,$$

in which we have the cross product of two vectors set equal to

zero. In coördinate form this equation becomes a statement that every coördinate of the vector $\mathbf{k} \times \mathbf{l}$ must vanish. Thus for $\mathbf{k} \equiv (k_1, k_2, k_3)$, $\mathbf{l} \equiv (l_1, l_2, l_3)$ we have,

$$k_2 l_3 - k_3 l_2 = k_3 l_1 - k_1 l_3 = k_1 l_2 - k_2 l_1 = 0.$$

If neither $\mathbf{k}$ nor $\mathbf{l}$ is zero we may conveniently write this as the two scalar equations,

$$\frac{k_1}{l_1} = \frac{k_2}{l_2} = \frac{k_3}{l_3},$$

it being understood that a zero in any numerator requires a zero in the corresponding denominator and vice versa. Thus for $\mathbf{p} \equiv (x, y, z)$, $\mathbf{a} \equiv (a_1, a_2, a_3)$, $\mathbf{m} \equiv (m_1, m_2, m_3)$ equation (4) becomes,

$$\frac{x - a_1}{m_1} = \frac{y - a_2}{m_2} = \frac{z - a_3}{m_3}.$$

Similarly equation (5) takes the form,

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

and (6) becomes,

$$\frac{x - a_1}{u_1} = \frac{y - a_2}{u_2} = \frac{z - a_3}{u_3}$$

Here u_1, u_2, u_3, being the cosines of the angles which the unit vector $\mathbf{u}$ makes with the coördinate axes, are known as the *direction cosines* of the line.

Equation of a Plane. Using the above notation we may write out at once the equation of a plane passing through a given point A and perpendicular to a given vector $\mathbf{m}$. The vector $AP = \mathbf{p} - \mathbf{a}$ will be perpendicular to $\mathbf{m}$ when and only when P is on the plane so that the desired equation in *point, direction form* is,

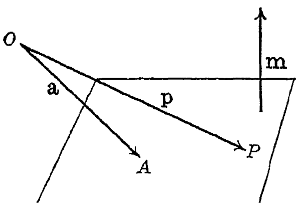

Fig. 27

(7) $$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0.$$

To get the equation of the plane through the three points, A, B, C we substitute $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ for $\mathbf{m}$ obtaining the *three point form,*

(8) $$[(\mathbf{p} - \mathbf{a})\ (\mathbf{b} - \mathbf{a})\ (\mathbf{c} - \mathbf{a})] = 0.$$

This may be readily reduced to the symmetric form,

$$(8') \qquad [\mathbf{p\,b\,c}] + [\mathbf{p\,c\,a}] + [\mathbf{p\,a\,b}] = [\mathbf{a\,b\,c}].$$

If for the vector $\mathbf{m}$ of equation (7) we use a unit vector $\mathbf{u}$ the equation becomes,

$$(9) \qquad (\mathbf{p} - \mathbf{a}) \cdot \mathbf{u} = 0,$$

and is called the *normal form* of the equation of the plane. It possesses an important property which we shall develop shortly.

Continuing with the above notation for the coördinates of the various vectors, equations (7), (8), (9) easily yield the familiar coördinate forms,

$$m_1 (x - a_1) + m_2 (y - a_2) + m_3 (z - a_3) = 0,$$

$$\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} = 0,$$

$$u_1 (x - a_1) + u_2 (y - a_2) + u_3 (z - a_3) = 0.$$

Distance from a Point to a Line. To find the distance r from a given point B of radius vector $\mathbf{b}$ to a given line whose equation in normal form is $(\mathbf{p} - \mathbf{a}) \times \mathbf{u} = 0$ we first draw a plane through B perpendicular to the line and let $\mathbf{r}$ be the projection of $\mathbf{b} - \mathbf{a}$ on this plane. Then evidently we may write,

$$r = |\mathbf{r}| = |\mathbf{r} \times \mathbf{u}| = |(\mathbf{b} - \mathbf{a}) \times \mathbf{u}|$$

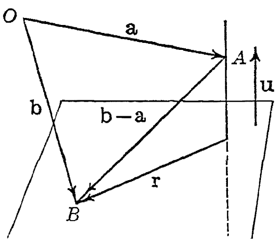

FIG. 28

the last equation coming from the fundamental theorem of the vector product which states that the vector product of two vectors is unaltered if either vector is replaced by its projection upon a plane perpendicular to the other. The desired distance is thus given by the formula,

$$(10) \qquad r = |(\mathbf{b} - \mathbf{a}) \times \mathbf{u}|$$

which we may conveniently describe by the statement,

The distance from a given point to a given line is the length of the vector obtained by substituting the radius vector of the given point in the first member of the normal form of the equation of the line.

This is the fundamental property of the normal form of the equation of a line.

Distance from a Point to a Plane. To find the distance r from a given point B of radius vector $\mathbf{b}$ to a given plane whose equation in the normal form is $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{u} = 0$ we draw a line through B perpendicular to the plane and let $\mathbf{r}$ be the projection of $\mathbf{b} - \mathbf{a}$ on this line. Then evidently we may write,

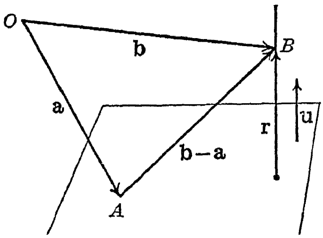

FIG. 29

$$r = |\mathbf{r}| = |\mathbf{r} \cdot \mathbf{u}| = |(\mathbf{b} - \mathbf{a}) \cdot \mathbf{u}|,$$

the last equation coming from the fundamental theorem of the scalar product which states that the scalar product of two vectors is unaltered if either vector is replaced by its projection upon the other. The desired distance is thus given by the formula,

$$(11) \qquad r = |(\mathbf{b} - \mathbf{a}) \cdot \mathbf{u}|,$$

which we may conveniently describe by the statement,

The distance from a given point to a given plane is the absolute value of the scalar obtained by substituting the radius vector of the given point in the first member of the normal form of the equation of the plane.

This is the fundamental property of the normal form of the equation of a plane. The reader will observe its close analogy to that for the line.

Distance between Two Skew Lines. To find the distance r between two skew lines whose equations are $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$ and $(\mathbf{p} - \mathbf{b}) \times \mathbf{n} = 0$ we may proceed by finding the distance from the point B on the second line to the plane containing the first line and parallel to the second. A unit vector $\mathbf{u}$ perpendicular to this plane will be perpendicular to both the given lines and is therefore given by $\mathbf{u} = \mathbf{m} \times \mathbf{n}/|\mathbf{m} \times \mathbf{n}|$. Then the equation of the plane in normal form is $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{u} = 0$ and by equation (10) the desired distance is,

$$(12) \qquad r = |(\mathbf{b} - \mathbf{a}) \quad \mathbf{u}| \qquad \text{where} \qquad \mathbf{u} = \frac{\mathbf{m} \times \mathbf{n}}{|\mathbf{m} \times \mathbf{n}|}$$

EXERCISES

1. Determine by vector methods the lengths of the sides, the cosines of the interior angles and the area of the triangle having the vertices,

   (*a*) $A = (-1, 2, 0)$, $B = (0, 4, 2)$, $C = (2, 0, 6)$
   (*b*) $A = (-8, -5, 2)$, $B = (0, 3, 6)$, $C = (6, 0, 0)$
   (*c*) $A = (0, -1, 1)$, $B = (2, 1, 0)$, $C = (2, -2, 3)$

   Ans. (*a*) 3, 6, 7; $\frac{11}{21}$, $-\frac{1}{9}$, $\frac{19}{21}$; 8.944

2. How far is the point P from the plane determined by the three points A, B, C?

   (*a*) $P = (0, 5, 1)$,
   $A = (-2, 6, -6)$, $B = (-3, 10, -9)$, $C = (-5, 0, -6)$
   (*b*) $P = (-1, 1, -2)$,
   $A = (1, 3, 2)$, $B = (4, -1, 1)$, $C = (-2, 1, 0)$
   (*c*) $P = (7, 2, 1)$,
   $A = (2, 3, 2)$, $B = (-6, 3, 1)$, $C = (-10, 2, 0)$

   Ans. (*a*) 3

3. How far is the point P from the line determined by the two points A, B?

   (*a*) $P = (-1, 2, 6)$, $A = (2, 3, -4)$, $B = (8, 6, -8)$
   (*b*) $P = (-1, 3, 0)$, $A = (0, 1, 9)$, $B = (0, 3, 10)$
   (*c*) $P = (5, -1, 4)$, $A = (1, 1, 8)$, $B = (4, 13, 5)$

   Ans. (*a*) 7

4. How far is the line through the points A, B from the line through the points C, D?

   (*a*) $A = (-16, 6, 4)$, $B = (-1, 2, -3)$,
   $C = (1, -1, 3)$, $D = (4, 9, 7)$
   (*b*) $A = (-9, -1, 4)$, $B = (3, 2, 1)$,
   $C = (6, 3, -1)$, $D = (-10, -2, 2)$
   (*c*) $A = (1, 0, 0)$, $B = (-1, 5, 6)$,
   $C = (1, 2, -1)$, $D = (1, -1, -4)$

   Ans. (*a*) 7

5. Derive a formula for the vector $\mathbf{r}$ running perpendicularly from the plane $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$ to the point with radius vector $\mathbf{b}$.

   $$Ans.\quad \mathbf{r} = \left\{ \frac{(\mathbf{b} - \mathbf{a})\cdot\mathbf{m}}{\mathbf{m}^2} \right\}\mathbf{m}$$

6. Derive a formula for the vector $\mathbf{r}$ running perpendicularly from the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$ to the point with radius vector $\mathbf{b}$.

   $$Ans.\quad \mathbf{r} = \frac{\mathbf{m} \times \{(\mathbf{b} - \mathbf{a}) \times \mathbf{m}\}}{\mathbf{m}^2}$$

7. Derive a formula for the vector $\mathbf{r}$ running along the common perpendicular from the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$ to the line

   $$(\mathbf{p} - \mathbf{b}) \times \mathbf{n} = 0.$$

   $$Ans.\quad \mathbf{r} = \frac{[(\mathbf{b} - \mathbf{a})\,\mathbf{m}\,\mathbf{n}]}{(\mathbf{m} \times \mathbf{n})^2}\,\mathbf{m} \times \mathbf{n}$$

8. Find the radius vector $\mathbf{q}$ of the point of intersection of the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$ and the plane $(\mathbf{p} - \mathbf{b}) \cdot \mathbf{n} = 0$.

$$\textit{Ans.} \quad \mathbf{q} = \mathbf{a} + \left\{ \frac{(\mathbf{b} - \mathbf{a})\cdot\mathbf{n}}{\mathbf{m} \cdot \mathbf{n}} \right\} \mathbf{m}$$

9. Find the radius vector $\mathbf{q}$ of the point of intersection of the three planes $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$, $(\mathbf{p} - \mathbf{b}) \cdot \mathbf{n} = 0$, $(\mathbf{p} - \mathbf{c}) \cdot \mathbf{r} = 0$.

$$\textit{Ans.} \quad \mathbf{q} = \frac{\mathbf{a} \cdot \mathbf{m}}{[\mathbf{m\,n\,r}]} \mathbf{n} \times \mathbf{r} + \frac{}{[\mathbf{m\,n\,r}]} \mathbf{r} \times \mathbf{m} + \frac{}{[\mathbf{m\,n\,r}]} \mathbf{m} \times \mathbf{n}$$

10. What is the equation of the line of intersection of the two planes $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$ and $(\mathbf{p} - \mathbf{b}) \cdot \mathbf{n} = 0$?

$$\textit{Ans.} \quad (\mathbf{p} - \mathbf{c}) \times (\mathbf{m} \times \mathbf{n}) = 0 \qquad \text{where}$$

$$\mathbf{c} = \frac{\mathbf{a} \cdot \mathbf{m}}{(\mathbf{m} \times \mathbf{n})^2} \mathbf{n} \times (\mathbf{m} \times \mathbf{n}) + \frac{\mathbf{b} \cdot \mathbf{n}}{(\mathbf{n} \times \mathbf{m})^2} \mathbf{m} \times (\mathbf{n} \times \mathbf{m})$$

11. Find the radius vector $\mathbf{q}$ of the foot of the perpendicular let fall from the point B upon the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$.

$$\textit{Ans.} \quad \mathbf{q} = \mathbf{a} + \left\{ \frac{(\mathbf{b} - \mathbf{a})\cdot\mathbf{m}}{\mathbf{m}^2} \right\} \mathbf{m}$$

12. Find the radius vector $\mathbf{q}$ of the foot of the perpendicular let fall from the point B upon the plane $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$.

$$\textit{Ans.} \quad \mathbf{q} = \mathbf{b} + \left\{ \frac{(\mathbf{a} - \mathbf{b})\cdot\mathbf{m}}{\mathbf{m}^2} \right\} \mathbf{m}$$

23. Centroid of a Set of Particles.

If we have given a set of n points $P_1, P_2, \cdots, P_n$, we may assign to each point P_i a scalar m_i. Each point and its associated scalar may then be thought of as a particle, the scalar m_i being regarded as the mass of the particle at the point P_i. Evidently a set of such particles will be completely characterized by their radius vectors $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ together with their masses $m_1, m_2, \cdots, m_n$. If two or more such sets of particles are given, then the *sum* of these sets is defined to be the set consisting of all the particles of all the given sets.

If we have such a set of n particles $P_1, P_2, \cdots, P_n$ with radius vectors $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ and corresponding masses $m_1, m_2, \cdots, m_n$ then we define the *centroid* of the set to be the particle P_0 with radius vector $\mathbf{p}_0$ and mass m_0 where,

$$\mathbf{p}_0 = \frac{m_1 \mathbf{p}_1 + m_2 \mathbf{p}_2 + \cdots + m_n \mathbf{p}_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m_i \mathbf{p}_i}{\Sigma m_i},$$

$$m_0 = m_1 + m_2 + \cdots + m_n = \Sigma m_i.$$

Consequently we always have,

$$m_0 \mathbf{p}_0 = \Sigma m_i \mathbf{p}_i.$$

An important property of the centroid is given by the theorem,

The centroid of the sum of two or more given sets of particles is the centroid of the set consisting of the centroids of the given sets.

To simplify the notation we shall prove the theorem only for the case of a set which is the sum of two given sets, but the proof is clearly capable of immediate extension to a set consisting of any finite number of subsets. Let the particles P_i of one given set have radius vectors $\mathbf{p}_i$ and masses m_i while the particles Q_i of the other set have radius vectors $\mathbf{q}_i$ and masses n_i. Then it follows from the definitions of the corresponding centroids P_0 and Q_0 that,

$$\Sigma m_i \mathbf{p}_i = m_0 \mathbf{p}_0, \qquad \Sigma n_i \mathbf{q}_i = n_0 \mathbf{q}_0.$$

The radius vector $\mathbf{r}_0$ and mass l_0 of the centroid of the sum of these two sets will be by definition,

$$\mathbf{r}_0 = \frac{\Sigma m_i \mathbf{p}_i + \Sigma n_i \mathbf{q}_i}{\Sigma m_i + \Sigma n_i}; \qquad l_0 = \Sigma m_i + \Sigma n_i,$$

and by the above equations these may be written,

$$\mathbf{r}_0 = \frac{m_0 \mathbf{p}_0 + n_0 \mathbf{q}_0}{m_0 + n_0}, \qquad l_0 = m_0 + n_0.$$

But these are clearly also the radius vector and mass of the centroid of the set consisting of the two particles P_0 and Q_0. The theorem is thus proven for this case.

Some geometric properties of the centroid are given in the following problems. The mechanical properties will be developed later.

EXERCISES

1. Find the centroids of the following sets of particles.

(a) $\mathbf{p}_1 = (3, 1, 5)$, $m_1 = 2$; $\mathbf{p}_2 = (1, 3, -4)$, $m_2 = 3$; $\mathbf{p}_3 = (0, 4, 5)$, $m_3 = 4$

(b) $\mathbf{p}_1 = (3, -3, -1)$, $m_1 = 1$; $\mathbf{p}_2 = (2, -1, -2)$, $m_2 = 2$; $\mathbf{p}_3 = (0, 3, -4)$, $m_3 = 4$

(c) $\mathbf{p}_1 = (-1, 0, 2)$, $m_1 = 4$; $\mathbf{p}_2 = (2, 1, -3)$, $m_2 = 2$; $\mathbf{p}_3 = (0, -2, -2)$, $m_3 = 1$

Ans. (a) $\mathbf{p}_0 = (1, 3, 2)$, $m_0 = 9$

2. Prove that if all the particles of a set are on a straight line, then the centroid will be on that line. Prove that if all the particles of a set are on a plane, then the centroid will be on that plane.

3. Prove that the centroid of two particles will be on the line joining them and at distances from them inversely proportional to their masses.
4. Prove that if all the particles of a set with positive masses are on or within a sphere, then the centroid will be on or within that sphere.
5. If three particles with equal masses are placed at the vertices of a triangle, prove that the centroid will be at the point of intersection of the medians of that triangle.
6. If three particles be placed at the vertices of a triangle and have masses proportional to the lengths of the opposite sides of the triangle, prove that the centroid will be at the point of intersection of the bisectors of the interior angles of the triangle.
7. Find a construction for the position of the centroid of a set of three particles which are placed at the mid-points of the sides of a triangle and have masses proportional to the lengths of the sides on which they lie.
8. If three particles be placed at the vertices of a triangle and have masses proportional to the tangents of the interior angles at those vertices, prove that the centroid will be at the point of intersection of the altitudes of the triangle.
9. If three particles be placed at the vertices of a triangle, prove that the three lines joining their centroid to the three vertices divide the triangle into parts whose areas are proportional to the masses of the opposite particles.
10. If four particles be placed at the vertices of a tetrahedron, prove that the six planes joining their centroid to the six edges divide the tetrahedron into parts whose volumes are proportional to the masses of the opposite particles.

24. Vector-Scalar Functions.

If x and y are two variables which may be either scalars or vectors independently and if these variables are related in such a way that there is a value of y assigned to each of a certain set of values of x, then we say that y is a single-valued function of x for this set of values.* The most familiar case of this is that in which x and y are both scalars and y is expressed by a formula in terms of x which yields a value corresponding to each value of x in a certain range. Thus for example if we have,

$$y = \sqrt{1 - x^2}, \qquad -1 \leq x \leq +1,$$

* Following Dirichlet's definition.

then y is a scalar function of the scalar x in the given range. There are however many ways in which y might be given as a function of x without it being possible to express y in terms of x by means of the ordinary symbols of analysis.

If, as in the above example, both x and y are scalars we say that y is a *scalar-scalar* function of x. This is the case discussed in the calculus. If y takes on only vector values while x remains a scalar then y is said to be a *vector-scalar* function of x. Similarly we have *scalar-vector* and *vector-vector* functions. The case which interests us at present is that of the vector-scalar function. We let the independent variable which is to remain a scalar be represented by the letter t and the dependent vector variable by $\mathbf{p}$ and express the functional relationship by the form $\mathbf{p} = \mathbf{p}(t)$, it being usually understood that t is to remain in some limited range of scalar values. As in the scalar-scalar example given above we may be able to write down a formula expressing $\mathbf{p}$ explicitly in terms of t such as,

$$\mathbf{p} \equiv (t^3, 3t^2, 6t),$$

but often $\mathbf{p}$ may be such a function of t that this can not be done by means of the ordinary symbols of analysis.

In the calculus we found it an aid in the study of a scalar-scalar function to consider the graph of the function which represents the relation of y to x by means of a plane curve drawn relative to the Cartesian coördinate system. In the study of a vector-scalar function we find it an aid to regard t as the measure of the time elapsed since some definite instant and to plot $\mathbf{p}$ as a radius vector running from some fixed point O to a moving point P. The *motion* of the point P then gives us an interpretation of the way in which $\mathbf{p}$ varies as a function of t.

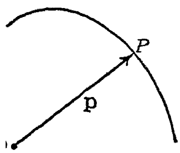

FIG. 30

If we consider a set of values of t differing by successively smaller amounts from some fixed value t_0 it will often happen that the corresponding values of $\mathbf{p}$ will ultimately all differ very little from some definite vector $\mathbf{p}_0$. In fact if we are given any positive scalar ϵ then it may always be possible to find a corresponding positive scalar δ such that we will have $|\mathbf{p} - \mathbf{p}_0| < \epsilon$ whenever we have $0 < |t - t_0| < \delta$. If this is the case we express the fact briefly by saying that as t approaches the limit t_0, $\mathbf{p}$ approaches

the limit $\mathbf{p}_0$. We write this in the form,

$$\mathbf{p} \to \mathbf{p}_0 \quad \text{for} \quad t \to t_0, \quad \text{or} \quad \lim_{t \to t_0} \mathbf{p} = \mathbf{p}_0.$$

It is here required that there be a value of $\mathbf{p}$ corresponding to every value of t sufficiently near to t_0 but nothing at all is implied concerning what happens when t actually takes on the value t_0. The reader will observe that all this is highly analogous to the case of the limit of scalar-scalar functions as discussed in the calculus.

If $\mathbf{p}$ and $\mathbf{q}$ are any two vector functions of the scalar t while m is a scalar function of t and if we have,

$$\mathbf{p} \to \mathbf{p}_0, \quad \mathbf{q} \to \mathbf{q}_0, \quad m \to m_0 \quad \text{for} \quad t \to t_0,$$

then it may be shown that we also have,

$$\mathbf{p} + \mathbf{q} \to \mathbf{p}_0 + \mathbf{q}_0, \quad m\,\mathbf{p} \to m_0\,\mathbf{p}_0, \quad \mathbf{p}\cdot\mathbf{q} \to \mathbf{p}_0\cdot\mathbf{q}_0,$$
$$\mathbf{p} \times \mathbf{q} \to \mathbf{p}_0 \times \mathbf{q}_0 \quad \text{for} \quad t \to t_0.$$

The proofs are left as an exercise for the reader.

It may happen that the limit which $\mathbf{p}$ approaches as t approaches t_0 is the value which $\mathbf{p}$ takes on when t takes on the value t_0. That is,

$$\lim_{t \to t_0} \mathbf{p} = \mathbf{p}(t_0).$$

When this is the case we say that $\mathbf{p}$ is a *continuous function* of t for $t = t_0$. If $\mathbf{p}$ is thus continuous for every value of t throughout a certain range we say that $\mathbf{p}$ is *continuous in the range.* It follows from the above theorems on limits that if $\mathbf{p}$, $\mathbf{q}$ and m are continuous functions of t in a given range then $\mathbf{p} + \mathbf{q}$, $m\,\mathbf{p}$, $\mathbf{p}\cdot\mathbf{q}$, $\mathbf{p} \times \mathbf{q}$ are likewise continuous functions of t in that range.

25. Derivative of a Vector-Scalar Function.

Let us assume that we have the vector $\mathbf{p}$ defined as a function $\mathbf{p}(t)$ of the scalar t for every value of t in some range $a < t < b$. Let t_0 be some value of t in this range and let $\mathbf{p}_0 = \mathbf{p}(t_0)$ be the corresponding value of $\mathbf{p}$. Then consider any other value of t in the range with its corresponding value of $\mathbf{p}$ and form the ratio,

$$\frac{\Delta\mathbf{p}}{\Delta t} \quad \text{where,} \quad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 \quad \text{and,} \quad \Delta t = t - t_0.$$

If we now let this second value of t approach the limit t_0 the denominator of the above fraction will approach zero and for that

reason alone we might expect the vector $\Delta\mathbf{p}/\Delta t$ to become infinitely long. But if $\mathbf{p}$ is a continuous function of t for $t = t_0$ we will also have $\mathbf{p}$ approaching the limit $\mathbf{p}_0$ and $\Delta\mathbf{p}$ will approach zero, for which reason alone we might expect the vector $\Delta\mathbf{p}/\Delta t$ to become very short. Since however both of these operations take place, it is quite possible for $\Delta\mathbf{p}/\Delta t$ to approach some definite vector as a limit. When this is the case we call this limit vector $\mathbf{p}_0'$ *the derivative of* $\mathbf{p}$ *with respect to* t for $t = t_0$.

$$\mathbf{p}_0' = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{p}}{\Delta t}.$$

If this limit vector $\mathbf{p}_0'$ exists for every value of t_0 in a certain range then these values of $\mathbf{p}_0'$ form a new vector-scalar function of t, $\mathbf{p}'(t)$ which we call the derived function or simply the derivative of $\mathbf{p}$ with respect to t. This function $\mathbf{p}'(t)$ may likewise possess a derivative with respect to t which we indicate by $\mathbf{p}''(t)$, and continuing in this fashion derivatives of successively higher orders may be obtained.

The reader will observe that formally the above definition of derivative is highly analogous to that for a scalar-scalar function, although as a matter of fact the analogy is more formal than real, for it must not be overlooked that the subtraction, division, and limiting process here involved are operations on vectors and are not the same as those bearing the same name and symbol for scalars. Following this analogy we have no difficulty in deriving suitable formulas for use in formal differentiation. If $\mathbf{p}$ and $\mathbf{q}$ are vector functions of the scalar t and m is a scalar function of t, then whenever the derivatives in the second members exist we have the relations,

(1) $\mathbf{a}' = 0$ (a constant)

(2) $(\mathbf{p} + \mathbf{q})' = \mathbf{p}' + \mathbf{q}'$

(3) $(m\,\mathbf{p})' = m\,\mathbf{p}' + m'\,\mathbf{p}$

(3′) $(k\,\mathbf{p})' = k\,\mathbf{p}'$ (k constant)

(3″) $(m\,\mathbf{a})' = m'\,\mathbf{a}$ (a constant)

(4) $(\mathbf{p}\cdot\mathbf{q})' = \mathbf{p}\cdot\mathbf{q}' + \mathbf{p}'\cdot\mathbf{q}$

(4′) $(\mathbf{a}\cdot\mathbf{p})' = \mathbf{a}\cdot\mathbf{p}'$ (a constant)

(4″) $(\mathbf{p}^2)' = 2\,\mathbf{p}\cdot\mathbf{p}'$

(5) $(\mathbf{p}\times\mathbf{q})' = \mathbf{p}\times\mathbf{q}' + \mathbf{p}'\times\mathbf{q}$

(5′) $(\mathbf{a}\times\mathbf{p})' = \mathbf{a}\times\mathbf{p}'$ (a constant)

We present the proof of formula (4) which gives a sufficient indication of how the others may be proven. Let $\mathbf{p}_0$ and $\mathbf{q}_0$ be the values of $\mathbf{p}$ and $\mathbf{q}$ for $t = t_0$ and let $\Delta(\mathbf{p}\cdot\mathbf{q})$, $\Delta\mathbf{p}$, $\Delta\mathbf{q}$ be the increments in the indicated functions when t changes from t_0 to $t_0 + \Delta t$. Then by definition we have,

$$\Delta(\mathbf{p}\cdot\mathbf{q}) = (\mathbf{p}_0 + \Delta\mathbf{p})\cdot(\mathbf{q}_0 + \Delta\mathbf{q}) - \mathbf{p}_0\cdot\mathbf{q}_0,$$

and consequently,

$$\frac{\Delta(\mathbf{p}\cdot\mathbf{q})}{\Delta t} = \mathbf{p}_0 \quad \frac{\Delta\mathbf{q}}{\Delta t} + \frac{\Delta\mathbf{p}}{\Delta t}\cdot\mathbf{q}_0 + \frac{\Delta\mathbf{p}}{\Delta t}\cdot\frac{\Delta\mathbf{q}}{\Delta t}\Delta t.$$

Applying the familiar theorems on the limit of the sum and the limit of the product of scalar variables together with our theorem on the limit of the scalar product of two vector variables, the above equation becomes in the limit,

$$(\mathbf{p}\cdot\mathbf{q})' = \mathbf{p}\cdot\mathbf{q}' + \mathbf{p}'\cdot\mathbf{q} + (\mathbf{p}'\cdot\mathbf{q}')\,0,$$

which proves formulas (4), (4′), (4″).

If the vector $\mathbf{p}$ *be given as a function of* t *by having its three coördinates* x, y, z *given as scalar functions of* t, *then if these scalar functions possess derivatives with respect to* t, *these derivatives form the coördinates of the derivative* $\mathbf{p}'$ *of* $\mathbf{p}$ *with respect to* t.

For we have $\mathbf{p} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and on applying formulas (2) and (3″) we find $\mathbf{p}' = x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k}$ as stated.

When $\mathbf{p}$ is a vector function of t then of course the length p of $\mathbf{p}$ is a scalar function of t. Its derivative is given by the formula,

$$p' = \frac{\mathbf{p}\cdot\mathbf{p}'}{p}.$$

For we have $p^2 = \mathbf{p}^2$ and differentiating both members gives $2pp' = 2\,\mathbf{p}\cdot\mathbf{p}'$ from which the above formula follows at once. This of course fails whenever $\mathbf{p}'$ does not exist and whenever $p = 0$.

If as in § 24 we regard t as the measure of the time elapsed since some definite instant and $\mathbf{p}$ as the radius vector from some fixed point O to a moving point P, then the derivative $\mathbf{p}'$ is also an easily visualized vector. In fact we shall see in § 29 that $\mathbf{p}'$ has the direction of the tangent to the path of P at P and points in the sense in which P is moving, while the length of $\mathbf{p}'$ is the

absolute value of the scalar velocity with which P is tracing out its path.

EXERCISES

1. Show analytically that if a vector-scalar function $\mathbf{p}(t)$ maintains a constant angle with each of two constant vectors $\mathbf{a}$ and $\mathbf{b}$ where $\mathbf{a} \times \mathbf{b} \neq 0$ then if $\mathbf{p}'$ exists, $\mathbf{p} \times \mathbf{p}' = 0$.

2. If $\mathbf{p}(t)$ is a vector-scalar function possessing a derivative throughout a certain interval and vanishing in that interval only for $t = t_0$ and if $u(t)$ is a scalar function of t given by the formulas,

$$u(t) = +|\mathbf{p}| \quad \text{for} \quad t \geqq t_0, \qquad u(t) = -|\mathbf{p}| \quad \text{for} \quad t \leqq t_0,$$

show that u' exists throughout the above interval and is given by the formulas,

$$u'(t) = \mathbf{p}\cdot\mathbf{p}'/u \quad \text{for} \quad t \neq t_0, \qquad u' = |\mathbf{p}'| \quad \text{for} \quad t = t_0.$$

26. Taylor's Theorem for Vector-Scalar Functions.

If $f(x)$ is any scalar function of the scalar variable x, continuous together with its first n derivatives in the interval $a \leqq x \leqq b$, then if x_0 and $x_0 + \Delta x$ are any distinct values of x in this interval and if we define the quantity ϵ by the equation,

$$(1) \quad f(x_0 + \Delta x) = f(x_0) + (\Delta x) f'(x_0) + \frac{(\Delta x)^2}{2!} f''(x_0) + \cdots + \frac{(\Delta x)^n}{n!} \{f^{(n)}(x_0) + \epsilon\},$$

then it is possible to show that,

$$\lim_{\Delta x \to 0} \epsilon = 0.$$

This is one form of Taylor's Theorem for real functions of a real variable substantially as developed in the calculus.

An analogous theorem may be proven for vector functions of a scalar. Let $\mathbf{p}(t)$ be any vector function of the scalar t, continuous together with its first n derivatives in the interval $a \leqq t \leqq b$ and let t_0 and $t_0 + \Delta t$ be any distinct values of t in this interval. Then if the vector $\boldsymbol{\varepsilon}$ is defined by the equation,

$$(2) \quad \mathbf{p}(t_0 + \Delta t) = \mathbf{p}(t_0) + (\Delta t)\, \mathbf{p}'(t_0) + \frac{(\Delta t)^2}{2!} \mathbf{p}''(t_0) + \cdots + \frac{(\Delta t)^n}{n!} \{\mathbf{p}^{(n)}(t_0) + \boldsymbol{\varepsilon}\},$$

we may show that,

$$\lim_{\Delta t \to 0} \boldsymbol{\varepsilon} = 0.$$

To show this we let $\mathbf{a}$ be any constant vector and consider the scalar function $f(t) = \mathbf{a} \cdot \mathbf{p}(t)$. Since $f'(t) = \mathbf{a} \cdot \mathbf{p}'(t)$, $f''(t) = \mathbf{a} \cdot \mathbf{p}''(t)$, etc., it is clear that $f(t)$ will be continuous together with its first n derivatives in the interval considered and we may apply to it Taylor's theorem for scalar functions of a scalar as given above. If then in equation (1) we substitute $\mathbf{a} \cdot \mathbf{p}(t)$ for $f(x)$ throughout and dot-multiply the members of equation (2) through by $\mathbf{a}$, it appears on comparison of the two that $\epsilon = \mathbf{a} \cdot \boldsymbol{\varepsilon}$ and we may consequently write,

$$\lim_{\Delta t \to 0} \mathbf{a} \cdot \boldsymbol{\varepsilon} = 0.$$

But since $\mathbf{a}$ is any vector we care to choose it follows that we have,

$$\lim_{\Delta t \to 0} \boldsymbol{\varepsilon} = 0,$$

as we wished to prove. In fact if we take $\mathbf{a}$ successively equal to the three coördinate vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we have the desired result at once from the fact that,

$$\boldsymbol{\varepsilon} = (\mathbf{i} \cdot \boldsymbol{\varepsilon})\, \mathbf{i} + (\mathbf{j} \cdot \boldsymbol{\varepsilon})\, \mathbf{j} + (\mathbf{k} \cdot \boldsymbol{\varepsilon})\, \mathbf{k}.$$

27. Some Vector-Scalar Differential Equations.

When the vector $\mathbf{p}$ is a function $\mathbf{p}(t)$ of the scalar t the manner in which $\mathbf{p}$ depends on t may be restricted by means of an equation involving certain of the derivatives of $\mathbf{p}$ with respect to t. We call this a *vector-scalar differential equation.* If such an equation is given we may be able to write an equation free from derivatives and restricting $\mathbf{p}$ in such a way that it will satisfy the differential equation. We then say that the new equation is a *solution* of the given differential equation.

Probably the simplest vector-scalar differential equation is

$$\mathbf{p}'(t) = \mathbf{q}(t),$$

in which $\mathbf{q}(t)$ is a given vector-scalar function of t. If we call $\mathbf{p}'(t) \equiv (x', y', z')$ and $\mathbf{q}(t) \equiv (f, g, h)$ then the given equation is equivalent to the three scalar differential equations,

$$x' = f(t), \qquad y' = g(t), \qquad z' = h(t),$$

the general solutions of which may be at once written down,

$$x = \int_{t_0}^{t} f(t)\,dt + x_0, \qquad y = \int_{t_0}^{t} g(t)\,dt + y_0,$$

$$z = \int_{t_0}^{t} h(t)\,dt + z_0,$$

t_0, x_0, y_0, z_0 being scalar constants. We may write these as,

$$\mathbf{p} = \int_{t_0}^{t} \mathbf{q}(t)\,dt + \mathbf{p}_0,$$

$\mathbf{p}_0$ being a constant vector. This is the general solution of the given equation.

If the given differential equation is,

(1) $$\mathbf{p}\cdot\mathbf{p}' = 0,$$

we may easily show that every solution of this equation is given by the equation,

(2) $$|\mathbf{p}| = c,$$

in which c is a scalar constant, positive or zero, that we are at liberty to choose. To show that equation (2) is a solution of equation (1) for every such choice of c we note that from (2) we have $\mathbf{p}^2 = |\mathbf{p}|^2 = c^2$. Differentiating with respect to t gives us $2\,\mathbf{p}\cdot\mathbf{p}' = 0$ from which equation (1) follows. Conversely if equation (1) be given it follows that $\mathbf{p}^2$ is a positive constant which we may call c^2 and hence $|\mathbf{p}| = c$.

If we have given the differential equation,

(3) $$\mathbf{p} \times \mathbf{p}' = 0,$$

we may show that every solution of this equation is given by the equation,

(4) $$\mathbf{p} \times \mathbf{c} = 0,$$

in which $\mathbf{c}$ is a non-zero constant vector which we are at liberty to choose. To show that equation (4) is a solution of equation

(3) for every such choice of $\mathbf{c}$ we note that from (4) we have $0 = \mathbf{c} \times (\mathbf{p} \times \mathbf{c}) = \mathbf{c}^2\,\mathbf{p} - (\mathbf{c}\cdot\mathbf{p})\,\mathbf{c}$ or $\mathbf{c}^2\,\mathbf{p} = (\mathbf{c}\cdot\mathbf{p})\,\mathbf{c}$. Differentiating with respect to t yields $\mathbf{c}^2\,\mathbf{p}' = (\mathbf{c}\cdot\mathbf{p}')\,\mathbf{c}$ and consequently $\mathbf{c}^4\,\mathbf{p} \times \mathbf{p}' = 0$ from which equation (3) follows. Conversely if equation (3) be given we first observe that if $\mathbf{p}$ is identically zero equation (4) follows at once. If $\mathbf{p}$ is not identically zero we may consider any interval in which it does not vanish and let $\mathbf{u}$ be a variable unit vector remaining always parallel to $\mathbf{p}$. Then,

$$\mathbf{p} \times \mathbf{u} = 0, \qquad \mathbf{u}^2 = 1, \qquad \mathbf{u}\cdot\mathbf{u}' = 0,$$

and we find successively,

$$0 = \mathbf{u} \times (\mathbf{p} \times \mathbf{u}) = \mathbf{u}^2\,\mathbf{p} - (\mathbf{p}\cdot\mathbf{u})\,\mathbf{u}, \qquad \mathbf{p} = (\mathbf{p}\cdot\mathbf{u})\,\mathbf{u},$$
$$\mathbf{p}' = (\mathbf{p}\cdot\mathbf{u})\,\mathbf{u}' + (\mathbf{p}\cdot\mathbf{u})'\,\mathbf{u}, \qquad 0 = \mathbf{p} \times \mathbf{p}' = (\mathbf{p}\cdot\mathbf{u})^2\,\mathbf{u} \times \mathbf{u}'.$$

Now since $\mathbf{p}\cdot\mathbf{u}$ can not vanish in the interval considered because $\mathbf{p}$ would then vanish also it follows that,

$$0 = \mathbf{u} \times (\mathbf{u} \times \mathbf{u}') = (\mathbf{u}\cdot\mathbf{u}')\,\mathbf{u} - \mathbf{u}^2\,\mathbf{u}', \qquad \mathbf{u}' = 0.$$

Thus $\mathbf{u}$ is a constant and our equation $\mathbf{p} \times \mathbf{u} = 0$ is of the form $\mathbf{p} \times \mathbf{c} = 0$ which we desired to prove.

We have observed that the differential equation (3) possesses the singular solution $\mathbf{p} = 0$ and in this case this is also a branch point for all the other solutions. Thus while always satisfying equation (3), $\mathbf{p}$ might vary in such a way as to satisfy equation (4) for some value of $\mathbf{c}$ and then on passing through the value zero, $\mathbf{p}$ might again vary satisfying equation (4) for another value of $\mathbf{c}$. Thus the vector $\mathbf{c}$ in equation (4) must be regarded as a parameter retaining a constant value as long as $\mathbf{p}$ is not zero but capable of changing to another constant value when $\mathbf{p}$ passes through the value zero.

An equation of the form,

$$(5) \qquad T_0(t)\,s'' + T_1(t)\,s' + T_2(t)\,s + T_3(t) = 0,$$

in which t and s are scalar variables and the $T_i(t)$ are given functions of t places a certain restriction on the way in which s may vary as a function of t. To solve such a linear differential equation of the second order we may proceed by first finding the general solution of the equation obtained from equation (5) by replacing $T_3(t)$ by zero. This is of the form $c_1\,\varphi_1(t) + c_2\,\varphi_2(t)$

where c_1 and c_2 are any constants and is known as the *complementary function.* Then to get the *general solution* of equation (5) we add to this complementary function any *particular solution* of equation (5). Thus if $f(t)$ is any solution of equation (5) the general solution will be,

$$s = f(t) + c_1\,\varphi_1(t) + c_2\,\varphi_2(t). \tag{6}$$

Analogous to equation (5) we have the equation,

$$T_0(t)\,\mathbf{p}'' + T_1(t)\,\mathbf{p}' + T_2(t)\,\mathbf{p} + T_3(t)\,\mathbf{a} = 0, \tag{7}$$

in which t is the independent scalar variable, $\mathbf{p}$ is a dependent vector variable and $\mathbf{a}$ is a vector constant. This is a *linear vector differential equation* of the second order and is equivalent to the three scalar equations we would obtain by setting the three coordinates of its first member equal to zero. Thus if $\mathbf{p} \equiv (x, y, z)$ and $\mathbf{a} \equiv (a_1, a_2, a_3)$ it appears that equation (7) is equivalent to the three equations obtained by replacing s in *the associated equation* (5) by x/a_1, y/a_2, z/a_3, respectively. In other words x/a_1, y/a_2, z/a_3 must be three solutions of equation (5) and therefore given by formula (6) for some choices of the constants c_1 and c_2. We may therefore write,

$$\frac{x}{a_1} = f(t) + k_1\,\varphi_1(t) + k_2\,\varphi_2(t),$$

$$\frac{y}{a_2} = f(t) + l_1\,\varphi_1(t) + l_2\,\varphi_2(t),$$

$$\frac{z}{a_3} = f(t) + m_1\,\varphi_1(t) + m_2\,\varphi_2(t),$$

in which $k_1, \cdots, m_2$ are arbitrary scalar constants. These three equations are however equivalent to the single vector equation,

$$\mathbf{p} = f(t)\,\mathbf{a} + \mathbf{c}_1\,\varphi_1(t) + \mathbf{c}_2\,\varphi_2(t), \tag{8}$$

in which $\mathbf{c}_1$ and $\mathbf{c}_2$ are constant vectors whose value we may assign at pleasure. Equation (8) thus gives us the general solution of equation (7) whenever equation (6) is the general solution of the associated equation (5).

CHAPTER IV

CURVILINEAR MOTION OF A PARTICLE

28. Determination of the Motion.

Having completed a brief survey of the elements of vector analysis in the preceding chapter we are now in a position to use this analysis in the study of further topics in theoretical mechanics. We shall first take up the curvilinear motion of a particle and in this study we shall find that vectors may be used in a way strikingly similar to that in which scalars were used in the study of the rectilinear motion of a particle. Just as the position of a point on a line may be determined by a scalar s giving its distance and sense from some point O regarded as fixed, so we shall let the position of a point in space be determined by a vector $\mathbf{p}$ running to it from the origin of a coördinate system which we shall regard as fixed. It will be observed that in both cases it is really only the position of the point *relative* to the origin or coördinate system that we express by s or $\mathbf{p}$, and hence we must first make an agreement as to the origin or coördinate system to be employed. In fact it is very doubtful if any valuable meaning could be assigned to the words, "absolute position."

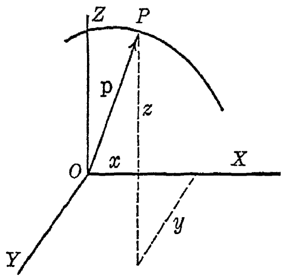

FIG. 31

The instant at which the point P occupies any given position may, as before, be expressed by a scalar t as soon as an agreement has been reached as to the instant to be used as the origin from which t is measured and the unit of time and sense to be employed.

If then a point or particle P trace out some path, its motion sets up a correspondence between the time t at which it occupies each position of its path and the vector $\mathbf{p}$ from the origin to that position. There being thus a value of $\mathbf{p}$ corresponding to each value of t we say that $\mathbf{p}$ is a function of t and write,

$$\mathbf{p} = \mathbf{p}(t).$$

Conversely, if $\mathbf{p}$ be given as a vector function of the scalar t we may, employing the above agreements, interpret this functional relation by a corresponding motion of a point P.

It might be well to state explicitly at this time some assumptions which were made tacitly when we undertook the study of rectilinear motion of a particle. As the point P moves about, its coördinates (x, y, z) which are likewise the coördinates of its radius vector $\mathbf{p}$ must be measured by some kind of a measuring stick. We assume that the length of this stick does not change as we take it from place to place. The instant at which P occupies its various positions must also be observed by an observer provided with some sort of a clock. We shall assume that the times observed are independent of the particular observer and clock employed and of the manner in which they move about as the observations are carried out. In other words, we assume that *distances and times are absolute.* These assumptions appear at first thought obviously justified and in fact mechanics was studied for centuries without any question arising concerning them. In the modern relativity mechanics however, these matters are gone into very carefully and other somewhat less simple hypotheses replace the above assumptions. The conclusions resulting are theoretically quite different from those which we shall reach but quantitatively the differences are very slight.

29. Vector Velocity.

If the motion of a point or particle P in tracing out a curve be determined by the equation,

$$(1) \qquad \mathbf{p} = \mathbf{p}(t),$$

in which the radius vector $\mathbf{p} = OP$ is given as a function of the time t then we define the *velocity* of the point P at any instant as the value of the derivative of $\mathbf{p}$ with respect to t at that instant. Thus we write,

$$(2) \qquad \mathbf{v} = \mathbf{p}'(t).$$

The velocity at each instant is thus a vector. Although this vector, as usual when not otherwise specified, is a free vector with no fixed position, still it is usually convenient to think of it placed with its initial point at the point P under consideration.

Let us consider in detail the nature of this vector velocity. At any particular instant t_0 the radius vector $\mathbf{p}$ will have some value which we shall designate by $\mathbf{p}_0$. If we now allow t to change by the amount Δt this produces a change in $\mathbf{p}$ which we shall call $\Delta \mathbf{p}$. Thus we have,

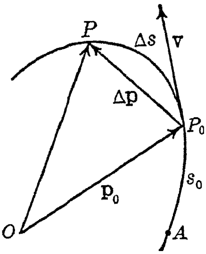

FIG. 32

$$\mathbf{p}_0 = \mathbf{p}(t_0), \qquad \mathbf{p}_0 + \Delta \mathbf{p} = \mathbf{p}(t_0 + \Delta t).$$

If we indicate by P_0 and P the positions of the moving point at the instants t_0 and $t_0 + \Delta t$ we evidently have $\Delta \mathbf{p} = P_0 P$. Then the velocity $\mathbf{v}$ is by definition,

$$\text{(3)} \qquad \mathbf{v} = \mathbf{p}' = \lim_{\Delta t \to 0} \left. \frac{\Delta \mathbf{p}}{\Delta t} \right) = \lim_{\Delta t \to 0} \frac{P_0 P}{\Delta t}$$

During the limiting process as Δt approaches zero P slides along the path towards P_0 and the vector $\Delta \mathbf{p}$ becomes shorter but at the same time Δt is also getting smaller and the vector $\Delta \mathbf{p}/\Delta t$ will in general not approach zero. Since $\Delta \mathbf{p}/\Delta t$ has constantly the direction of $P_0 P$ and has the sense of $P_0 P$ for Δt positive, it follows that its direction becomes in the limit that of the tangent to the path at P_0 and its sense that in which t increases along the path. We thus have the theorem,

The direction of the vector velocity is that of the tangent to the path and its sense that in which t increases along the path.

In other words, the sense of the velocity is that in which P is moving along its path.

To visualize the length of $\mathbf{p}'$ we find it convenient to introduce two new concepts, the *scalar velocity* and the *vector tangent.* Let us pick any point A on the path as an origin and either sense along the path as positive. Then just as in the case of a straight line we may express by a scalar s the distance and sense of P from A along the path. As P moves along its path it determines s as a function of t and the derivative of s with respect to t is defined as the *scalar velocity* s'. So if we indicate by Δs the increment of s from P_0 to P, then by definition we have,

$$s' = \lim_{\Delta t \to 0} \left. \frac{\Delta s}{\Delta t} \right) .$$

At each point of the curve we consider a vector $\mathbf{t}$ called the

vector tangent and defined as follows. It is a vector one unit long, tangent to the curve at the point and pointing in the sense in which the above scalar s increases along the curve. We shall see that the following formula is exactly equivalent to the above definition.

$$\mathbf{t} = \lim_{\Delta t \to 0} \left(\frac{\Delta \mathbf{p}}{\Delta s} \right). \tag{4}$$

For firstly, since $\Delta \mathbf{p}/\Delta s$ lies constantly along $\Delta \mathbf{p} = P_0 P$, it must in the limit lie along the tangent at P_0. Secondly, the formula must yield a unit vector, for $|\Delta \mathbf{p}|$ is a chord and $|\Delta s|$ is its arc and the limit of their ratio must be one. Finally, if Δs is positive, s increases from P_0 to P and $\Delta \mathbf{p}/\Delta s$, being the same in sense as $\Delta \mathbf{p}$, points in the sense of increasing s; while if Δs is negative s decreases from P_0 to P and $\Delta \mathbf{p}/\Delta s$, being opposite in sense to $\Delta \mathbf{p}$, points in the sense of increasing s. Thus $\Delta \mathbf{p}/\Delta s$ points constantly in the sense of increasing s and the same must be true of its limit $\mathbf{t}$. Thus the value of $\mathbf{t}$ given by formula (4) agrees in length, sense and direction with the definition.

Now starting with the identity,

$$\frac{\Delta \mathbf{p}}{\Delta t} = \frac{\Delta s}{\Delta t} \frac{\Delta \mathbf{p}}{\Delta s},$$

we have on proceeding to the limit, remembering the above definitions,

$$\mathbf{p}' = s' \, \mathbf{t}. \tag{5}$$

This important equation furnishes a verification of our statement that the velocity lies along the tangent and it also shows that,

The length of the vector velocity is the numerical value of the scalar velocity.

The reader will observe that the above discussion has been carried through only for the case in which all the derivatives mentioned exist.

We may now set up the following formulas useful for purposes of computation.

$$\mathbf{v} = \mathbf{p}', \qquad s' = \pm \, |\mathbf{p}' \qquad \mathbf{t} = \pm \frac{\mathbf{p}'}{|\mathbf{p}'|}. \tag{6}$$

The first is the definition of $\mathbf{v}$. The second is obtained from equation (5) by taking the lengths of both members. The last is obtained from equation (5) by replacing s' by its value given by the second. The ambiguous sign is in each case positive when s and t increase in the same sense along the curve and negative when they increase in opposite senses. The above discussion fails in part in the case in which $\mathbf{p}'$ vanishes at the point considered. We shall discuss this case further in § 31.

30. Vector Acceleration.

In the preceding section we defined the vector velocity, $\mathbf{v}$ for each instant of the motion. The velocity thus constitutes a vector function of the scalar t and may in turn possess a derivative with respect to t. We call this derivative the *vector acceleration* $\mathbf{j}$ at the instant. Thus we have,

$$\mathbf{j} = \mathbf{v}' = \mathbf{p}''. \tag{1}$$

The acceleration is a free vector but we often think of the acceleration, like the velocity, with its initial point at the point P under consideration. The acceleration being rather more difficult to visualize than the velocity, we shall discuss in this section only its simplest properties, leaving the others to be brought out in the next two sections.

We recall equation (5) of the preceding section,

$$\mathbf{p}' = s'\,\mathbf{t},$$

and differentiating with respect to t we have,

$$\mathbf{j} = \mathbf{p}'' = s'\,\mathbf{t}' + s''\,\mathbf{t}. \tag{2}$$

The acceleration, $\mathbf{j}$ is thus broken up or resolved into two terms, one lying along the tangent to the curve and one perpendicular to it; for $\mathbf{t}'$ must be perpendicular to $\mathbf{t}$ since $\mathbf{t}$ is of constant length. We call these two terms respectively the *tangential and normal components of the acceleration.* Just as we called s' the scalar velocity, so it is natural to call s'' the *scalar acceleration,* and with this agreement we have at once the theorem,

The length of the tangential component of the vector acceleration is the absolute value of the scalar acceleration.

Thus we see that the tangential component of the acceleration is that portion of it which tells us how the velocity is changing

in length, and a little thought will convince one that, on the other hand, the normal component of the acceleration tells how the velocity is changing in direction. We shall examine this matter in greater detail in § 31.

It is important to observe that in general the length of the vector acceleration is not the absolute value of the scalar acceleration. In fact equation (2) shows that this can happen when and only when s' or $\mathbf{t}'$ is zero; that is when the particle comes momentarily to rest or its path is momentarily straight.

For purposes of computation we now have the following additional formulas,

$$(3) \qquad \mathbf{j} = \mathbf{p}''$$

The first is the definition of $\mathbf{j}$. To obtain the second we form the dot-product of both members of equation (2) with $\mathbf{t}$ obtaining $s'' = \mathbf{p}''\cdot\mathbf{t}$ and then replace $\mathbf{t}$ by its value, $\mathbf{p}'/|\mathbf{p}'|$. The ambiguous sign is positive when s and t increase along the curve in the same sense and negative when they increase in opposite senses.

31. The Tangent Line and Osculating Plane.

The velocity $\mathbf{p}'$ and the acceleration $\mathbf{p}''$ of the point or particle P are closely associated with the *tangent line* and *osculating plane* to the path of P at P. To derive the equation of the tangent line to the path of P at the point P_0 which it occupies at the instant t_0 we proceed as follows. We assume as before that the position of P at each instant is determined by having its radius vector $\mathbf{p}$ from the origin given as a function $\mathbf{p} = \mathbf{p}(t)$. Let $\mathbf{p}_0$ be the radius vector of P_0 and let $\mathbf{p}_0 + \Delta\mathbf{p}$ be the radius vector of P at another instant $t_0 + \Delta t$. Then the equation in point-direction form of the line through these two points is,

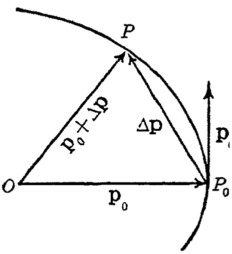

FIG. 33

$$(1) \qquad (\mathbf{p} - \mathbf{p}_0) \times \Delta\mathbf{p} = 0.$$

But by Taylor's theorem for vector-scalar functions (§ 26) we

have,

$$(2) \qquad \mathbf{p}(t_0 + \Delta t) = \mathbf{p}(t_0) + \Delta t\,\{\mathbf{p}'(t_0) + \boldsymbol{\varepsilon}\}, \qquad \lim_{\Delta t \to 0} \boldsymbol{\varepsilon} \quad 0.$$

Thus $\Delta\mathbf{p} = \Delta t\,\{\mathbf{p}_0' + \boldsymbol{\varepsilon}\}$, where $\mathbf{p}_0' = \mathbf{p}'(t_0)$, and hence equation (1) may be written,

$$\Delta t\,(\mathbf{p} - \mathbf{p}_0) \times (\mathbf{p}_0' + \boldsymbol{\varepsilon}) = 0.$$

If we do not assign the value zero to Δt we may drop this scalar factor from the first member and write the equation of the line through P_0 and P as,

$$(3) \qquad (\mathbf{p} - \mathbf{p}_0) \times (\mathbf{p}_0' + \boldsymbol{\varepsilon}) = 0.$$

If now we let the time interval Δt approach zero the point P will approach P_0 along the path and the above line will become in the limit the tangent line to the path at P_0. But while this is taking place the only change in equation (3) is that by Taylor's theorem $\boldsymbol{\varepsilon}$ will approach zero as a limit. Thus the equation of the tangent line at P_0 is

$$(4) \qquad (\mathbf{p} - \mathbf{p}_0) \times \mathbf{p}_0' = 0.$$

This brings out once more the fact already established that $\mathbf{p}'$ has the direction of the tangent to the path at P.

In case $\mathbf{p}_0' = 0$ it is necessary to carry the development by Taylor's theorem in equation (2) to higher derivatives of $\mathbf{p}$ and we obtain the more general equation for the tangent line,

$$(4') \qquad (\mathbf{p} - \mathbf{p}_0) \times \mathbf{p}_0^{(k)} = 0,$$

k being the order of the lowest ordered derivative of $\mathbf{p}$ which does not vanish for $t = t_0$.

The derivation of the equation of the osculating plane to the path at P_0 is along similar lines. We first write the equation of the plane containing the tangent line at P_0 and the point P. This being a plane through the point P_0 and containing the two vectors $\mathbf{p}_0'$ and $\Delta\mathbf{p}$, will have the equation,

$$(5) \qquad (\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{p}_0' \times \Delta\mathbf{p} = 0.$$

Taylor's theorem now gives us,

$$(6) \quad \mathbf{p}(t_0 + \Delta t) = \mathbf{p}(t_0) + \Delta t\,\mathbf{p}'(t_0) + \frac{(\Delta t)^2}{2!}\,\{\mathbf{p}''(t_0) + \boldsymbol{\varepsilon}\}, \lim_{\Delta t \to 0} \boldsymbol{\varepsilon} = 0.$$

That is,

$$\Delta \mathbf{p} = \Delta t\, \mathbf{p}_0' + \frac{(\Delta t)^2}{2!}\,(\mathbf{p}_0'' + \boldsymbol{\varepsilon}).$$

Substituting this value in equation (5) gives,

$$(7) \qquad (\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{p}_0' \times \left\{ \Delta t\, \mathbf{p}_0' + \frac{(\Delta t)^2}{2!}\,(\mathbf{p}_0'' + \boldsymbol{\varepsilon}) \right\} = 0.$$

The term $\Delta t\, \mathbf{p}_0'$ may be dropped from the brace and, dividing out the factor $(\Delta t)^2/2!$ to which we shall not assign the value zero, we have for the equation of the above plane,

$$(8) \qquad (\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{p}_0' \times (\mathbf{p}_0'' + \boldsymbol{\varepsilon}) = 0.$$

If now we let the time interval Δt approach zero, the point P will approach P_0 along the path and the plane determined by the tangent line at P_0 and the point P will by definition become in the limit the *osculating plane* to the path at P_0. But while this is going on the only change in equation (8) is that by Taylor's theorem $\boldsymbol{\varepsilon}$ will approach zero as a limit. Thus the equation of the osculating plane is,

$$(9) \qquad (\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{p}_0' \times \mathbf{p}_0'' = 0.$$

It thus appears that both the velocity $\mathbf{p}_0'$ and the acceleration $\mathbf{p}_0''$ lie in the osculating plane to the path at P_0. In case $\mathbf{p}_0' \times \mathbf{p}_0'' = 0$ it is necessary to carry the development by Taylor's theorem in equation (6) to higher derivatives of $\mathbf{p}$ and we obtain the more general equation for the osculating plane,

$$(9') \qquad (\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{p}_0^{(k)} \times \mathbf{p}_0^{(l)} = 0,$$

where k is the smallest number for which $\mathbf{p}_0^{(k)} \neq 0$ and, k being now fixed, l is the smallest number for which $\mathbf{p}_0^{(k)} \times \mathbf{p}_0^{(l)} \neq 0$.

The osculating plane at P_0 may be equally well determined as the limiting position of the plane determined by P_0 and two other points P and Q on the path as the points P and Q simultaneously approach P_0 along the path. The analytic treatment is, however somewhat more involved and it is necessary to specify that the three arcs P_0P, P_0Q, PQ must remain infinitesimals of the same order during the limiting process.

32. Curvature and Torsion; Frenet Formulas.

We have seen that the unit vector tangent to the path of P at the point P is given by the formula, § 29, (4)

$$\mathbf{t} = \frac{d\mathbf{p}}{ds}. \tag{1}$$

Now $\mathbf{t}$ being a unit vector its derivative is perpendicular to it and if we lay off a unit vector, $\mathbf{n}$ in the direction and sense of $d\mathbf{t}/ds$, we have what is called the *principal normal* or simply the normal to the curve at the point.

$$\mathbf{n} = \frac{d\mathbf{t}}{ds} \div \tag{2}$$

The unit vector $\mathbf{b}$ drawn perpendicular to both $\mathbf{t}$ and $\mathbf{n}$ and such that the triple of vectors $\mathbf{t},\mathbf{n},\mathbf{b}$ has a positive sense of rotation is called the *binormal* to the curve at the point. These three vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are thus mutually perpendicular and form what is known as the *trihedral* at the point. Evidently we have,

$$\mathbf{t} = \mathbf{n} \times \mathbf{b}, \qquad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{3}$$

The formulas for computing these three vectors are,

$$\mathbf{t} = \pm \frac{\mathbf{p}'}{|\mathbf{p}'|} \qquad \mathbf{n} = \frac{\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')}{|\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')|} \tag{4}$$

$$\mathbf{b} = \pm \frac{\mathbf{p}' \times \mathbf{p}''}{|\mathbf{p}' \times \mathbf{p}''|}$$

The first of these formulas we have already derived, § 29, (6). If we differentiate both members of this with respect to t we have,

$$\mathbf{t}' = \pm \frac{\mathbf{p}'^2\, \mathbf{p}'' - (\mathbf{p}' \cdot \mathbf{p}'')\, \mathbf{p}'}{|\mathbf{p}'|^3} = \pm \frac{\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')}{|\mathbf{p}'|^3}$$

the last member coming from the fundamental identity, § 19, (5). Thus we have,

$$\frac{d\mathbf{t}}{ds} = \frac{\mathbf{t}'}{s'} = \frac{\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')}{s'^4} \tag{5}$$

and to obtain the above formula for $\mathbf{n}$ we have merely to reduce this to a unit vector by dividing it by its own length. The

formula for **b** then follows by forming the cross-product of the formulas for **t** and **n**.

It is noteworthy that there is no ambiguity of sign in the formula for the normal. It follows that the sense of the normal in no way depends on the sense chosen as positive along the curve nor on the sense in which the particle is tracing out the curve. Furthermore the presence of two successive vector multiplications in the formula shows that the sense of rotation of the coördinate axes does not affect the sense of **n**. It thus appears that,

The vector normal at any point of the curve is an intrinsic property of the curve at the point.

Formulas (4) make it at once apparent that the tangent and normal lie in the osculating plane,

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{p}_0' \times \mathbf{p}_0'' = 0,$$

while the binormal is perpendicular to it.

The length of the normal vector $\mathbf{t}'$ at any point of the path is the *rate of turning of the particle* on its path at this point. To see this, let $\mathbf{t}_0$ be the tangent at the point P_0 and $\mathbf{t}_0 + \Delta\mathbf{t}$ be the tangent after an interval of time Δt. Let us lay off these two unit vectors from some common origin O. We then draw a unit circle with center at O and having $\mathbf{t}_0$ and $\mathbf{t}_0 + \Delta\mathbf{t}$ as two of its radii. Since this is a unit circle the angle between the two vectors and the arc of the circle intercepted between them will be numerically equal and we may represent both of them by the positive quantity $\Delta\theta$. Since $\Delta\theta$ is the angle between two tangents to the path and Δt is the time elapsed while the tangent was turning from one position to the other we may properly call the limit of their ratio, as Δt approaches zero, the *rate of turning* of the particle at the point,

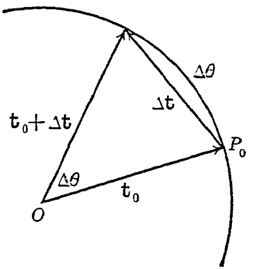

FIG. 34

$$\theta' = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}.$$

But in the unit circle here considered $\Delta\theta$ is a certain arc and $|\Delta\mathbf{t}|$ is its chord so that the limit of their ratio is one and we

may therefore replace $\Delta\theta$ by $|\Delta\mathbf{t}|$ in the above expression and write,

$$\theta' = \lim_{\Delta t \to 0} \frac{|\Delta\mathbf{t}|}{\Delta t} = |\mathbf{t}'$$

If the rate of turning θ' be divided by the absolute value of the scalar velocity s' we have a quantity which is independent of the scalar velocity with which the curve is traced out and which is known as the *curvature* of the curve. It is the *space rate of turning,* that is the number of radians that the tangent is turning per unit length of the curve. Its reciprocal ρ is the *radius of curvature.* For purposes of computation we have the formula,

$$(6) \qquad \frac{1}{\rho} = \frac{v}{|s'|} = \frac{|\mathbf{t}'|}{\mathbf{p}'|} = \frac{|\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')|}{|\mathbf{p}'|^4} = \frac{|\mathbf{p}' \times \mathbf{p}''|}{|\mathbf{p}'|^3}$$

It should be noted that the curvature is always positive or zero.

The binormal $\mathbf{b}$ remains perpendicular to the osculating plane and so the rate of turning of the osculating plane is also the rate of turning $|\mathbf{b}'|$ of the binormal. This is called the *rate of twisting of the particle* on its path at this point. If this rate of twisting be divided by the absolute value of the scalar velocity s' we have a quantity which is independent of the scalar velocity with which the curve is traced out and which is known as the *torsion* of the curve. It is the *space rate of twisting,* that is the number of radians that the osculating plane is turning per unit length of the curve. Its reciprocal τ is the *radius of torsion.*

$$(7) \qquad \frac{|\mathbf{b}'|}{|s'|} \qquad \frac{d\mathbf{b}}{ds}$$

We have considered here only the absolute value of the torsion. Unlike the curvature, the torsion is regarded as sometimes positive and sometimes negative or zero. Its sign will be considered at the end of this article.

The three unit vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ being mutually perpendicular form a trihedral on which we may project other vectors and determine their components in these three directions. As an aid to doing this we have three important formulas known as the *Frenet Formulas* (1850) which give the components of the derivatives of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ with respect to s in the directions of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. They

are as follows,

$$(8) \qquad \frac{d\mathbf{t}}{ds} = \frac{1}{\rho}\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\frac{1}{\rho}\mathbf{t} - \frac{1}{\tau}\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = \frac{1}{\tau}\mathbf{n}.$$

These may be tabulated as follows,

	$\mathbf{t}$	$\mathbf{n}$	$\mathbf{b}$
$\frac{d\mathbf{t}}{ds}$	0	$\frac{1}{\rho}$	0
$\frac{d\mathbf{n}}{ds}$	$-\frac{1}{\rho}$	0	$-\frac{1}{\tau}$
$\frac{d\mathbf{b}}{ds}$	0	$\frac{1}{\tau}$	0

where each of the nine numbers in the table gives the measure of the vector at the left of its row on the vector at the top of its column. The proof of the first formula is immediate. Since,

$$\mathbf{n} = \frac{d\mathbf{t}}{ds} \div \left|\frac{d\mathbf{t}}{ds}\right|, \qquad \frac{1}{\rho} = \left|\frac{\mathbf{t}'}{s'}\right| = \left|\frac{d\mathbf{t}}{ds}\right|$$

it follows that,

$$\frac{d\mathbf{t}}{ds} = \frac{1}{\rho}\mathbf{n}.$$

To prove the third formula we recall that,

$$\mathbf{b}^2 = 1, \qquad \mathbf{t}\cdot\mathbf{b} = 0$$

and on differentiating with respect to s these give,

$$\mathbf{b}\cdot\frac{d\mathbf{b}}{ds} = 0, \qquad \frac{d\mathbf{b}}{ds} + \mathbf{b}\cdot\frac{d\mathbf{t}}{ds} = \mathbf{t}\;\;\frac{d\mathbf{b}}{ds} + \frac{1}{\rho}\mathbf{b}\cdot\mathbf{n} = \mathbf{t}\cdot\frac{d\mathbf{b}}{ds} = 0.$$

Thus $d\mathbf{b}/ds$ being perpendicular to both $\mathbf{b}$ and $\mathbf{t}$ must lie along $\mathbf{n}$ and its length being $1/\tau$ we have the third Frenet formula established except for sign. But we now agree to so choose the sign of $1/\tau$ that the third Frenet formula holds. The second Frenet formula now follows easily.

$$\frac{d\mathbf{n}}{ds} = \frac{d(\mathbf{b}\times\mathbf{t})}{ds} = \frac{1}{\rho}\mathbf{b}\times\mathbf{n} + \frac{1}{\tau}\mathbf{n}\times\mathbf{t} = -\frac{1}{\rho}\mathbf{t} - \frac{1}{\tau}\mathbf{b}.$$

The Frenet formulas are thus established and it only remains to derive a formula for the computation of the torsion under the

above assumption as to its sign. Recalling that,

$$\mathbf{b} = \pm \frac{\mathbf{p}' \times \mathbf{p}''}{|\mathbf{p}' \times \mathbf{p}''|},$$

we find by differentiation and reduction that,

$$\mathbf{b}' = \mp \frac{[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']}{|\mathbf{p}' \times \mathbf{p}''|^3}\, \mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}').$$

Hence,

$$\frac{d\mathbf{b}}{ds} = \pm \frac{\mathbf{b}'}{|\mathbf{p}'|} = - \frac{[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']}{(\mathbf{p}' \times \mathbf{p}'')^2} \frac{\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')}{|\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')|} = - \frac{[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']}{(\mathbf{p}' \times \mathbf{p}'')^2}\,\mathbf{n}.$$

Comparing this with the third of the Frenet formulas gives the result,

$$\frac{1}{\tau} = - \frac{[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']}{(\mathbf{p}' \times \mathbf{p}'')^2}. \tag{9}$$

Unlike the curvature, the torsion is a rational function of the coördinates of P and their derivatives.

A point of the curve at which the curvature vanishes, i.e.,

$$\mathbf{p}' \times \mathbf{p}'' = 0, \qquad \mathbf{p}' \neq 0,$$

is a *point of inflection* of the curve. A point at which the torsion vanishes, i.e.,

$$[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}'''] = 0, \qquad \mathbf{p}' \times \mathbf{p}'' \neq 0,$$

is a *plane point* or *stall* of the curve.

EXERCISES

1. Express $\mathbf{p}'$, $\mathbf{p}''$, $\mathbf{p}'''$ in terms of the scalars s', ρ, τ and their derivatives with respect to the time and the vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. Compute the value of $[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']$. *Ans.* $[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}'''] = -s'^6/(\rho^2\tau)$

2. By application of the formula,

$$\frac{1}{\rho} = \frac{|\mathbf{p}' \times \mathbf{p}''|}{|\mathbf{p}'|^3}$$

to the motion $\mathbf{p} \equiv (t, y, 0)$ derive the usual formula of the calculus for the curvature of a plane curve,

$$\frac{1}{\rho} = \frac{|y''|}{(1 + y'^2)^{3/2}}.$$

3. A point P traces out the twisted cubic, $\mathbf{p} \equiv (6t, 3t^2, t^3)$. Determine for the instant t the scalar velocity and acceleration, the curvature and the torsion. Find the equations of the tangent line and the osculating plane. Evaluate all these for the instant, $t = 2$.

   Ans. $(t = 2)$ $18, 12, 1/54, -1/54, \dfrac{x-12}{1} = \dfrac{y-12}{2} = \dfrac{z-8}{2}$,
$$2x - 2y + z - 8 = 0$$

4. A point P traces out the helix, $\mathbf{p} \equiv (2\cos 2t, 2\sin 2t, 3t)$. Determine for the instant t the scalar velocity and acceleration, the curvature and the torsion. Find the equations of the tangent line and the osculating plane. Evaluate all these for the instant, $t = 0$.

   Ans. $(t = 0)$ $5, 0, 8/25, -6/25, \dfrac{x-2}{0} = \dfrac{y}{4} = \dfrac{z}{3}, 3y - 4z = 0$

5. Show that for a plane curve the unit normal vector $\mathbf{n}$ as drawn from the point P always points toward the concave side of the curve.

6. If $\mathbf{p}$ is a linear function of t, $\mathbf{p} = \mathbf{a} + \mathbf{b}\,t$ $(\mathbf{b} \neq 0)$, show that the motion of P is uniform rectilinear. ($\mathbf{a}$, $\mathbf{b}$ constant.)

7. If $\mathbf{p}$ is a quadratic function of t, $\mathbf{p} = \mathbf{a} + \mathbf{b}\,t + \mathbf{c}\,t^2$ $(\mathbf{c} \neq 0)$, show that the motion of P is on a parabola with axis parallel to $\mathbf{c}$. Show that the scalar velocity is least for $t = -\mathbf{b}\cdot\mathbf{c}/(2\,\mathbf{c}^2)$. ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ constant.)

8. The radius vector $\mathbf{p}$ of the point P is given in terms of the arc s by the formula, $\mathbf{p} = \mathbf{a}\,s + \mathbf{a} \times \mathbf{q}(s)$, where $\mathbf{a}$ is a constant vector and $\mathbf{q}(s)$ is an arbitrary vector function of s. Show that the tangent makes a constant angle with $\mathbf{a}$, that the principal normal is always perpendicular to $\mathbf{a}$ and that the ratio of the curvature to the torsion is a constant.

9. If $\mathbf{p}$ is given by the formula,

$$\begin{aligned} &(a) \quad \mathbf{p} = \mathbf{a}\cos\mu t + \mathbf{b}\sin\mu t \\ \text{or } &(b) \quad \mathbf{p} = \mathbf{a}\cosh\mu t + \mathbf{b}\sinh\mu t \end{aligned} \qquad (\mathbf{a}, \mathbf{b}, \mu \text{ const.})$$

   show that the acceleration is parallel and proportional to the radius vector $\mathbf{p}$. Name the path in each case and state the relation of the vectors $\mathbf{a}$ and $\mathbf{b}$ to it.

10. If we set,

$$\boldsymbol{\omega} = \frac{1}{\rho}\mathbf{b} - \frac{1}{\tau}\mathbf{t},$$

   show that we have,

$$\frac{d\mathbf{t}}{ds} = \boldsymbol{\omega} \times \mathbf{t}, \qquad \frac{d\mathbf{n}}{ds} = \boldsymbol{\omega} \times \mathbf{n}, \qquad \frac{d\mathbf{b}}{ds} = \boldsymbol{\omega} \times \mathbf{b}.$$

33. Rectilinear and Plane Motion.

If we examine the formulas for curvature and torsion,

$$\frac{1}{\rho} = \frac{|\mathbf{p}' \times \mathbf{p}''|}{|\mathbf{p}'|^3}, \qquad \frac{1}{\tau} = -\frac{[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']}{(\mathbf{p}' \times \mathbf{p}'')^2},$$

of the path of the point P it might at first appear that for $\mathbf{p}' \times \mathbf{p}''$ identically zero throughout an interval the curvature would be constantly zero and the path consequently a straight line, while for $[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']$ identically zero the torsion would be constantly zero and the path consequently a plane curve. These conclusions are however contingent on the *continued existence* of the curvature and torsion respectively and may fail if the denominators of the respective formulas vanish in the interval considered. In fact we have the two theorems,

I. *A necessary condition that the path of P be a straight line is that* $\mathbf{p}' \times \mathbf{p}''$ *vanish identically throughout the interval considered. This condition will also be sufficient if* $\mathbf{p}'$ *does not vanish in the interval.*

II. *A necessary condition that the path of P be a plane curve is that* $[\mathbf{p}'\,\mathbf{p}''\,\mathbf{p}''']$ *vanish identically throughout the interval considered. This condition will also be sufficient if* $\mathbf{p}' \times \mathbf{p}''$ *does not vanish in the interval.*

To prove Theorem I we first observe that if the motion is constantly along the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$, $(\mathbf{m} \neq 0)$ then by differentiation we have $\mathbf{p}' \times \mathbf{m} = 0$. This is an equation of the type, § 27, (4), $\mathbf{p} \times \mathbf{c} = 0$ except that $\mathbf{p}$ and $\mathbf{c}$ in that equation have been replaced by $\mathbf{p}'$ and $\mathbf{m}$. The facts proven concerning this last equation show at once that if we have $\mathbf{p}' \times \mathbf{m} = 0$ throughout an interval then $\mathbf{p}' \times \mathbf{p}'' = 0$ throughout that interval, while if $\mathbf{p}' \times \mathbf{p}'' = 0$, $\mathbf{p}' \neq 0$ throughout an interval then there exists a non-zero constant vector $\mathbf{m}$ such that $\mathbf{p}' \times \mathbf{m} = 0$ throughout the interval. Integrating this we have $\mathbf{p} \times \mathbf{m} = \mathbf{c}$ where $\mathbf{c}$ is any constant vector perpendicular to $\mathbf{m}$. For $\mathbf{a} = \mathbf{m} \times \mathbf{c}/\mathbf{m}^2$ this becomes $(\mathbf{p} - \mathbf{a}) \times \mathbf{m} = 0$, which being the equation of a straight line, completes the proof of the theorem. If the auxiliary condition $\mathbf{p}' \neq 0$ were not imposed we might have $\mathbf{p}' \times \mathbf{p}'' = 0$ without the motion being rectilinear throughout. It might in fact be a zigzag motion consisting of several straight segments within which $\mathbf{p}' \times \mathbf{p}''$ would be zero joined by points at which $\mathbf{p}'$ would be zero.

To prove Theorem II we first observe that if the motion is constantly in the plane $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$, $(\mathbf{m} \neq 0)$ then by differentiation we have $\mathbf{p}' \cdot \mathbf{m} = \mathbf{p}'' \cdot \mathbf{m} = 0$ and consequently,

$$(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{m} = (\mathbf{p}' \cdot \mathbf{m})\, \mathbf{p}'' - (\mathbf{p}'' \cdot \mathbf{m})\, \mathbf{p}' = 0.$$

Conversely if we have $(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{m} = 0$, $\mathbf{p}' \times \mathbf{p}'' \neq 0$ throughout a certain interval then by the above equation $(\mathbf{p}' \cdot \mathbf{m})\, \mathbf{p}' \times \mathbf{p}'' = 0$ and $\mathbf{p}' \cdot \mathbf{m} = 0$ throughout the interval. Integrating this gives $\mathbf{p} \cdot \mathbf{m} = c$, where c is an arbitrary scalar constant. For $\mathbf{a} = c\, \mathbf{m}/\mathbf{m}^2$ this becomes $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$, which being the equation of a plane completes the proof of the following lemma.

A necessary condition that the motion of P be in the plane $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{m} = 0$ *is that* $(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{m} = 0$. *This condition will also be sufficient if* $\mathbf{p}' \times \mathbf{p}''$ *does not vanish in the interval considered.*

The equation $(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{m} = 0$, $(\mathbf{m} \neq 0)$ is of the type, § 27, (4), $\mathbf{p} \times \mathbf{c} = 0$ mentioned above and consequently if $(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{m} = 0$ throughout a certain interval we also have,

$$0 = (\mathbf{p}' \times \mathbf{p}'') \times (\mathbf{p}' \times \mathbf{p}''') = [\mathbf{p}'\, \mathbf{p}''\, \mathbf{p}''']\, \mathbf{p}',$$

while if $[\mathbf{p}'\, \mathbf{p}''\, \mathbf{p}'''] = 0$, $\mathbf{p}' \times \mathbf{p}'' \neq 0$ throughout a certain interval then there exists a non-zero vector $\mathbf{m}$ such that $(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{m} = 0$ throughout the interval. The lemma proven above is thus equivalent to the theorem to be proven. If the auxiliary condition $\mathbf{p}' \times \mathbf{p}'' \neq 0$ were not imposed we might have $[\mathbf{p}'\, \mathbf{p}''\, \mathbf{p}'''] = 0$ without the motion being plane throughout. It might in fact be a skew motion consisting of several plane segments within which $[\mathbf{p}'\, \mathbf{p}''\, \mathbf{p}''']$ would be zero joined by points at which $\mathbf{p}' \times \mathbf{p}''$ would be zero for both of the adjacent segments. This peculiar case was first noticed by Peano.

EXERCISES

1. What will be the nature of the motion of the point P if $[\mathbf{p}\, \mathbf{p}'\, \mathbf{p}'']$ is identically zero throughout the interval considered?

2. Show that if $\mathbf{p} \times \mathbf{p}''$ is a constant vector throughout a certain interval of time, then the motion determined by $\mathbf{p}$ is plane motion in a plane through the origin.

34. Various Coördinate Systems.

The motion of the point P has been determined by having its radius vector from some fixed point O given as a function of the

time and this may be accomplished by giving the Cartesian coördinates of this vector $\mathbf{p}$ as scalar functions of the time. But there are various other ways of determining the motion of P and these often involve expressing the distances from P to certain fixed points, lines or planes as functions of the time. In any such case we would wish to express the velocity, acceleration, curvature, etc., in terms of these distances and their successive derivatives with respect to the time. An important theorem in this connection is the following.

If r is the distance from any fixed point, line or plane to the moving point P, then the derivative r' of r with respect to t equals the measure of the velocity $\mathbf{p}'$ *of P on the perpendicular drawn from the fixed point, line or plane to P.*

Of course in the case of the fixed point the above perpendicular is merely the line from the fixed point to P. This theorem is readily proven by the aid of the concept of the *gradient* of a scalar-point function (§ 99) but may be proven directly as follows. Let Q be the foot of the perpendicular let fall from the moving point P upon the given fixed line or plane, or in the case of a given fixed point A, let Q coincide with A. Let O be the fixed origin and call $OP = \mathbf{p}$, $OQ = \mathbf{q}$, $QP = \mathbf{r}$. Then we have $\mathbf{p} = \mathbf{q} + \mathbf{r}$ and differentiating with respect to the time gives,

$$\mathbf{p}' = \mathbf{q}' + \mathbf{r}'.$$

Since Q is confined to the fixed point, line or plane its velocity $\mathbf{q}'$ is either zero or confined to the line or plane and in any case we have $\mathbf{r} \cdot \mathbf{q}' = 0$. Thus dot-multiplying the above equation through by $\mathbf{r}$ yields,

$$\mathbf{r} \cdot \mathbf{p}' = \mathbf{r} \cdot \mathbf{r}'.$$

Since the distance r of the theorem is the length of the vector $\mathbf{r}$ we have $\mathbf{r}^2 = r^2$ and $2\,\mathbf{r} \cdot \mathbf{r}' = 2rr'$ which reduces the above equation to $\mathbf{r} \cdot \mathbf{p}' = rr'$ or,

$$r' = \frac{\mathbf{r}}{r} \cdot \mathbf{p}'.$$

The second member is the measure of the velocity of P on $\mathbf{r}$ and the theorem is proven.

A special case of this proposition for the case of the distance to a fixed plane has in effect been discussed in the derivation of the formula for the coördinates of the derivative of a vector-

scalar function. For we saw there that if $\mathbf{p} \equiv (x, y, z)$ then $\mathbf{p}' \equiv (x', y', z')$ and our proposition thus holds for the coördinate planes, since x' is both the derivative of the distance from the YZ-plane to the point P and also the measure of the velocity $\mathbf{p}'$ upon the axis along which this distance is measured.

The above theorem gives us at once a simple construction for the velocity to many motions and is especially convenient in the case of motion along a plane curve. For example, suppose the point P moves in a plane so that its distance h to a fixed point A of that plane remains e times as great as its distance k to another point B of the plane; e being a positive constant. The bipolar equation of its motion would then be,

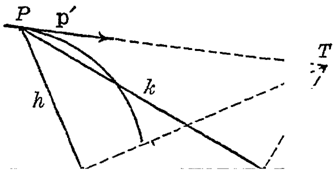

FIG. 35

$$h = ek,$$

and differentiating with respect to the time we have,

$$h' = ek',$$

and consequently,

$$\frac{h'}{h} = \frac{k'}{k}.$$

Now from the above theorem we know that the measures of the velocity of P along the lines h and k are equal to h' and k' and hence in this case are proportional to h and k. If then at A and B we erect perpendiculars to the lines h and k respectively, their intersection T will be a point on the velocity or on the velocity produced, i.e. on the tangent to the path at P. It is an interesting exercise in Euclidean geometry to continue this argument and show that for $e \neq 1$ the path of P is a circle. It is known as the circle of Apollonius, 260–200 B.C. Our theorem likewise yields simple constructions for the tangent at a given point of an ellipse, hyperbola or Cassini's oval whose foci are given.

If the particle P move in a plane we may employ a plane coördinate system for the determination of its motion. If for example the plane of the motion be the XY-plane of our Cartesian system we may use a polar coördinate system in that plane with coördinates p and θ. If the initial line of the polar system

is taken along the positive X-axis and the pole at the origin then evidently the radius vector $\mathbf{p}$ has the Cartesian coördinates,

$$\begin{aligned}
\mathbf{p} &\equiv (p \cos \theta,\ p \sin \theta,\ 0),\\
\mathbf{v} = \mathbf{p}' &\equiv (p' \cos \theta - p \sin \theta\ \theta',\ p' \sin \theta + p \cos \theta\ \theta',\ 0),\\
\mathbf{j} = \mathbf{p}'' &\quad (p'' \cos \theta - 2p' \sin \theta\ \theta' - p \cos \theta\ \theta'^2 - p \sin \theta\ \theta'',\\
&\quad p'' \sin \theta + 2p' \cos \theta\ \theta' - p \sin \theta\ \theta'^2 + p \cos \theta\ \theta'',\ 0).
\end{aligned}$$

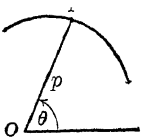

Fig. 36

To get the measure v_p of the velocity along the positive sense of $\mathbf{p}$ we form the dot-product of $\mathbf{v}$ with the unit vector in the direction of $\mathbf{p}$, $(\cos \theta, \sin \theta, 0)$ and to get the measure v_θ in the direction at right angles to this in the sense of increasing θ we form the dot-product of $\mathbf{v}$ with the unit vector in that direction $(-\sin \theta, \cos \theta, 0)$ thus obtaining the simple results,

$$v_p = p', \qquad v_\theta = p\theta'.$$

Similarly we find,

$$j_p = p'' - p\theta'^2, \qquad j_\theta = 2p'\ \theta' + p\theta''.$$

When the point P moves in space we sometimes find it convenient to determine its motion by expressing one distance p and two angles θ and φ as functions of the time. These three quantities are known as the polar coördinates of P and are conveniently related to its Cartesian coördinates as indicated in the figure. Evidently the radius vector $\mathbf{p}$ has the Cartesian coordinates,

$$\mathbf{p} \equiv (p \sin \theta \cos \varphi,\ p \sin \theta \sin \varphi,\ p \cos \theta),$$

from which the coördinates of the velocity and acceleration may be obtained by differentiation.

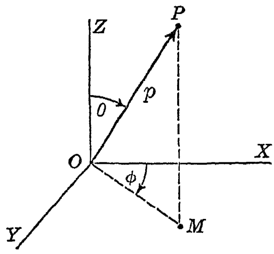

Fig. 37

EXERCISES

1. Employ the theorem of the text to prove that the tangent to a parabola at any point on it bisects the angle between the focal radius at that point and the perpendicular upon the directrix from that point.
2. Employ the theorem of the text to derive a construction for the tangent at a given point of an ellipse whose foci are given. Similarly for an hyperbola and Cassini's oval.

3. Employ the theorem of the text to prove that for any conic that portion of the tangent lying between the point of tangency and the directrix subtends a right angle at the focus.

4. If the point P moves in a plane with its polar coördinates p and θ expressed as functions of the time, show that the curvature is given by the formula,

$$\frac{1}{\rho} = \frac{|pp'\theta'' - pp''\theta' + 2p'^2\theta' + p^2\theta'^3|}{(p'^2 + p^2\theta'^2)^{3/2}}.$$

5. If P moves in space with its polar coördinates p, θ, φ expressed as functions of the time, find the measures of the velocity $\mathbf{v}$ and the acceleration $\mathbf{j}$ in the direction along OP; perpendicular to OP in the plane OPM in the sense of increasing θ; perpendicular to the plane OPM in the sense of increasing φ.

6. The point P moves in a certain plane and its perpendicular distances to two fixed lines in that plane are r_1 and r_2, the point lying in the angle α between the two lines. Show that the measures of the velocity along the two lines are respectively,

$$\begin{array}{cc} r_2' + r_1' \cos\alpha & r_1' + r_2' \cos\alpha \\ \sin\alpha & \sin\alpha \end{array}$$

7. The motion of a point in a plane is determined by its distances from two fixed points at a distance $2a$ from each other. Show that the scalar velocity of the point is,

$$s' = \qquad \overline{\cosh\lambda - \cos\varphi}\,,$$

where λ is the logarithm of the ratio of the two distances and φ is the angle between them.

The velocity of a moving point P is the sum of a component of length λ in the direction of its radius vector from a fixed point O and a component of length μ in a fixed direction. Show that P moves in a plane through O parallel to the fixed direction and that the corresponding components of its acceleration are of lengths,

$$\lambda' - \frac{\lambda\mu}{p}\cos\theta, \qquad \mu' + \frac{\lambda\mu}{p},$$

where θ is the angle between the radius vector and the fixed direction.

35. Areal and Angular Velocities.

As the point P moves along its path its radius vector $\mathbf{p}$ sweeps out various areas in various intervals of time. Let us seek an expression for the rate at which this area is being swept out by the radius vector at any particular instant t_0. We shall call

the radius vector of P at that instant $\mathbf{p}_0$ and the position of P, P_0. If now t increases an amount Δt, $\mathbf{p}$ will change by an amount which we shall call $\Delta\mathbf{p}$ while the area bounded by the two vectors $\mathbf{p}_0$ and $\mathbf{p}_0 + \Delta\mathbf{p}$ and the arc Δs over which P has traveled we shall call ΔA. Then we define the areal velocity A' at the instant t_0 as,

$$A' = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t}.$$

When Δt becomes very small Δs will become very short and nearly coincide with $\Delta\mathbf{p}$ so that we may assume as a postulate that,

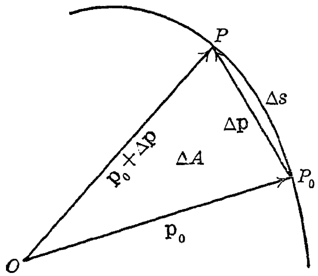

FIG. 38

$$\lim_{\Delta t \to 0} \frac{\Delta A}{T} = 1,$$

where T is the area of the triangle having $\mathbf{p}_0$ and $\mathbf{p}_0 + \Delta\mathbf{p}$ as two sides. Then since T is given by the formula, $T = \frac{1}{2}|\mathbf{p}_0 \times \Delta\mathbf{p}|$ we have,

$$A' = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta A}{T} \quad \frac{\frac{1}{2}|\mathbf{p}_0 \times \Delta\mathbf{p}|}{\Delta t} \right)$$
$$= \lim_{\Delta t \to 0} \frac{1}{2} \left| \mathbf{p}_0 \times \frac{\Delta\mathbf{p}}{\Delta t} \right| = \frac{1}{2}|\mathbf{p}_0 \times \mathbf{p}_0'|,$$

and in general,

$$(1) \qquad A' = \tfrac{1}{2}|\mathbf{p} \times \mathbf{p}'| = \tfrac{1}{2}|\mathbf{p} \times \mathbf{v}|.$$

If we differentiate both members of equation (1) with respect to the time we have for the areal acceleration,

$$(2) \qquad A'' = \frac{(\mathbf{p} \times \mathbf{p}')\cdot(\mathbf{p} \times \mathbf{p}'')}{2|\mathbf{p} \times \mathbf{p}'|}$$

This formula suggests the following interesting theorem.

If the acceleration of a particle is constantly directed along the radius vector to it from some fixed point, the areal velocity about that point is a constant and the particle remains in a fixed plane through the fixed point.

It is here implied that $\mathbf{p}$ never takes on the value zero. To prove the second part of the theorem we observe that since $\mathbf{p} \times \mathbf{p}''$

is the derivative of $\mathbf{p} \times \mathbf{p}'$ then for $\mathbf{p} \times \mathbf{p}'' = 0$ we must have $\mathbf{p} \times \mathbf{p}' = \mathbf{a}$ where $\mathbf{a}$ is a constant vector. If this constant vector $\mathbf{a}$ is not zero, then forming the dot-product of the members of this last equation with $\mathbf{p}$ gives us $\mathbf{p} \cdot \mathbf{a} = 0$ which is the equation of a plane through the origin and proves the last part of the theorem for this case. If the constant vector $\mathbf{a}$ is zero then we have $\mathbf{p} \times \mathbf{p}' = 0$ which differential equation has the solution $\mathbf{p} \times \mathbf{b} = 0$, (§ 27), where $\mathbf{b}$ is a non-zero constant vector. This is the equation of a straight line through the origin and proves the last part of our theorem for this case. The proof of the first part of the theorem is now immediate for we have by equation (1) $A' = \frac{1}{2}|\mathbf{p} \times \mathbf{p}'| = \frac{1}{2}|\mathbf{a}|$, and A' being half the length of a constant vector must be constant.

If the areal velocity remains constant equation (1) shows that $\mathbf{p} \times \mathbf{p}'$ is of constant length but does not require that it be of constant direction, and in fact $\mathbf{p}''$ may not lie along $\mathbf{p}$, although from equation (2) we see that if A' is constant but not zero then the plane of $\mathbf{p}$ and $\mathbf{p}''$ is perpendicular to the plane of $\mathbf{p}$ and $\mathbf{p}'$.

We saw in § 32 that if $\mathbf{t}$ is the unit tangent vector to the path of P then $|\mathbf{t}'|$ might properly be called the rate of turning of the curve since it is in fact the number of radians per unit time through which the tangent to the path of P is turning at P. An entirely similar argument applied to the unit vector $\mathbf{p}/|\mathbf{p}|$ shows that $|(\mathbf{p}/|\mathbf{p}|)'|$ may properly be called the *scalar angular velocity* ω of P about O since it is in fact the number of radians per unit time that the radius vector $\mathbf{p}$ is turning;

$$\omega = \left|\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right)'\right|. \tag{3}$$

We find it convenient also to define the *vector angular velocity* of P about O as the vector,

$$\boldsymbol{\omega} = \left(\frac{\mathbf{p}}{|\mathbf{p}|}\right)', \tag{4}$$

whose length has just been defined as the scalar angular velocity of P about O. Performing the indicated differentiation we find the formula,

$$\boldsymbol{\omega} = \frac{\mathbf{p} \times (\mathbf{p}' \times \mathbf{p})}{|\mathbf{p}|^3}, \tag{5}$$

from which we have at once the relation,

(6) $$A' = \tfrac{1}{2}|\mathbf{p}|\,|\omega \times \mathbf{p}|.$$

Let us consider the following problem.

The point P traces out the parabola $y^2 = 4x$, passing through the point $(0, 0)$ at the instant $t = 0$ and through the point $(1, 2)$ at the instant $t = 1$. If the areal velocity of P about the origin is a constant express the radius vector $\mathbf{p}$ of P as a function of t.

If we call $\lambda = y/2$ we have $\mathbf{p} = (\lambda^2, 2\lambda, 0)$ and consequently $A' = \tfrac{1}{2}|\mathbf{p} \times \mathbf{p}'| = \lambda^2 \lambda'$. Separating the variables and integrating between corresponding limits gives us,

$$A' \int^{t} dt = \int^{\lambda} \lambda^2\, d\lambda \qquad \text{or} \qquad A' t = \tfrac{1}{3}\lambda^3.$$

By the usual methods of the calculus we may easily find that during the interval from $t = 0$ to $t = 1$ the area swept out by the radius vector is $1/3$ and since A' is constant we have $A' = 1/3$. Consequently $\lambda = t^{1/3}$ and the position of P on its path at any instant is given by the radius vector,

$$\mathbf{p} = (t^{2/3}, 2t^{1/3}, 0).$$

EXERCISES

1. A point traces out the ellipse $x^2/a^2 + y^2/b^2 = 1$, passing through the point $(a, 0)$ at the times $t = 0, 2, 4, \cdots$. Determine the linear velocity v as a function of the time t so that the areal velocity of the point about the origin shall be constant. Hint: Let
$$\mathbf{p} \equiv (a \cos\theta, b \sin\theta, 0).$$
2. A point P traces on the conical surface $x^2/a^2 + y^2/a^2 = z^2/b^2$ the spiral curve, $OP = \mathbf{p} \equiv (a\theta\cos\theta, a\theta\sin\theta, b\theta)$ with a constant areal velocity k about the origin. Determine θ as a function of the time t, assuming that θ is zero for $t = 0$.
3. A point P traces the helix $OP = \mathbf{p} \equiv (\cos\theta, \sin\theta, \theta)$ with a constant areal velocity unity. Determine the functional relation between θ and the time t, assuming that θ is zero for $t = 0$.
4. If a particle moves in a plane curve with a constant areal velocity about some point in that plane, show that its acceleration must be constantly directed towards or from that fixed point.
5. Obtain formulas for the areal velocity A' and areal acceleration A'' of P about O in plane Cartesian and plane polar coördinates, i.e. for $\mathbf{p} \equiv (x, y, 0)$ and $\mathbf{p} \equiv (p\cos\theta, p\sin\theta, 0)$.

6. Describe in words the vector $\mathbf{q}$ whose derivative $\mathbf{q}'$ with respect to the time is $\mathbf{p} \times \mathbf{p}'$, where $\mathbf{p}$ is regarded as a vector with fixed initial point O and varying in a known manner with the time.

36. The Equation of Motion.

Newton's second law of motion states that,

The rate of change of momentum is proportional to the force acting and in the direction of its action.

In § 6 we discussed the interpretation of this and the first law in so far as they apply to the rectilinear motion of a particle. We found that for this case the laws may be formulated into the single equation,

$$\frac{d(mv)}{dt} = \lambda f,$$

in which m is the mass of the particle, t the time, v the velocity, f the force, and λ a positive constant depending only upon the units chosen for the various quantities. In this simple case the direction of the change in momentum mentioned in the law is made the same as that of the force by merely taking the proportionality factor λ positive.

In the interpretation of the second law for the curvilinear motion of a particle the force and the change in momentum are no longer confined to a fixed line and we must find some means of expressing the fact stated in the law that at each instant they have the same direction and sense. Since we have here to combine the concepts of amount, direction and sense it appears that we may conveniently do so by regarding the rate of change of momentum and the force both as vectors. Then the single vector equation,

$$\frac{d(m\,\mathbf{v})}{dt} = \lambda\,\mathbf{f}, \tag{1}$$

in which λ is a positive scalar constant, expresses the content of the second law since it states that the rate of change of momentum and the force are proportional in amount and the same in direction and sense. We have already defined the velocity $\mathbf{v}$ as the vector $\mathbf{p}'$; the *momentum* is the vector $m\,\mathbf{v}$ and this equation itself constitutes a definition of the force as a vector.

Here as in the case of rectilinear motion we assume the law to imply that forces are additive, but now in a vector sense. Thus if under certain conditions a force $\mathbf{f}_1$ acts on the particle and under

other conditions a force $\mathbf{f}_2$, then if both conditions are satisfied the force will be the vector sum $\mathbf{f}_1 + \mathbf{f}_2$.

In the vast majority of cases the mass of the particle is a constant so that equation (1) takes the form,

$$m\,\mathbf{j} = \lambda\,\mathbf{f}, \tag{2}$$

where $\mathbf{j}$ is the vector acceleration, $\mathbf{p}''$. If now the force $\mathbf{f}$ were known to be a certain vector function φ of the time, the position and the velocity of the particle, its value in terms of these quantities could be substituted in equation (2) and we would have,

$$m\,\frac{d^2\mathbf{p}}{dt^2} = \lambda\,\varphi\quad t,\,\mathbf{p},\,\frac{d\mathbf{p}}{dt} \tag{3}$$

This is a vector differential equation of the second order and is known as the *equation of motion* of the particle. If we express the equation in terms of the coördinates of $\mathbf{p}$ and their derivatives with respect to the time we obtain a system of three scalar differential equations of the second order. This system is consequently of the sixth order and its integration for the determination of the coördinates of $\mathbf{p}$ in terms of t forms, in any but the simplest cases, a rather difficult problem. We shall consider in later sections of this chapter certain cases in which the integration can be partially or wholly carried out.

37. The Hodograph.

If the velocity $\mathbf{p}'$ of the moving point P be drawn as a radius vector from some fixed origin O', then its terminus P' will trace

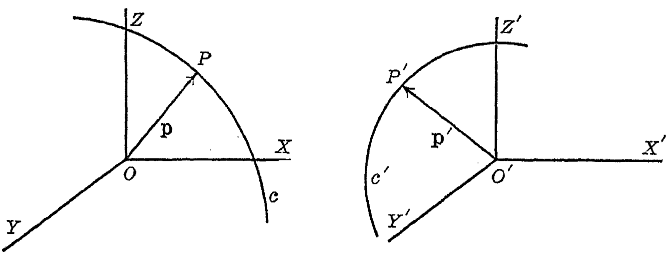

FIG. 39

out a motion called the *hodograph* of the original motion of P. The velocity of P' being the derivative of $\mathbf{p}'$ is then the accelera-

tion of P. If we were to imagine P to move with unit scalar velocity along its actual path c then $\mathbf{p}'$ would have a constant unit length and P' would trace out a curve c' lying on the surface of a sphere of unit radius. By thus assuming that P has a unit scalar velocity throughout its path the idea of time is in effect eliminated from the consideration since the arc s measured along the path may take its place. The hodograph thus becomes a purely geometric concept known as the *spherical indicatrix* of the original path c. Thus it may be shown in differential geometry that the tangent to the spherical indicatrix at P' is parallel to the osculating plane at P. This theorem is in fact merely another form of the remark already made in § 32 that the principal normal $\mathbf{n}$ at P which has by definition the direction and sense of $d\mathbf{t}/ds$ also lies in the osculating plane.

The hodographs of some of the simpler types of motion are immediately obvious. The hodograph of a rectilinear motion is rectilinear and along a line parallel to the original motion and passing through the origin O'. The hodograph of a plane motion is a plane motion in a plane parallel to the plane of the original motion and passing through the origin O'. The hodograph of a uniform circular motion is a uniform circular motion with center at the origin O'.

The hodograph was formerly a more useful concept than it is at present. When mechanics was studied by methods which were immediate extensions of the synthetic methods of Euclidean geometry the hodograph served as a means of visualizing and studying the acceleration of the point P by considering the relatively simple concept of the velocity of P'. But from our point of view, having once grasped the concept of the derivative of a vector-scalar function, the passage from $\mathbf{p}'$ to $\mathbf{p}''$ is made by differentiation with respect to the time without the necessity of visualizing $\mathbf{p}'$ as a radius vector determining a new motion. The hodograph retains its value as throwing an interesting side light on certain aspects of the motions studied but it is no longer a necessary tool for this study.

EXERCISES

Show that the hodograph of a simple harmonic motion is a simple harmonic motion. Compare the positions of P and P' on their paths at the same moment.

38. Motion with Constant Acceleration.

The motion of a particle subject to a constant acceleration is a study of great practical importance because it is to a considerable degree of approximation the motion of a small heavy body through the air near the surface of the earth. This was first discovered by Galileo at the age of 26 in experiments which he conducted from the leaning tower of Pisa about 1590. The curvature and rotation of the earth and the resistance of the air tend to cause variations in the acceleration of a body passing through the air and the study of the motion as so affected is an important branch of mechanics known as *ballistics*. The study of the case in which the acceleration is assumed to be constant is the first and simplest branch of ballistics.

Under the hypothesis that the acceleration is constant the equation of motion is,

$$\mathbf{p}'' = \mathbf{g}, \tag{1}$$

where $\mathbf{g}$ is a constant vector. This is a special case of the linear vector differential equation,

$$T_0(t)\,\mathbf{p}'' + T_1(t)\,\mathbf{p}' + T_2(t)\,\mathbf{p} + T_3(t)\,\mathbf{a} = 0,$$

discussed in § 27. We saw there that the general solution of such an equation is obtained by first finding the general solution of the associated scalar equation,

$$T_0(t)\,s'' + T_1(t)\,s' + T_2(t)\,s + T_3(t) = 0,$$

which is of the form,

$$s = f(t) + c_1\,\varphi_1(t) + c_2\,\varphi_2(t),$$

where c_1 and c_2 are arbitrary scalar constants. Then the general solution of the vector equation is,

$$p = f(t)\,\mathbf{a} + \mathbf{c}_1\,\varphi_1(t) + \mathbf{c}_2\,\varphi_2(t),$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are arbitrary vector constants. In the case (1) under consideration the associated scalar equation is simply, $s'' - 1 = 0$ the solution of which is obviously,

$$s = \tfrac{1}{2}t^2 + c_1\,t + c_2,$$

so that the solution of equation (1) is,

$$\mathbf{p} = \tfrac{1}{2}t^2\,\mathbf{g} + t\,\mathbf{v}_0 + \mathbf{p}_0, \tag{2}$$

$\mathbf{v}_0$ and $\mathbf{p}_0$ being arbitrary vector constants.

We may now distinguish two types of motion. If the vectors $\mathbf{g}$ and $\mathbf{v}_0$ are parallel so that $\mathbf{g} \times \mathbf{v}_0 = 0$, equation (2) yields at once,

$$(\mathbf{p} - \mathbf{p}_0) \times \mathbf{g} = 0.$$

This is the equation of a straight line through the terminus of $\mathbf{p}_0$ in the direction of $\mathbf{g}$. The motion is thus rectilinear. If however $\mathbf{g} \times \mathbf{v}_0 = \mathbf{m}$ is not zero, we have from equation (2),

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{m} = 0.$$

This is the equation of a plane through the terminus of $\mathbf{p}_0$ parallel to $\mathbf{g}$ and $\mathbf{v}_0$ and the motion is consequently plane motion.

In each of the two types of motion the path lies in a plane through the terminus of $\mathbf{p}_0$ and parallel to $\mathbf{g}$ so that we may conveniently choose that plane as the XY coördinate plane and the direction and sense of the vector $\mathbf{g}$ as that of the negative Y-axis. If we further take as the origin the position of the moving point at the instant $t = 0$, we may write the coördinates,

$$\mathbf{g} = (0, -g, 0), \qquad \mathbf{p}_0 = (0, 0, 0), \qquad \mathbf{v}_0 = (v_0 \cos \alpha, v_0 \sin \alpha, 0),$$

where g is a positive scalar constant, v_0 is the length of the initial velocity $\mathbf{v}_0$ and α is the angle which $\mathbf{v}_0$ makes with the X-axis. Expanding equation (2) in coördinates we have,

$$(3) \qquad \mathbf{p} = ((v_0 \cos \alpha)t, (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2, 0)$$

and consequently,

$$(3) \qquad \begin{aligned} \mathbf{p}' &= (v_0 \cos \alpha, v_0 \sin \alpha - gt, 0) \\ \mathbf{p}'' &= (0, -g, 0). \end{aligned}$$

The first theorem to be derived from equations (3) is,

The scalar velocity of the particle at any moment is that which it would have acquired in a rectilinear motion parallel to the Y axis from a point with ordinate $v_0^2/2g$ to its actual position at the moment, this motion to have the given constant acceleration.

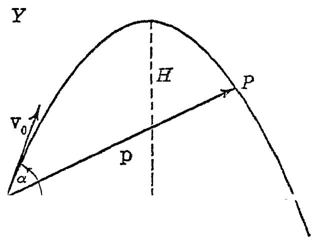

FIG. 40

To prove this we obtain from the second of equations (3),

$$v^2 = \mathbf{p}'^2 = v_0^2 (\cos^2 \alpha + \sin^2 \alpha) - 2g (v_0 \sin \alpha \, t - \tfrac{1}{2}gt^2)$$

or,

$$v^2 = v_0^2 - 2gy, \tag{4}$$

where y is the Y coördinate of P. If we recall from our study of uniformly accelerated rectilinear motion (§ 7) the equation, $v^2 - v_0^2 = 2j\,(s - s_0)$ and in this set $v_0 = 0$, $j = -g$, $s_0 = v_0^2/2g$, $s = y$ we obtain exactly equation (4) and find our theorem proven.

To get the equation of the path of the moving point P we eliminate t between the equations giving the coördinates of P,

$$x = (v_0 \cos \alpha)\, t, \qquad y = (v_0 \sin \alpha)\, t - \tfrac{1}{2}gt^2,$$

obtaining,

$$y = (\tan \alpha)\, x - \left(\frac{g}{2v_0^2 \cos^2 \alpha}\right) x^2. \tag{5}$$

If we rewrite this in the form,

$$\left(x - \frac{v_0^2}{g} \sin \alpha \cos \alpha\right)^2 = -\frac{2v_0^2 \cos^2 \alpha}{g}\left(y - \frac{v_0^2 \sin^2 \alpha}{2g}\right), \tag{5}$$

we recognize the curve as a parabola with a vertical axis opening downward, length of latus rectum $2v_0^2 \cos^2 \alpha/g$, vertex at the point, $(v_0^2 \sin \alpha \cos \alpha/g,\ v_0^2 \sin^2 \alpha/2g)$, focus the point, $(v_0^2 \sin \alpha \cos \alpha/g,\ v_0^2\,(\sin^2 \alpha - \cos^2 \alpha)/2g)$, and directrix the line, $y = v_0^2/2g$.

If we think of several particles leaving the origin with the same scalar velocity v_0 but at different angles α with the X-axis, then since the equation of the directrix of the path is $y = v_0^2/2g$ and does not involve α, it follows that the paths of all these particles will have the same directrix. This common directrix possesses an interesting property, for it is evident that we may reword the first theorem of this article to read,

The scalar velocity of the particle at any moment is that which it would have acquired in moving from rest subject to the given acceleration in a direction parallel to the Y-axis from the directrix to the actual position of the particle.

The Y coördinate of the vertex of the parabola gives us the *maximum height H* reached by the particle, while twice the X coördinate of the vertex gives the *range R* or horizontal distance covered by the particle before reaching the X-axis again. Thus we have,

$$H = v_0^2 \sin^2 \alpha/2g, \qquad R = 2v_0^2 \sin \alpha \cos \alpha/g. \tag{6}$$

We may write the formula for R as $R = v_0^2 \sin 2\alpha/g$, which makes it clear that R will have its maximum value for a given value of v_0 when $2\alpha = \pi/2$ or $\alpha = \pi/4$.

If we think of several particles leaving the origin simultaneously with the same scalar velocity v_0 but at different angles α with the X-axis then to get the equation of the locus of their positions at any given instant we eliminate α between the equations giving their coördinates and find,

$$x^2 + (y + gt^2/2)^2 = v_0^2\, t^2. \tag{7}$$

We recognize this as the equation of a circle with center at the point $(0, -gt^2/2)$ and radius $v_0\, t$. Thus at any given instant the particles all lie on a circle the center of which is constantly moving downward like a particle moving from rest from the origin under the given acceleration and the radius of which is increasing proportionally to the time.

If the initial scalar velocity v_0 of the particle be fixed and if any point (x, y) of the plane be given, we may inquire for what values of the angle α the path of the particle will pass through the given point. To answer this we set $\tan \alpha = m$ and find that equation (5) takes the form of a quadratic equation in m,

$$(gx^2)\, m^2 - (2v_0^2\, x)\, m + (2v_0^2\, y + gx^2) = 0. \tag{8}$$

From the well known properties of the roots of a quadratic equation we conclude that we shall have,

$$\begin{aligned} &2 \text{ real values of } m \text{ for } g^2 x^2 + 2gv_0^2\, y - v_0^4 < 0,\\ &1 \text{ real value of } m \text{ for } g^2 x^2 + 2gv_0^2\, y - v_0^4 = 0,\\ &0 \text{ real values of } m \text{ for } g^2 x^2 + 2gv_0^2\, y - v_0^4 > 0. \end{aligned}$$

The parabola, $g^2 x^2 + 2gv_0^2\, y - v_0^4 = 0$ is called the *parabola of safety* since if the point (x, y) were outside it the moving particle could not "hit" the point if projected from the origin with the scalar velocity v_0. The focus of this parabola is the origin and its vertex is on the directrix of all the parabolas of motion. The reader may readily prove that the parabola of safety is the envelope of all the parabolas of motion with the given value of v_0 and different values of α. The apparent outline of all the water thrown from a fountain or lawn sprinkler which is pointed upward furnishes a familiar example of the parabola of safety. The particles of water issue from the nozzle with about the same scalar

velocity but in different directions and thus approximate the theoretical conditions.

EXERCISES

1. A gunner operating an anti-aircraft gun having a muzzle velocity of 1600 feet per second observes an enemy balloon at an airline distance of 50,000 feet and an angle of elevation of arctan 3/4. Is the balloon in range? If so, at what angle must the gun be set to hit it? Disregard the air resistance and take $g = 32$. *Ans.* $\alpha = 63° 26'$

2. A piece of artillery A fires a projectile which after piercing a captive balloon B falls at C on the horizontal plane through A. Let D be the projection of B on this plane. If $AD = 2000$ ft., $DB = 1000$ ft., determine the initial vector velocity, $\mathbf{v}_0$.

   Ans. $v_0 = 357.8$, $\alpha = 45°$

3. A projectile is fired with an initial velocity v_0 over ground which in the line of fire makes an angle θ with the horizontal. Show that the range will be greatest when the angle of fire α bisects the angle between the ground and the vertical.

4. Show that for a given initial scalar velocity the maximum range of a gun fired over ground sloping upward at an angle of 30° is 2/3 of the maximum range over horizontal ground.

5. A gunner aims at a captive balloon and finds the angle of elevation to be θ, and that to hit the balloon he must increase this inclination of the gun by the amount ϵ. Show that there are two values of ϵ, ϵ_1 and ϵ_2 and that,

$$\theta + \epsilon_1 + \epsilon_2 = 90°.$$

39. Pendulum Motion.

Let us imagine a bead strung on a piece of smooth wire which has been formed into a circle and set up in a vertical plane. If the wire be supposed perfectly smooth and if the resistance of the air be negligible then the motion of the bead on the wire constitutes what is known as *pendulum motion*. We call such a motion a case of *constrained motion* because the bead is constrained to remain on the circular wire. In the general case of constrained motion of a particle it may be constrained to remain on any curve or surface which may be at rest or moving in a predetermined manner. We also sometimes consider *one sided constraint* in which the particle is free to move about in the space on one side of a surface but can not pass to the other side. Other more complicated types of constraint are also studied. The requirement that the particle remain on a certain curve or surface of course modifies its motion and we think of this as being brought about by

a force known as the *constraint* of the curve or surface on the particle. The other forces acting on the particle are then known as *applied forces*. When the curve or surface is spoken of as *perfectly smooth* it simply means that the constraint is constantly normal to the curve or surface at the particle.

To simplify the notation in the case of pendulum motion let us take the center of the circle as the origin and the plane of the circle as the XY-plane with the X-axis directed downward. If we represent by a the radius of the circle and by θ the angle from the X-axis to the radius vector $\mathbf{p}$ of the moving particle P, then evidently we have,

$$\begin{aligned} \mathbf{p} &\equiv (a\cos\theta,\ a\sin\theta,\ 0), \\ \mathbf{p}' &\equiv (-a\sin\theta\,\theta',\ a\cos\theta\,\theta',\ 0), \\ \mathbf{p}'' &\equiv (-a\sin\theta\,\theta'' - a\cos\theta\,\theta'^2,\ a\cos\theta\,\theta'' - a\sin\theta\,\theta'^2,\ 0). \end{aligned}$$

If we represent by s the length of the arc traced out by P in moving from its initial position P_0, then we have for the scalar velocity s',

$$s' = |\mathbf{p}'| = -a\theta',$$

the negative sign being due to the fact that we are here assuming in the figure that θ is decreasing and s increasing at the moment considered.

The only applied force acting on the particle is the force of gravity, $m\,\mathbf{g}$ acting downward and producing an acceleration $\mathbf{g} \equiv (g, 0, 0)$ where g is a positive scalar constant. The constraint of the smooth wire will be a force acting at right angles to it and consequently along the radius vector $\mathbf{p}$. We may therefore represent the acceleration which it produces by $-\lambda\,\mathbf{p}$ where λ is a variable scalar, positive when the wire is pressing inward on the particle and negative when the wire is pressing outward on it. Adding the accelerations together we have as the equation of motion,

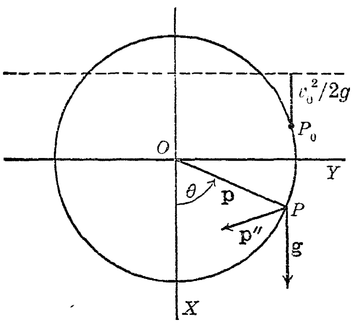

FIG. 41

(1) $$\mathbf{p}'' = \mathbf{g} - \lambda\,\mathbf{p}.$$

If we equate the coördinates of the two members of this equation we have the two scalar equations,

$$
(2) \qquad \begin{aligned} - a \sin\theta\, \theta'' - a \cos\theta\, \theta'^2 + a\lambda \cos\theta - g &= 0, \\ a \cos\theta\, \theta'' - a \sin\theta\, \theta'^2 + a\lambda \sin\theta \quad &= 0, \end{aligned}
$$

between which we may eliminate λ obtaining,

$$
(3) \qquad + g \sin\theta = 0.
$$

This is a second order differential equation for the determination of θ as a function of the time t. If we are concerned only with a motion in which θ remains very small we may get an approximation to the solution by replacing $\sin\theta$ by θ and then we easily find that the motion is approximately simple harmonic and consequently has approximately a constant period. But to integrate the equation as it stands we multiply through by $2\theta'$, thus rendering it exact, and integrate with respect to t from $t = 0$ to $t = t$ obtaining,

$$
\int_0^t 2a\theta'\theta''\, dt + \int_0^t 2g \sin\theta\, \theta'\, dt = 0
$$

or,

$$
(4) \qquad a\,(\theta'^2 - \theta_0'^2) = 2g\,(\cos\theta - \cos\theta_0),
$$

where θ_0 and θ_0' are the values assumed by θ and θ' for $t = 0$.

This first integral of the equation of motion gives us at once an interesting theorem. By the equation the scalar velocity which is $v = - a\theta'$ and the X coördinate of P which is $x = a \cos\theta$ are related as follows,

$$
v^2 - v_0^2 = 2g\,(x - x_0).
$$

But as we have seen (§ 7, (5)) if the body were falling freely under the action of gravity the same equation would hold. We therefore conclude that,

In pendulum motion the variation of the scalar velocity between any two positions of the particle is the same as if the particle had fallen freely through the vertical distance between the positions.

If we write the above equation in the form, $v^2 = v_0^2 + 2g\,(x - x_0)$, then since we must always have $v^2 \geqq 0$, it follows that, $x \geqq x_0 - v_0^2/2g$ which gives us the theorem,

The moving particle can not rise more than the vertical distance $v_0^2/2g$ *above its initial position.*

If we represent by c the constant scalar,

$$c = \frac{a\theta_0'^2}{2g} - \cos\theta_0,$$

equation (4) takes the form,

$$(4') \quad a\theta'^2 = 2g\,(c + \cos\theta) \qquad \text{or} \qquad dt = -\frac{d\theta}{2g\,\sqrt{c + \cos\theta}}$$

Integrating between corresponding limits yields for decreasing θ,

$$(5) \qquad = -\sqrt{\frac{a}{2g}}\int^{\theta} \frac{}{\sqrt{c + \cos\theta}}$$

If solved for θ this equation will give us θ in terms of the time t and the motion of the particle is completely determined. However the actual evaluation of the integral in the second member presents some difficulties and the result takes a different form for different values of c. We shall therefore discuss separately the three cases,

$$\text{I. } c > 1, \qquad \text{II. } c = 1, \qquad \text{III. } c < 1.$$

I. $c > 1$, *Continuous Motion.*

In this case we see from equation (4′) that $\theta'^2 \geqq 2g\,(c - 1)/a$ and hence θ' must always exceed a certain positive constant in absolute value and the particle must perform infinitely many circuits of the circle all in the same sense. We therefore call this the case of *continuous motion.* In this case the integral in equation (5) can not be evaluated in terms of any of the functions studied in the elementary calculus but it can be expressed in terms of a certain function studied in more advanced analysis known as Legendre's *elliptic integral of the first kind.* This function,

$$F(k, \varphi) = \int \frac{d\varphi}{\sqrt{1 - \quad \sin^2\varphi}}$$

is discussed and the values tabulated in Legendre's *Traité des fonctions elliptiques,* T.2, Paris (1826). A brief table of its values

is in Pierce's *A Short Table of Integrals*, Boston (1899) and extensive tables are to be found in *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Washington (1922). When the upper limit of the integral φ has the value $\pi/2$ then $F(k, \varphi)$ becomes the *complete elliptic integral of the first kind*,

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

likewise tabulated in the works refered to. If we call

$$2/(c + 1) = k^2, \qquad \theta/2 = \varphi, \qquad \theta_0/2 = \varphi_0$$

we easily find that,

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{c + \cos \theta}} = \sqrt{2}\, k \,\{F(k, \varphi) - F(k, \varphi_0)\},$$

which reduces equation (5) to,

$$t = -\sqrt{\frac{2a}{g\,(c + 1)}}\,\{F(k, \varphi) - F(k, \varphi_0)\}. \tag{6}$$

To find the time T of a complete circuit of the circle we may compute the time from $\theta = \pi$ to $\theta = 0$ and multiply by 2, obtaining,

$$T = 2\sqrt{\frac{2a}{g\,(c + 1)}}\,K(k). \tag{7}$$

II. $c = 1$, *Single Swing.*

In this case the integral in equation (5) may readily be evaluated in terms of the elementary functions and gives us for increasing θ,

$$t = \sqrt{\frac{a}{g}}\left\{\log \tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right) - \log \tan\left(\frac{\pi}{4} + \frac{\theta_0}{4}\right)\right\}. \tag{8}$$

It is clear from this result that it would require an infinite length of time for θ to reach the value π. In other words, if the particle were started upward on the circle with a velocity giving c the value 1 the particle would continually approach the top of the circle as a limit point without ever quite reaching it. We therefore call this the case of *single swing motion.*

III. $c < 1$, *Oscillation.*

In this case, as in Case I, the integral in equation (5) can not be evaluated in terms of the elementary functions but may be expressed in terms of Legendre's elliptic integral of the first kind, although with a different set of substitutions than formerly. We first introduce a new constant k by the relation $k^2 = (c + 1)/2$ and for increasing θ equation (5) takes the form,

$$t = \frac{1}{2}\sqrt{\frac{a}{g}}\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{k^2 - \sin^2 \theta/2}}.$$

This suggests that we introduce a new variable of integration φ such that,

$$k \sin \varphi = \sin \theta/2, \qquad k \sin \varphi_0 = \sin \theta_0/2.$$

Our equation then becomes,

$$t = \sqrt{\frac{a}{g}}\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

and it is clear that we may write this as,

$$(9) \qquad t = \sqrt{\frac{a}{g}}\,\{F(k, \varphi) - F(k, \varphi_0)\},$$

thus in effect expressing θ in terms of t.

If we examine the definition of the constant c we see that for Case III we have $-1 \leqq c < +1$ and consequently there always exists a constant angle γ such that $\cos \gamma = -c$, $0 \leqq \gamma < \pi$. It then appears from equation (4′) that since $\theta'^2 \geqq 0$ we have $-\gamma \leqq \theta \leqq +\gamma$ and the motion must all take place in the portion of the circle between these limits. The angle γ is not only thus the upper limit of the values of θ but is actually attained by θ in a finite time. For if in equation (5) we set $\theta = \gamma$, we find from the definitions of k and φ that,

$$k = \sqrt{\frac{1 - \cos \gamma}{2}} = \sin \gamma/2, \qquad \varphi = \pi/2,$$

and consequently by equation (9) θ in increasing from θ_0 will at-

tain the value γ in the time,

$$(10) \qquad t = \sqrt{\frac{a}{g}}\,\{K(\sin \gamma/2) - F(\sin \gamma/2, \varphi_0)\}.$$

It is evident from equation (4) that when θ takes on this value γ, θ' is zero and the particle comes momentarily to rest. The motion is thus an *oscillation* with θ varying between $-\gamma$ and $+\gamma$, the motion on one side of the X-axis being a reflection in the axis of that on the other side. To find the period of the motion we may find the time from $\theta = 0$ to $\theta = \gamma$ and multiply by 4 thus giving us by equation (10) the period,

$$(11) \qquad = 4\sqrt{\frac{a}{g}}\,K(\sin \gamma/2).$$

Having now in effect determined θ as a function of t for each of the three cases, we may return to equations (2) and compute the quantity $a\lambda$ which gives us the amount and sense of the acceleration produced by the constraint exerted on the bead by the circular wire. Eliminating θ'' between equations (2) and replacing θ'^2 by its value from equation (4′) gives us,

$$(12) \qquad a\lambda = g\,(2c + 3\cos\theta).$$

Thus for a continuous motion in which $c > 3/2$, $a\lambda$ remains constantly positive and the bead presses constantly outward on the wire. If $1 < c \leqq 3/2$, we will still have continuous motion but $a\lambda$ will be positive and the bead will press outward on the wire only while $\theta < \text{arc cos}\,(-2c/3)$. For single swing motion $c = 1$ and the bead will press outward only while $\theta < \text{arc cos} - 2/3$, ($\theta < 131° \, 49'$). For the case of oscillation we may replace c by $-\cos\gamma$ and discuss the equation,

$$(12') \qquad a\lambda = g\,(3\cos\theta - 2\cos\gamma).$$

If γ is less than a right angle, both $\cos\theta$ and $\cos\gamma$ are positive and since $\cos\theta$ is never less than $\cos\gamma$ it is clear that $3\cos\theta - 2\cos\gamma$ remains positive and the bead presses constantly outward on the wire. But if γ is even slightly larger than a right angle there will be an interval at the top of the motion when both $\cos\theta$ and $\cos\gamma$ will be negative with,

$$\cos\gamma \leqq \cos\theta < \tfrac{2}{3}\cos\gamma,$$

so that the bead will press inward on the wire.

So if one swings a full pail of water at arm's length in a vertical circle, the water will not even start to spill if the arm does not rise above the horizontal, but if the pail is swung higher than this it must be swung rapidly clear over if one is to avoid a spill.

EXERCISES

1. Determine the length of the second's pendulum, i.e. with period 2 seconds, at a point where $g = 981.$ cm./sec.^{2} if the pendulum is to swing 10° each side of the position of equilibrium.

   Ans. 99.018 cm.

2. A swing will hold a 200 pound man when at rest. Through how large an angle from the center position may a 100 pound boy safely swing? *Ans.* 60°

3. A bullet falls 2 feet and lands just within the rim of a smooth hemispherical bowl of diameter 4 feet having its rim in a horizontal plane. How long will the bullet remain in contact with the bowl? Take $g = 32.$ *Ans.* 0.4407 sec.

4. A weight of 1 pound is attached to one end of a string 4 feet long capable of holding a weight of 9 pounds. The other end of the string is fixed and the weight swung in a vertical circle. How many revolutions per second may it make without breaking the string?

   Ans. 1.079

5. I have a clock whose pendulum used to swing 10° each side of the middle. I have just oiled the clock and now the pendulum swings 20° each side of the middle. If the clock formerly kept correct time, how much will it now gain or lose in a true day?

   Ans. Lose 8 min. 11 sec.

6. With the data of Problem 3 determine how the pressure exerted by the bullet on the bowl as it passes through the bottom point compares with what it would be if the bullet were at rest there.

   Ans. Five times as great

7. Show that the period of a small swing of a simple pendulum is given by the formula,

$$T = 2\pi\sqrt{a/g}\left\{1 + \frac{1}{16}\gamma^2 + \frac{11}{3072}\gamma^4 + \cdots\right\},$$

   where γ is the angle of swing from the middle position. For a pendulum 2 feet long swinging 20° both sides of the middle compute the approximate period using one term of the series, two terms, three terms. *Ans.* 1.57080, 1.58276, 1.58284

40. Central Motion.

One very important and interesting type of curvilinear motion of a particle is that in which the acceleration is constantly directed

along the line joining the particle with some fixed point known as the *center of accelerations*. This kind of motion is called *central motion* and its importance lies chiefly in the fact that it is to a high degree of approximation the motion of the earth and the other planets and the comets about the sun as center. It is also studied in connection with investigations of the structure of the molecule and the atom. If we choose the center of accelerations as origin and as usual let the terminus P of the radius vector $\mathbf{p}$ be regarded as the moving particle under consideration, then the vector differential equation,

$$\mathbf{p} \times \mathbf{p}'' = 0, \tag{1}$$

expresses the above assumption that the acceleration of P has the same direction as the radius vector $\mathbf{p}$, although possibly the opposite sense.

Since we have as yet said nothing about the way in which the length and sense of $\mathbf{p}''$ vary it might seem that we could conclude very little concerning the nature of the motion, but three interesting facts are readily proven.

Every central motion in which the particle does not pass through the center of accelerations lies in a plane through the center of accelerations.

For by equation (1) $\mathbf{p} \times \mathbf{p}'' = (\mathbf{p} \times \mathbf{p}')' = 0$ and hence (§ 27)

$$\mathbf{p} \times \mathbf{p}' = \mathbf{c}, \tag{2}$$

where $\mathbf{c}$ is a constant vector. If $\mathbf{c}$ is not zero we form the dot-product of both members of this equation with $\mathbf{p}$ and have, $\mathbf{p} \cdot \mathbf{c} = 0$ which, being the equation of a plane through the origin, proves the theorem for this case. If however $\mathbf{c}$ is zero equation (2) becomes $\mathbf{p} \times \mathbf{p}' = 0$ which as we have seen (§ 27) is a statement that as long as $\mathbf{p}$ does not vanish it remains constant in direction. The particle P thus traces out a straight line through the origin and the theorem is proven for this case also. Evidently the vector $\mathbf{c}$ is perpendicular to the plane of the motion.

Another important theorem following at once from equation (2) is,

The areal velocity of the particle P about the center of accelerations is a constant.

To arrive at this conclusion we have merely to recall the formula for the areal velocity (§ 35), $A' = \frac{1}{2}|\mathbf{p} \times \mathbf{p}'|$ and to employ equa-

tion (2) to write it in the form, $A' = \frac{1}{2}|\mathbf{c}|$. Since equation (2) thus tells us that the areal velocity is a constant this equation is known as *the integral of areas* of the original differential equation (1).

Closely related to this theorem is the fact that,

The distance from the center of accelerations to the tangent to the path of the particle at P varies inversely as the scalar velocity of P.

For the normal form of the equation of the tangent to the path at the point P_0 is,

$$\frac{1}{|\mathbf{p}_0'|}(\mathbf{p} - \mathbf{p}_0) \times \mathbf{p}_0' = 0,$$

and the distance r from the center of accelerations to this line is evidently (§ 22),

$$r = \frac{|\mathbf{p}_0 \times \mathbf{p}_0'|}{|\mathbf{p}_0'|} = \frac{|\mathbf{c}|}{|s_0'|},$$

where s_0' is the scalar velocity at P_0. This proves the theorem.

These three theorems seem to exhaust the important conclusions we can reach concerning central motion in its most general form. We shall now further assume that the length and sense of the acceleration $\mathbf{p}''$ depend only on the distance $p = |\mathbf{p}|$ of the particle P from the center of accelerations O. This seems in many cases a very natural assumption. For instance it is hard to imagine any reason why the gravitational attraction between two particles in otherwise empty space should depend upon anything but their mutual distance and their masses. Under this new hypothesis our equation of motion takes the form,

$$\mathbf{p}'' = \frac{f(p)}{p}\mathbf{p}, \tag{3}$$

where $f(p)$ is an arbitrary scalar function of the positive scalar p which is evidently negative when P is attracted toward O and positive when P is repelled from O. We shall need to assume that $f(p)$ is an integrable function of p between any two positive values of p and that p does not vanish during the motion.

Some general conclusions follow at once from the fact that only $\mathbf{p}''$ and $\mathbf{p}$ are involved in this equation, neither the time t nor the velocity $\mathbf{p}'$ appearing. We first observe that if $\varphi(t)$ is any function, scalar or vector, of the scalar t then if the derivative

exists we have,

$$\frac{d\varphi}{dt}\frac{dt}{d(-t)} \qquad \frac{d\varphi}{dt}.$$

If in this equation we replace φ by its successive derivatives with respect to $-t$ we readily see that,

A change in the sign of the independent variable changes the sign of the derivatives of odd order but leaves the derivatives of even order unaffected.

It appears then that since the equation of motion involves only $\mathbf{p}$ and $\mathbf{p}''$, any motion satisfying this equation would continue to satisfy it if t were replaced by $-t$. This amounts to saying that the particle P could proceed either way along the same path while still satisfying the given equation (3), the radius vector $\mathbf{p}$ and the acceleration $\mathbf{p}''$ being the same at the same point of the path for both motions, but the velocity $\mathbf{p}'$ being in one case the negative of what it is in the other. Thus if the particle P while moving subject to the given equation (3), passes through the point P_0 with a velocity $\mathbf{p}_0'$, it will be tracing out the same path as if it passed through P_0 with the velocity $-\mathbf{p}_0'$, but the sense in which the path is traced will be different in the two motions.

A point of the path at which the velocity is perpendicular to the radius vector, i.e., $\mathbf{p}\cdot\mathbf{p}' = 0$, is called an *apse* and the line through an apse and the center of accelerations is called an *apsidal line*, while the distance from the center to the apse is called an *apsidal distance*. We may now establish that,

In central motion the path of the particle is symmetric with respect to every apisdal line.

For if P pass through an apse P_0 with the velocity $\mathbf{p}_0'$, it is apparent that its motion will be the reflection in the apsidal line of the motion it would have if it passed through P_0 with the velocity $-\mathbf{p}_0'$. But on the other hand, as pointed out above, the two motions will differ only in the sense in which the path is traced. It thus appears that the path constitutes its own reflection in the apsidal line, as stated in the theorem. From this it follows that, although there may be many apsides,

In any central motion there can not be more than two apsidal distances.

For let us consider any three successive apsides along the path. The first and third must by the previous theorem be symmetrically

placed relative to the apsidal line for the second and therefore be at the same apsidal distance. And thus the entire succession of alternate apsides are at the same apsidal distance, while the ones that lie between them must likewise lie at some fixed apsidal distance. There are consequently at most two apsidal distances on the path.

If the equation of motion (3) be given together with the radius vector $\mathbf{p}_0$ and the velocity $\mathbf{p}'_0$ for the instant t_0, we may complete the integration of the equation and express $\mathbf{p}$ as a function of the time. As a first integral we have the integral of areas, $\mathbf{p} \times \mathbf{p}' = \mathbf{c} = \mathbf{p}_0 \times \mathbf{p}'_0$, which was previously developed and we get a second integral by forming the scalar product of both members of equation (3) with $2\,\mathbf{p}'$ obtaining,

$$2\,\mathbf{p}'\cdot\mathbf{p}'' = 2f(p)\,\frac{\mathbf{p}\cdot\mathbf{p}'}{p}.$$

Recalling that $p' = \mathbf{p}\cdot\mathbf{p}'/p$, we see that this equation is exact and may be written,

$$(\mathbf{p}'^2)' = 2f(p)\,p'.$$

Integrating between corresponding limits gives,

$$\mathbf{p}'^2 - \mathbf{p}'^2_0 = \int_{p_0}^{p} 2f(p)\,dp$$

or,

$$(4) \qquad \mathbf{p}'^2 = g(p) \qquad \text{where} \qquad g(p) = \int_{p_0}^{p} 2f(p)\,dp + \mathbf{p}'^2_0.$$

This equation gives at once the theorem,

The scalar velocity of a particle in a given central motion depends only on its distance p from the center of accelerations.

For in a given central motion the function $g(p)$ is determined and the scalar velocity s' is given as a function of p by,

$$s' = \sqrt{\mathbf{p}'^2} = \sqrt{g(p)}.$$

If m is the mass of the particle the quantity $T = \frac{1}{2}m\,\mathbf{p}'^2$ is known as its *kinetic energy* and under the circumstances of our

problem, the quantity,

$$V = - \int^{p} m\, f(p)\, dp,$$

is known as the *potential energy* of the particle; p_1 being any fixed value of p. Equation (4) may therefore be written,

$$T + V = T_0 + V_0,$$

thus stating that,

The total energy, kinetic plus potential, remains constant during the motion.

For this reason equation (4) is known as the *energy integral* of the given differential equation (3). The principle just stated is a special case of the general *Principle of the Conservation of Energy*, of great importance not only in mechanics but in all the exact sciences. It should be noted that although the kinetic energy of the particle is uniquely defined, the potential energy depends on the choice of the lower limit of integration p_1. However we are usually, as here, concerned only with the variations in the potential energy and not with its actual value.

To complete the integration of the differential equation (3) it will be convenient to introduce the variable unit vector $\mathbf{u}$ having the direction and sense of $\mathbf{p}$. Then we have,

$$\mathbf{p} = p\, \mathbf{u}, \qquad \mathbf{p}' = p\, \mathbf{u}' + p'\, \mathbf{u},$$

and our integrals (2) and (4) become respectively,

$$(2') \qquad p^2\, \mathbf{u} \times \mathbf{u}' = \mathbf{c}, \qquad (4') \quad p^2\, \mathbf{u}'^2 + \mathbf{p}'^2 = g(p).$$

Squaring both members of (2′) yields, $p^4\, \mathbf{u}'^2 = \mathbf{c}^2$ which reduces (4′) to the form,

$$(5) \qquad p' = g(p) - \frac{\mathbf{c}^2}{p^2}$$

or,

$$dt = h(p)\, dp \qquad \text{where} \qquad h(p) = \pm \left\{ g(p) - \frac{\mathbf{c}^2}{p^2} \right\}^{-1/2},$$

the positive or negative sign being chosen according as p is increasing or decreasing at the instant considered. Integrating

between corresponding limits then yields,

$$t - t_0 = \int_{p_0}^{p} h(p)\, dp. \tag{6}$$

We have here t expressed as a function of p which in effect gives us p as a function of t. It only remains to determine $\mathbf{u}$ as a function of t and the solution of equation (3) will be complete. Cross-multiplying both members of equation (2′) by $\mathbf{u}$ gives us the differential equation,

$$p^2\, \mathbf{u}' = \mathbf{c} \times \mathbf{u}, \tag{7}$$

for the determination of $\mathbf{u}$, it being now permissible to regard p^2 as a known function of t. We shall show that this equation possesses one and only one solution $\mathbf{u}(t)$ taking on at the instant t_0 the given initial value $\mathbf{u}_0$. In fact this solution is,

$$\mathbf{u} = \cos(\theta - \theta_0)\, \mathbf{u}_0 + \frac{1}{c} \sin(\theta - \theta_0)\, \mathbf{c} \times \mathbf{u}_0, \tag{8}$$

where,

$$c = \qquad\qquad = \int^{t} \frac{c}{p^2}\, dt, \qquad \theta_0 = \theta(t_0).$$

For evidently $\theta' = c/p^2$ and we have,

$$p^2\, \mathbf{u}' = - c \sin(\theta - \theta_0)\, \mathbf{u}_0 + \cos(\theta - \theta_0)\, \mathbf{c} \times \mathbf{u}_0 = \mathbf{c} \times \mathbf{u},$$

so that formula (8) gives a solution of equation (7) and evidently takes on the value $\mathbf{u}_0$ for $t = t_0$. To see that this solution is unique we assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are any two solutions of equation (7) both taking on the value $\mathbf{u}_0$ for $t = t_0$. Substituting these in equation (7) and subtracting the results we find,

$$p^2\, (\mathbf{u}_2 - \mathbf{u}_1)' = \mathbf{c} \times (\mathbf{u}_2 - \mathbf{u}_1).$$

Dot-multiplying both members by $\mathbf{u}_2 - \mathbf{u}_1$ and dividing out the non-vanishing factor p^2 we have,

$$(\mathbf{u}_2 - \mathbf{u}_1) \cdot (\mathbf{u}_2 - \mathbf{u}_1)' = 0.$$

Thus $\mathbf{u}_2 - \mathbf{u}_1$ is a vector of constant length and since this vector

vanishes at the instant t_0, it must be identically zero. This establishes the uniqueness of the solution (8). Since we have in equations (6) and (8) formulas giving p and $\mathbf{u}$ in terms of t the solution of equation (3) is now complete.

Since by equation (8) we have $\mathbf{u} \cdot \mathbf{u}_0 = \cos(\theta - \theta_0)$ it appears that $\theta - \theta_0$ is the angle which $\mathbf{u}$ makes with its initial position $\mathbf{u}_0$ and we may therefore regard p and θ as the polar coördinates of P in the plane of the motion. The equation of the path of P in these coördinates may be readily obtained directly from equation (5) and the definition of θ. Eliminating dt between these equations gives the desired equation,

(9) $$- \theta_0 = \int^{p} \frac{c\,h(p)}{p^2}\,dp.$$

It frequently happens that by solving this equation for p in terms of θ and substituting back in the definition of θ we obtain an equation,

(10) $$= \int^{\theta} \frac{p^2}{c}\,d\theta,$$

giving us the relation between θ and t more readily than by the direct use of the definition of θ.

We have seen above that when the length and sense of the central acceleration are given by the function $f(p)$ and when the initial position and velocity of P are given, then equation (9) gives the path of P in plane polar coördinates. As a sort of converse of this problem we may ask;

If the equation of a plane curve be given in polar coördinates, what will be the form of the function $f(p)$ determining a central acceleration from the origin capable of producing this curve as the path of P?

From the given equation of the path we find $\frac{d\theta}{dp}$, preferably expressing the result in terms of p. Then by equation (9) we have,

$$\frac{d\theta}{dp} = \frac{c\,h(p)}{p^2} \qquad \text{or} \qquad h = \frac{p^2}{c}\frac{d\theta}{dp},$$

and from equation (4) we then get the desired result,

$$f(p) = \frac{1}{2}\frac{dg}{dp} \qquad \text{where} \qquad g = \frac{1}{h^2} + \frac{c^2}{p^2}.$$

The problem is thus solved by two differentiations, but the work may be much facilitated by the introduction of a new variable $q = 1/p$. Then from the above formulas we find,

$$\frac{dq}{d\theta} = \frac{dq}{dp}\frac{dp}{d\theta} = -\frac{1}{p^2}\frac{dp}{d\theta} = -\frac{1}{ch},$$

and hence,

$$g = c^2 q^2 + c^2\left(\frac{dq}{d\theta}\right)^2.$$

If $\dfrac{dq}{d\theta}$ be expressed in terms of q, we then have f very conveniently by the formula,

$$f = -\frac{q^2}{2}\frac{dg}{dq},$$

which follows from the fact that,

$$f = \frac{1}{2}\frac{dg}{dp} = \frac{1}{2}\frac{dg}{dq}\frac{dq}{dp}.$$

If desired we may also write this as,

$$f = -c^2 q^3 - c^2 q^2 \frac{dq}{d\theta}\frac{d^2q}{d\theta^2}\frac{d\theta}{dq},$$

which yields the simple formula,

$$f = -c^2 q^2\left(q + \frac{d^2q}{d\theta^2}\right).$$

EXERCISES

1. The particle P traces the following plane curves subject to an acceleration directed along the radius vector p and with a constant areal velocity $c/2$. Express the amount of the acceleration as a function $f(p)$, negative for an attraction and positive for a repulsion.

   (*a*) A circle through the origin, $p = 2a\cos\theta$
   (*b*) The lemniscate, $p^2 = a^2\cos 2\theta$
   (*c*) The logarithmic spiral, $p = e^\theta$
   (*d*) The hyperbolic spiral, $p = 1/\theta$

(e) A conic with focus at the origin, $p = k/(1 - e \cos \theta)$
(f) The cardioid, $p = a(1 + \cos \theta)$
(g) An ellipse with center at the origin,

$$p^2 = \frac{a^2 b^2}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}$$

(h) The rectangular hyperbola, $p^2 = c^2 \sec 2\theta$
Ans. (a) $f = -8a^2 c^2/p^5$, (b) $f = -3a^4 c^2/p^7$, (h) $f = p/c^2$

2. The motion of the particle P is given by the formula,

$$\mathbf{p} = \mathbf{a} \cos \mu t + \mathbf{b} \sin \mu t,$$

in which $\mathbf{a}$ and $\mathbf{b}$ are vector constants, $\mathbf{a} \times \mathbf{b} \neq 0$, and μ is a positive scalar constant.
(a) Show that the motion is a central motion with the amount of the acceleration given by $f(p) = -\mu^2 p$.
(b) Show that the areal velocity is the constant,

$$A' = \frac{\mu}{2} \sqrt{(\mathbf{a} \times \mathbf{b})^2}.$$

(c) Show that the path is an ellipse and determine its semi-axes.
(d) Show that the hodograph is an ellipse similar to the original ellipse and similarly placed.

3. The particle P has the motion, $\mathbf{p} = \mathbf{a} \cos t + \mathbf{b} \sin t$ where

$$\mathbf{a} \equiv (13/5, -2/5, 14/5), \qquad \mathbf{b} \equiv (2/15, -58/15, 31/15).$$

Determine the radius vector of P when it is at the terminus of the major axis and the instant when it first occupies that position.
Ans. $(5/3, -10/3, 10/3)$ $t = 0.9273$

4. The particle P having an initial position and velocity,

$$\mathbf{p}_0 \equiv (2, 0, 0), \qquad \mathbf{p}_0' \equiv (0, 1/4, 0)$$

is attracted toward the origin with an acceleration of amount $2/p^5$. Find the equation of the path in the XY-plane and the location of the particle π seconds and 2π seconds after the initial instant. Determine the period of the motion.
Ans. $x^2 + y^2 - 2x = 0$, For $t = \pi$, $p = 1.82954$, $\theta = 23° 49' 37''$

5. The particle P having an initial position and velocity,

$$\mathbf{p}_0 \equiv (2, 0, 0), \qquad \mathbf{p}_0' \equiv (0, 2, 0)$$

is attracted toward the origin with an acceleration of amount $4p$. Find the equation of the path in the XY-plane and the location of the particle $\pi/8$ seconds and $\pi/4$ seconds after the initial instant. Determine the period of the motion.
Ans. $x^2/4 + y^2/1 = 1$, For $t = \pi/8$, $p = 1.58114$, $\theta = 26° 33' 54''$

6. The particle P having an initial position and velocity,

$$\mathbf{p}_0 \equiv (2, 0, 0), \qquad \mathbf{p}_0' \equiv (0, 1, 0)$$

is repelled from the origin with an acceleration of amount $p/4$. Find the equation of the path in the XY-plane and the location of the particle (log 2) seconds after the initial instant.

Ans. $x^2 - y^2 = 4$, For $t = \log 2$, $p = 2.23607$, $\theta = 18° 26' 6''$

41. Planetary Motion.

The most important case of central motion is *planetary motion* in which the amount of the acceleration varies inversely as the square of the distance of the moving particle from the center of accelerations. Historically this arose as follows. The German astronomer Kepler (1571–1630) made an exhaustive study of the motions of the planets using the observations collected by his master Tycho Brahe and as a result published in 1609 and 1619 the following known as,

Kepler's Laws of Planetary Motion.

I. *The orbit of each planet is an ellipse with the sun at one focus* (1609).

II. *The areal velocity of each planet about the sun remains constant* (1609).

III. *The squares of the periods of the motions of the different planets are proportional to the cubes of the major axes of their orbits* (1619).

These laws were published by Kepler as three independent statements of the facts of nature, although he did make some attempt to unify and explain them by a certain theory of vortices due to Descartes. About 1679 Sir Isaac Newton (1642–1727) showed that Kepler's laws will follow if it be assumed that the planets are attracted toward the sun with an acceleration inversely proportional to the square of the distance. His demonstration, which is purely geometric, is given in the first book of his *Principia* published in 1687. Probably the first satisfactory analytic treatment is that given by Euler in 1744 in his *Theoria motuum planetarum et cometarum.*

The discussion of central motion given in the preceding article is of course applicable to planetary motion but a more satisfactory treatment may be obtained by making a fresh start and obtaining

some integrals of the equation of motion peculiar to this case. The equation of motion (3) of the preceding article becomes in this case,

$$(1) \qquad \mathbf{p}'' = -\frac{k^2}{p^3}\,\mathbf{p} \qquad \text{where} \qquad p = |\mathbf{p}|,$$

and in which k^2 is a constant, positive for an attraction and negative for a repulsion. We shall discuss particularly the attraction.

Equation (1) yields at once the result $\mathbf{p} \times \mathbf{p}'' = 0$ which integrates into the integral of areas,

$$(2) \qquad \mathbf{p} \times \mathbf{p}' = \mathbf{c},$$

giving us the three theorems proven from it in the preceding article. It will be observed that Kepler's second law is among them. It also follows as before that the path of the particle is symmetric with respect to every apsidal line and that there can not be more than two apsidal distances.

To get a second integral we interchange the two members of equation (2) and cross-multiply member for member with equation (1) obtaining,

$$\mathbf{p}'' \times \mathbf{c} = \frac{k^2}{p^3}\{\mathbf{p} \times (\mathbf{p}' \times \mathbf{p})\} = k^2\left(\frac{p\,\mathbf{p}' - p'\,\mathbf{p}}{p^2}\right) = k^2\left(\frac{\mathbf{p}}{p}\right)'.$$

This differential equation integrates at once into the *velocity integral,*

$$(3) \qquad \mathbf{p}' \times \mathbf{c} = \frac{k^2}{p}\,\mathbf{p} + \mathbf{d},$$

in which the vector constant of integration $\mathbf{d}$ evidently lies in the plane of the motion. From this important integral all the remaining properties of the motion are readily obtainable.

If we dot-multiply both members of equation (3) by $\mathbf{p}$ we have,

$$\mathbf{p} \cdot \mathbf{p}' \times \mathbf{c} = k^2\,p + \mathbf{d}\cdot\mathbf{p}$$

and on reducing the first member by equation (2) we find,

$$(4) \qquad c^2 - k^2\,p = \mathbf{d}\cdot\mathbf{p}.$$

If we indicate by θ the angle which $\mathbf{p}$ makes with the fixed vector

$\mathbf{d}$ and call $|\mathbf{d}| = d$ this equation takes the form,

$$(5)\qquad p = \frac{c^2/k^2}{1 + (d/k^2)\cos\theta}.$$

This is the equation of the orbit in the plane polar coördinates (p, θ) and we recognize it as the equation of a conic with focus at the origin, with d/k^2 as eccentricity, and c^2/k^2 as semi-latus rectum. The vector $\mathbf{d}$ runs in the direction in which p takes on its minimum value, the corresponding point on the path being known as the *perihelion*. The angle θ is called the *true anomaly* of the particle P. It will be observed that we have thus estabished Kepler's first law.

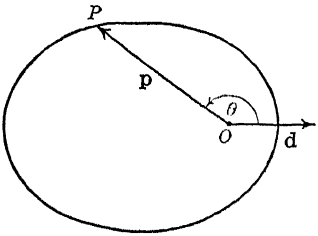

FIG. 42

Returning to the velocity integral (3) we cross-multiply both members by $\mathbf{c}$ and remembering that $\mathbf{c}$ is perpendicular to the plane of the motion we find,

$$c^2\,\mathbf{p}' = \frac{k^2}{p}\,\mathbf{c}\times\mathbf{p} + \mathbf{c}\times\mathbf{d},$$

and taking the lengths of both members after transposition of a term yields,

$$(6)\qquad \left|\mathbf{p}' - \frac{\mathbf{c}\times\mathbf{d}}{c^2}\right| \qquad k^2$$

Thus if the vectors $\mathbf{p}'$ and $\mathbf{c}\times\mathbf{d}/c^2$ be laid off from the origin O, the terminus of $\mathbf{p}'$ will always lie on a circle with center at the terminus of $\mathbf{c}\times\mathbf{d}/c^2$ and radius k^2/c. In other words, the *hodograph* of the motion is on a circle having its center on the latus rectum of the orbit at a distance d/c from the focus and having a radius k^2/c.

Returning again to the velocity integral (3) we cross-multiply both members by $\mathbf{p}$ and find after reduction,

$$(7)\qquad (\mathbf{p}\cdot\mathbf{p}')\,\mathbf{c} = \mathbf{d}\times\mathbf{p}.$$

If we take the lengths of both members and recall that $p' = \mathbf{p}\cdot\mathbf{p}'/p$ we have,

$$(8)\qquad p' = \frac{d}{c}\sin\theta,$$

showing that the radial distance p increases while θ is in the first two quadrants; decreases while θ is in the last two quadrants; and is changing fastest when P is at the extremities of the latus rectum.

From the astronomical point of view the most important question is that of determining p and θ as functions of the time t. We accomplish this by first squaring equations (4) and (7) and adding them member for member which gives, $(c^2 - k^2 p)^2 + (\mathbf{p}\cdot\mathbf{p}')^2 c^2 = d^2 p^2$, or since $p' = \mathbf{p}\cdot\mathbf{p}'/p$,

$$(9) \qquad dt = \pm \frac{cp\, dp}{\sqrt{d^2 p^2 - (c^2 - k^2 p)^2}}.$$

To complete the integration of this equation it will be convenient to consider the following three cases separately.

$$\text{I. } d - k^2 < 0, \qquad \text{II. } d - k^2 = 0, \qquad \text{III. } d - k^2 > 0.$$

I. $d - k^2 < 0$. For this case we may simplify the notation by introducing the three new constants,

$$a = \frac{k^2 c^2}{k^4 - d^2}, \qquad \mu^2 = \frac{(k^4 - d^2)^3}{k^4 c^6}, \qquad e^2 = \frac{d^2}{k^4},$$

which causes equation (9) to take the form,

$$\mu\, dt = \pm \frac{p\, dp}{a\sqrt{a^2 e^2 - (p - a)^2}}.$$

If we count time from the instant of perihelion passage when p takes on the value $c^2/(k^2 + d) = a(1 - e)$, the above integrates into,

$$\mu t = \int_{a(1-e)}^{p} \frac{p\, dp}{a\sqrt{a^2 e^2 - (p - a)^2}}.$$

To evaluate the integral in the second member we now introduce a new variable of integration φ defined by the relation,

$$(10) \qquad p = a(1 - e\cos\varphi),$$

which reduces our equation to the form,

$$\mu t = \int_0^{\varphi} (1 - e\cos\varphi)\, d\varphi,$$

and gives at once,

$$\mu t = \varphi - e \sin \varphi. \tag{11}$$

This important equation is known as *Kepler's Equation* and the quantities μt and φ are known respectively as the *mean anomaly* and the *eccentric anomaly.* The solution of Kepler's equation for φ, when the time t is given and the constants μ and e are known, is an interesting problem upon which literally hundreds of papers have been written. However in any given numerical case the solution is not difficult. A few trials will soon find a value of φ which approximately satisfies the equation and this may be rapidly corrected to any desired degree of accuracy by Newton's method. Thus if φ_0 is an approximate value of φ, then Newton's method shows that under certain conditions,

$$\varphi_1 = \frac{\mu t + e (\sin \varphi_0 - \varphi_0 \cos \varphi_0)}{1 - e \cos \varphi_0},$$

will be a better approximation. Repeated application of this formula soon gives a value of φ correct to within the limitations of the tables employed in the computation.

When φ has thus been determined by equation (11) we have p at once by equation (10) and then θ by equation (5). In terms of our new constants equation (5) appears as,

$$p = \frac{a (1 - e^2)}{1 + e \cos \theta}, \tag{5'}$$

showing that for Case I the orbit is an ellipse with semi-major axis a and eccentricity e. To get θ directly in terms of φ we may eliminate p between equations (5′) and (10) obtaining,

$$\cos \theta = \frac{\cos \varphi - e}{1 - e \cos \varphi} \tag{12}$$

or the equivalent,

$$\tan \theta/2 = \sqrt{\frac{1 + e}{1 - e}} \tan \varphi/2. \tag{12'}$$

From equation (11) it appears that if t increase from any given value by the amount $2\pi/\mu$, then φ and consequently θ will increase by 2π and p will resume its former value. The motion is thus periodic with period $T = 2\pi/\mu$. The definitions of a and μ^2 show

that $a^3 \mu^2 = k^2$ so that we have,

$$T = \frac{2\pi}{k},$$

which proves Kepler's third law.

This completes the analytic discussion of planetary motion for the case in which the orbit is an ellipse. Before leaving this case we shall briefly present the geometric derivation of Kepler's equation from Kepler's first and second laws of planetary motion. This is of historical interest and shows the geometric significance of the eccentric anomaly φ. The equation of any ellipse in standard form is $x^2/a^2 + y^2/b^2 = 1$ and the equation of its major auxiliary circle is $x^2 + y^2 = a^2$. If P is any point on the ellipse and Q is the point on the circle in the same quadrant and having the same X coördinate OM, then by substitution of OM for x in the equations we find $MP/b = MQ/a$. If F is the focus of the ellipse we have $OF = ae$ and from the figure,

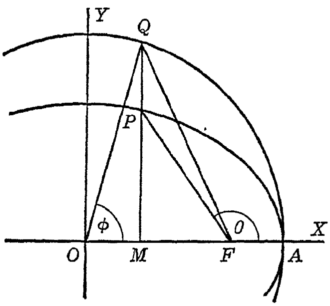

FIG. 43

$$a \cos \varphi = OM = OF - MF = ae + p \cos \theta,$$

$$b \sin \varphi = \frac{b}{a}(a \sin \varphi) = \frac{b}{a} MQ = MP = p \sin \theta.$$

Squaring and adding gives,

$$a^2 (\cos \varphi - e)^2 + b^2 \sin^2 \varphi = p^2 \cos^2 \theta + p^2 \sin^2 \theta = p^2.$$

and since $b^2 = a^2 (1 - e^2)$

$$p^2 = a^2 (1 - 2e \cos \varphi + e^2 \cos^2 \varphi)$$

or

$$p = a (1 - e \cos \varphi).$$

On comparing this with equation (10) we find that the angle φ of the figure is the eccentric anomaly of the particle P.

Furthermore since the relation, $MP/b = MQ/a$ holds for every such pair of points P and Q the same ratio must hold between the

areas of the semi-sectors AMP and AMQ and between the areas of the triangles FMP and FMQ and likewise between the area E of the entire ellipse and the area C of the entire circle. Thus we have,

$$\frac{AMP}{b} = \frac{AMQ}{a}, \qquad \frac{FMP}{b} = \frac{FMQ}{a}, \qquad \frac{E}{b} = \frac{C}{a},$$

and by subtraction,

$$\frac{AFP}{b} = \frac{AFQ}{a}.$$

With these and by Kepler's second law we have,

$$\frac{\mu t}{2\pi} = \frac{t}{T} = \frac{AFP}{E} = \frac{AFQ}{C} = \frac{AOQ - FOQ}{C}$$
$$= \frac{\frac{1}{2}a^2\,\varphi - \frac{1}{2}a^2\,e\sin\varphi}{\pi a^2},$$

from which Kepler's equation follows,

$$\mu t = \varphi - e\sin\varphi.$$

We now turn to the discussion of Case,

II. $d - k^2 = 0$. To simplify the notation of equation (9) for this case we introduce the two new constants,

$$a = c^2/2k^2, \qquad \mu^2 = 4k^8/c^6,$$

reducing equation (9) to the form,

$$\mu\,dt = \pm\frac{p\,dp}{2a\sqrt{ap - a^2}}.$$

If we count time from the instant of perihelion passage when p takes on the value $c^2/2k^2 = a$, the above integrates into,

$$\mu t = \int_a^p \frac{p\,dp}{2a\sqrt{ap - a^2}}.$$

To evaluate the integral in the second member we introduce a new variable of integration φ by the relation,

$$(13) \qquad p = a\,(1 + \varphi^2),$$

reducing our equation to the form,

$$\mu t = \int_0^\varphi (1 + \varphi^2)\, d\varphi,$$

which gives as the analog of Kepler's equation the cubic,

$$\mu t = \varphi + \tfrac{1}{3}\varphi^3. \tag{14}$$

When φ has been determined from this equation we have p from equation (13) and then θ from equation (5). In terms of our new constants equation (5) appears as,

$$p = \frac{2a}{1 + \cos\theta} = a \sec^2 \frac{\theta}{2}, \tag{5''}$$

showing that for Case II the orbit is a parabola with focus at the origin and latus rectum $4a$. To get θ directly in terms of φ we eliminate p between equations (5″) and (13) obtaining,

$$\cos\theta = \frac{1 - \varphi^2}{1 + \varphi^2} \quad \text{or,} \quad \tan\frac{\theta}{2} = \varphi. \tag{15}$$

The discussion of Case III is along lines similar to those followed in Case I. The development of the details is left as an exercise for the reader.

EXERCISES

1. Show that the cubic equation (14) of the text, $\mu t = \varphi + \frac{1}{3}\varphi^3$ may be solved for φ by the trigonometric substitutions,

$$\cot\beta = \tfrac{3}{2}\mu t, \qquad \cot\alpha = \sqrt[3]{\cot\beta/2}, \qquad \varphi = 2\cot 2\alpha.$$

(Employ the identity, $2\cot 2x = \cot x - \tan x$.)

2. Show that in planetary motion the center of accelerations lies within the circular hodograph for elliptic orbits, on the hodograph for parabolic orbits, and outside for hyperbolic orbits.

3. Discuss the integration of equation (9) of the text for Case III, $d - k^2 > 0$, showing that if we let,

$$a = \frac{k^2 c^2}{d^2 - k^4}, \quad \mu^2 = \frac{(d^2 - k^4)^3}{k^4 c^6}, \quad e^2 = \frac{d^2}{k^4}, \quad p = a\,(e\cosh\varphi - 1),$$

the equation integrates into, $\mu t = e \sinh \varphi - \varphi$. Show that,

$$p = \frac{a\,(e^2 - 1)}{1 + e \cos \theta}, \quad \cos \theta = \frac{e - \cosh \varphi}{e \cosh \varphi - 1}, \quad \tan \frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}} \tanh \frac{\varphi}{2}.$$

4. Show that for elliptic orbits,

$$\frac{1}{a} = \frac{2}{p} - \frac{\mathrm{p}'^2}{k^2},$$

and that consequently if the constant k^2 is given, the period of the motion depends only upon the initial radial distance p_0 and the initial scalar velocity s_0'.

5. In elliptic orbits the semi-major axis a of the ellipse is known in astronomy as the *mean distance* of the planet from the sun. Show that the actual average distance relative to the time is not a but $a\,(1 + e^2/2)$.

6. A particle is attracted toward the origin with an acceleration of amount $32/9p^2$. At the instant $t = 1$ it has the position and velocity $\mathrm{p}_0 = (0, 2, 0)$, $\mathrm{p}_0' = (-4/3, 4/3, 0)$. Find the equation of the path in the XY-plane and the location of the particle for $t = 0$ and $t = 7/2$.

Ans. $y^2 = -4\,(x - 1)$, For $t = 7/2$, $\mathrm{p} \equiv (-3, 4, 0)$

7. A particle is attracted toward the origin with an acceleration of amount $16/p^2$. At the instant $t = 0$ it has the position and velocity, $\mathrm{p}_0 = (2, 0, 0)$, $\mathrm{p}_0' = (0, 2\sqrt{3}, 0)$. Find the equation of the path in the XY-plane and the location of the particle π seconds and 2π seconds after the initial instant.

Ans. $p = 3/(1 + \frac{1}{2} \cos \theta)$, For $t = \pi$, $p = 4.87026$,
$\theta = 140° \, 10' \, 39''$

GENERAL EXERCISES ON CHAPTER IV

1. A cycloid is generated by a point on a circle of radius a which rolls on the lower side of a horizontal line. If a wire be curved along this cycloid and a bead slide from rest along this wire, determine the position of the bead at any instant, assuming that the wire is smooth and that the resistance of the air is proportional to the speed of the bead and is $2k$ units of force per unit mass of the bead at unit speed. Show that if the bead reaches the vertex of the cycloid, the time required is independent of the initial position of the bead on the wire. If the radius of the above circle is 1 foot and the resistance of the air is 4 poundals per pound of bead at a speed of 1 foot per second, compute the above time. $(g = 32)$ *Ans.* $T = 1.1781$

2. A projectile is fired into the air with such a speed that the air resistance may be regarded as proportional to the speed. Determine the position of the projectile in terms of the time and the initial

position and velocity. If several such projectiles be fired simultaneously from the same point in the same direction but with different initial speeds, show that at any instant they are all on the same straight line parallel to the initial velocities and that this line moves downward with a speed independent of the initial velocities.

3. If a unit particle is acted upon by a force whose tangential measure in the sense of the motion is $f(t)$, show that the distance covered in the time T is,

$$s = \int_0^T (T - t)\, f(t)\, dt \qquad \text{or} \qquad s = \int_0^T - t\, f(t)\, dt,$$

according as the initial or terminal velocity of the particle is zero. Apply to some of the types of motion previously discussed.

CHAPTER V

DISPLACEMENTS AND MOTION OF A RIGID BODY

42. Definition of a Rigid Body.

In elementary geometry we set up a postulate to the effect that,

Any figure may be moved from one place to another without altering its size or shape.

This is known as the *congruence postulate.* In classical mechanics we retain all the assumptions of geometry and the congruence postulate among them, but in mechanics we assume further that not merely the figures of geometry may move without change of size or shape, but figures having a mass may likewise do so without altering the mass distribution. The figures which so move are known as *rigid bodies.* The principal property of a rigid body is contained in the following definition.

A rigid body is a collection of particles such that the distances between any two of them remain constant for every motion of the body.

Two points should be noted in connection with this definition. In the first place it is here implied that as the rigid body moves it is possible in some way to trace each individual particle, for unless the particles retained their individuality during the motion the definition would be meaningless. Furthermore we are here assuming, as heretofore, that the particles move in a continuous manner so that, for instance, they do not appear at one point at a certain instant and at a distant point immediately thereafter. Without this last assumption we might reflect a set of particles in a plane without their ceasing to form the particles of the same rigid body, since their mutual distances would be retained by such a reflection.

Although we shall accept the definition of a rigid body which is given above, still it is by no means certain that such bodies can actually exist in our space. This may be readily appreciated if we consider the corresponding situation on a surface. Figures drawn on the surface of a sphere may be freely moved about on that sphere without change of size or shape, but if the sphere were

very slightly altered to form an ellipsoid with three unequal axes, figures drawn on its surface could not be moved about on that surface and retain their size and shape. We have no reason to suppose that our space might not be like the surface of the ellipsoid in this respect.

Another difficulty presents itself in connection with our definition. It is evident that in formulating the definition the idea of the distance between two points is regarded as a previously known concept. But if we attempt to actually determine whether any given object is rigid the only convenient means for seeing if the distances between its points remain constant is to apply some sort of a measure to them and this measure should in its turn be a rigid body. We have thus no theoretical justification for the assumption that any rigid bodies exist in nature and no means of determining whether a given body is rigid or not. Nevertheless a large part of theoretical mechanics is built up on the basis of this definition and the conclusions reached are certainly in very close agreement with the experimental facts.

43. Immediate Consequences of the Definition.

The above definition of a rigid body tells us that if O and P be any two points of a rigid body, then the length $|\mathbf{p}|$ of the vector $OP = \mathbf{p}$ remains constant during any motion of the body. There are several immediate consequences of this definition which it will be convenient for us to have available in analytic form. If the body be given a displacement in which $\mathbf{p}_1$ and $\mathbf{p}_2$ are the initial and terminal values assumed by the vector $\mathbf{p}$, then by the above definition $|\mathbf{p}_2| = |\mathbf{p}_1|$ and consequently

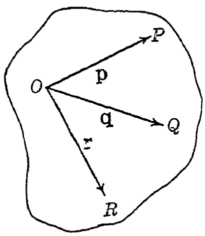

Fig. 44

(1) $$\mathbf{p}_2^2 = \mathbf{p}_1^2$$

If R be another point of the rigid body and $\mathbf{r}_1$ and $\mathbf{r}_2$ be the initial and terminal values of the vector $OR = \mathbf{r}$, we have as before $(\mathbf{r}_2 - \mathbf{p}_2)^2 = (\mathbf{r}_1 - \mathbf{p}_1)^2$ and on expanding these expressions and simplifying by means of equation (1) we find,

(2) $$\mathbf{p}_2 \cdot \mathbf{r}_2 = \mathbf{p}_1 \cdot \mathbf{r}_1.$$

Similarly we have,

(3) $$(\mathbf{p}_2 \times \mathbf{r}_2)^2 = (\mathbf{p}_1 \times \mathbf{r}_1)^2,$$

since on expanding the two members we find the equation to read,

$$\mathbf{p}_2^2\,\mathbf{r}_2^2 - (\mathbf{p}_2\cdot\mathbf{r}_2)^2 = \mathbf{p}_1^2\,\mathbf{r}_1^2 - (\mathbf{p}_1\cdot\mathbf{r}_1)^2,$$

which holds as a consequence of equations (1) and (2). If Q be another point of the body with $OQ = \mathbf{q}$, then we may likewise show that,

$$(4) \qquad [\mathbf{p}_2\,\mathbf{q}_2\,\mathbf{r}_2] = [\mathbf{p}_1\,\mathbf{q}_1\,\mathbf{r}_1],$$

but we must here make use of the considerations of continuity mentioned in the previous article. A reflection of the body in any plane would maintain the mutual distances of its points but would change the sign of $[\mathbf{p}\,\mathbf{q}\,\mathbf{r}]$. The further details of the proof are left to the reader.

The mechanical interpretation of the above four equations is not difficult. The fact that $\mathbf{p}\cdot\mathbf{r}$ remains constant during any motion of the rigid body is equivalent to the statement that if the three vertices of a triangle are points of a rigid body, then its angles are constant. Equation (3) states in effect that if the vertices of a parallelogram are fixed in the rigid body, its area is a constant, while equation (4) makes a similar statement about the volume of a parallelopiped.

By means of the above equations we may readily prove the important theorem,

A displacement of a rigid body is completely determined by the displacements of any three of its points, if the points are not collinear.

This means in effect that if the initial and terminal positions of any three non-collinear points of the body are given, we may determine the terminal position of any fourth point of the body, if its initial position is known. Thus if A, B, C, P are any four points of the rigid body with A, B, C non-collinear, having radius vectors from a fixed point O which are $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, $\mathbf{p}_1$ and $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{c}_2$, $\mathbf{p}_2$ before and after the displacement respectively, then we may express $\mathbf{p}_2$ in terms of the other seven vectors as follows. Call,

$$\begin{aligned} &\mathbf{p}_1 - \mathbf{a}_1 = \mathbf{r}_1, \quad &&\mathbf{b}_1 - \mathbf{a}_1 = \mathbf{k}_1, \quad &&\mathbf{c}_1 - \mathbf{a}_1 = \mathbf{l}_1, \quad &&\mathbf{m}_1 = \mathbf{k}_1 \times \mathbf{l}_1,\\ &\mathbf{p}_2 - \mathbf{a}_2 = \mathbf{r}_2, \quad &&\mathbf{b}_2 - \mathbf{a}_2 = \mathbf{k}_2, \quad &&\mathbf{c}_2 - \mathbf{a}_2 = \mathbf{l}_2, \quad &&\mathbf{m}_2 = \mathbf{k}_2 \times \mathbf{l}_2. \end{aligned}$$

By the identity, § 21, Prob. 1, *f*,

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\,\mathbf{d} = (\mathbf{d}\cdot\mathbf{a})\,\mathbf{b}\times\mathbf{c} + (\mathbf{d}\cdot\mathbf{b})\,\mathbf{c}\times\mathbf{a} + (\mathbf{d}\cdot\mathbf{c})\,\mathbf{a}\times\mathbf{b}$$

we may write,

$$[\mathbf{k}_2\,\mathbf{l}_2\,\mathbf{m}_2]\,\mathbf{r}_2 = (\mathbf{r}_2\cdot\mathbf{k}_2)\,\mathbf{l}_2 \times \mathbf{m}_2 + (\mathbf{r}_2\cdot\mathbf{l}_2)\,\mathbf{m}_2 \times \mathbf{k}_2 + (\mathbf{r}_2\cdot\mathbf{m}_2)\,\mathbf{k}_2 \times \mathbf{l}_2,$$

and on applying equations (2), (3), (4) to the scalar coefficients in this equation we have,

$$(5)\quad [\mathbf{k}_1\,\mathbf{l}_1\,\mathbf{m}_1]\,\mathbf{r}_2 = (\mathbf{r}_1\cdot\mathbf{k}_1)\,\mathbf{l}_2 \times \mathbf{m}_2 + (\mathbf{r}_1\cdot\mathbf{l}_1)\,\mathbf{m}_2 \times \mathbf{k}_2 + (\mathbf{r}_1\cdot\mathbf{m}_1)\,\mathbf{k}_2 \times \mathbf{l}_2.$$

This give us $\mathbf{r}_2$ because $[\mathbf{k}_1\,\mathbf{l}_1\,\mathbf{m}_1] = (\mathbf{k}_1 \times \mathbf{l}_1)^2$ can not vanish since A, B, C are non-collinear, and then we have at once $\mathbf{p}_2 = \mathbf{r}_2 + \mathbf{a}_2$.

Thus far we have considered only the *displacements* of a rigid body; i.e. we have been concerned only with the initial and terminal positions of the body without considering *how* the body passed from the one to the other. But now suppose that O is any fixed point and P and Q are any points of a moving rigid body. Let $\mathbf{p}_0$ and $\mathbf{q}_0$ be the values of OP and OQ at some given instant t_0, and $\mathbf{p}$ and $\mathbf{q}$ the values of these vectors at any instant t. Then we may regard $\mathbf{p}$ and $\mathbf{q}$ as vector functions of the scalar t and by equation (1) write $(\mathbf{p} - \mathbf{q})^2 = (\mathbf{p}_0 - \mathbf{q}_0)^2$. Differentiating with respect to the time gives us $2\,(\mathbf{p}' - \mathbf{q}')\cdot(\mathbf{p} - \mathbf{q}) = 0$ and if we represent by $\mathbf{u}$ a unit vector having the direction and sense of $\mathbf{p} - \mathbf{q}$, we have on dividing by $|\mathbf{p} - \mathbf{q}|$ and multiplying by $\mathbf{u}$, the result,

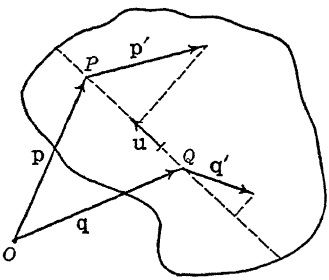

FIG. 45

$$(6)\qquad (\mathbf{p}'\cdot\mathbf{u})\,\mathbf{u} = (\mathbf{q}'\cdot\mathbf{u})\,\mathbf{u}.$$

We have thus proven the important theorem,

The projections of the velocities of any two points of a rigid body on the line joining them are equal.

If at any moment in the motion of a rigid body the velocities of three of its points are known, we may determine from them the velocity of any other point of the body, provided the first three points are not collinear. To see this let O be any fixed point and A, B, C, P any four points of the rigid body, where A, B, C are

non-collinear. At any instant t let the points have radius vectors from O, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{p}$ and let us call,

$$\mathbf{p} - \mathbf{a} = \mathbf{r}, \qquad \mathbf{b} - \mathbf{a} = \mathbf{k}, \qquad \mathbf{c} - \mathbf{a} = \mathbf{l}, \qquad \mathbf{k} \times \mathbf{l} = \mathbf{m}.$$

Then by the identity (§ 21, Prob. 1, f) quoted above we may write,

$$[\mathbf{k}\,\mathbf{l}\,\mathbf{m}]\,\mathbf{r} = (\mathbf{r}\cdot\mathbf{k})\,\mathbf{l}\times\mathbf{m} + (\mathbf{r}\cdot\mathbf{l})\,\mathbf{m}\times\mathbf{k} + (\mathbf{r}\cdot\mathbf{m})\,\mathbf{k}\times\mathbf{l}.$$

But now by equations (2) and (3) the scalar quantities $[\mathbf{k}\,\mathbf{l}\,\mathbf{m}]$, $\mathbf{r}\cdot\mathbf{k}$, $\mathbf{r}\cdot\mathbf{l}$, $\mathbf{r}\cdot\mathbf{m}$ remain unchanged during any motion of the body, so that if we differentiate both members of the above equation with respect to the time, we may treat these scalars as constants and obtain,

$$(7)\quad [\mathbf{k}\,\mathbf{l}\,\mathbf{m}]\,\mathbf{r}' = (\mathbf{r}\cdot\mathbf{k})\,(\mathbf{l}'\times\mathbf{m} + \mathbf{l}\times\mathbf{m}') \\ + (\mathbf{r}\cdot\mathbf{l})\,(\mathbf{m}'\times\mathbf{k} + \mathbf{m}\times\mathbf{k}') + (\mathbf{r}\cdot\mathbf{m})\,(\mathbf{k}'\times\mathbf{l} + \mathbf{k}\times\mathbf{l}'),$$

or upon expansion,

$$(8)\quad (\mathbf{k}\times\mathbf{l})^2\,\mathbf{r}' = (\mathbf{r}\cdot\mathbf{k})\,(\mathbf{l}^2\,\mathbf{k}' - \mathbf{k}\cdot\mathbf{l}\,\mathbf{l}') \\ + (\mathbf{r}\cdot\mathbf{l})\,(\mathbf{k}^2\,\mathbf{l}' - \mathbf{k}\cdot\mathbf{l}\,\mathbf{k}') + [\mathbf{r}\,\mathbf{k}\,\mathbf{l}]\,(\mathbf{k}\times\mathbf{l}' + \mathbf{k}'\times\mathbf{l}).$$

Since the points A, B, C are non-collinear the coefficient $(\mathbf{k} \times \mathbf{l})^2$ can not vanish and we have $\mathbf{r}'$ and hence $\mathbf{p}'$ expressed in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{p}$ and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$. This establishes our proposition. If the three points A, B, C are collinear it will not be possible to determine from their velocities the velocity of a point P not on their line because clearly we could change the velocity of P by rotating the body about the line of A, B, C without affecting the velocity of A, B and C.

EXERCISES

1. If O, P, Q, R be any four points of a rigid body and if $OP = \mathbf{p}$, $OQ = \mathbf{q}$, $OR = \mathbf{r}$, show that

$$[\mathbf{p}_2\,\mathbf{q}_2\,\mathbf{r}_2] = [\mathbf{p}_1\,\mathbf{q}_1\,\mathbf{r}_1],$$

where $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{r}_1$ and $\mathbf{p}_2$, $\mathbf{q}_2$, $\mathbf{r}_2$ are the values of $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ respectively before and after any displacement of the body.

2. A rigid body is given a displacement in which its points A, B, C have the following initial and terminal positions.

$$A_1 = (5, -1, 1), \qquad B_1 = (-1, 0, 2), \qquad C_1 = (-1, 2, 1),$$
$$A_2 = (2, 3, -1), \qquad B_2 = (-1, -2, 1), \qquad C_2 = (0, -2, 3).$$

What is the terminal position of the point P whose initial position is (x_1, y_1, z_1)? Also $(4, 2, 0)$? *Ans.* $(3, 2, 2)$

3. Using the notation of equation (5) show that if the point P is coplanar with A, B, C we may determine $\mathbf{r}_2$ by the equation,

$$\begin{vmatrix} \mathbf{r}_2 & \mathbf{k}_2 & \mathbf{l}_2 \\ \mathbf{k}_1\cdot\mathbf{r}_1 & \mathbf{k}_1\cdot\mathbf{k}_1 & \mathbf{k}_1\cdot\mathbf{l}_1 \\ \mathbf{l}_1\cdot\mathbf{r}_1 & \mathbf{l}_1\cdot\mathbf{k}_1 & \mathbf{l}_1\cdot\mathbf{l}_1 \end{vmatrix} = 0.$$

4. If the three points A, B, P of a rigid body are collinear and have radius vectors from a fixed point O which are $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{p}_1$ and $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{p}_2$ before and after a displacement respectively, show that $\mathbf{p}_2$ may be expressed in terms of the other vectors by the equation,

$$(\mathbf{b}_1 - \mathbf{a}_1)^2\,(\mathbf{p}_2 - \mathbf{a}_2) = \{(\mathbf{b}_1 - \mathbf{a}_1)\cdot(\mathbf{p}_1 - \mathbf{a}_1)\}\,(\mathbf{b}_2 - \mathbf{a}_2).$$

5. Using the notation of equation (8) show that if the point P is coplanar with A, B, C, we may determine $\mathbf{r}'$ by the equation,

$$\begin{vmatrix} \mathbf{r}' & \mathbf{k}' & \mathbf{l}' \\ \mathbf{k}\cdot\mathbf{r} & \mathbf{k}\cdot\mathbf{k} & \mathbf{k}\cdot\mathbf{l} \\ \mathbf{l}\cdot\mathbf{r} & \mathbf{l}\cdot\mathbf{k} & \mathbf{l}\cdot\mathbf{l} \end{vmatrix} = 0.$$

6. If the three points A, B, P of a rigid body are collinear and have radius vectors from a fixed point O, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$ respectively, show that the velocity $\mathbf{p}'$ of P is given by the formula,

$$(\mathbf{b} - \mathbf{a})^2\,(\mathbf{p}' - \mathbf{a}') = \{(\mathbf{b} - \mathbf{a})\cdot(\mathbf{p} - \mathbf{a})\}\,(\mathbf{b}' - \mathbf{a}').$$

7. If any two points of a rigid body have the same velocity, show that all points of the body on their line have this same velocity.

8. The XY-plane moves as a rigid body, the velocities of the following points on it being as here given.

$$\begin{array}{lll} A = (0, 0, 0), & B = (2, 0, 0), & C = (1, 1, 0), \\ \mathbf{a}' = (0, 0, 1), & \mathbf{b}' = (0, 0, 2), & \mathbf{c}' = (0, 0, \tfrac{3}{2}). \end{array}$$

What is the velocity of the point $P \equiv (x, y, z)$? Are there any points with zero velocity in the XY-plane?

9. The XY-plane moves as a rigid body, the velocities of the following points on it being as here given.

$$\begin{array}{lll} A \equiv (1, 2, 0), & B \equiv (0, 0, 0), & C \equiv (2, 1, 0), \\ \mathbf{a}' \equiv (6, -8, 4), & \mathbf{b}' \equiv (4, -7, 6), & \mathbf{c}' \equiv (5, -9, 8). \end{array}$$

What is the velocity of the point $P \equiv (x, y, z)$? Are there any points with zero velocity in the XY-plane?

10. The sphere $x^2 + y^2 + z^2 = 81$ moves as a rigid body, the velocities of the following points on it being as here given.

$$\begin{array}{lll} A \equiv (9, 0, 0), & B \equiv (8, 4, 1), & C \equiv (6, 3, 6), \\ \mathbf{a}' \equiv (4, 16, 12), & \mathbf{b}' \equiv (-7, 14, 9), & \mathbf{c}' \equiv (-13, 11, 6). \end{array}$$

What is the velocity of the point $P \equiv (x, y, z)$? Are there any points on the sphere with zero velocity?

44. Rodrigues' Formula.

Let us consider the displacement of the points of a rigid body brought about by rotating the body through a certain angle about a certain fixed axis. As in any displacement, we are concerned only with the initial and terminal positions of the particles and not with their velocities nor intermediate positions. We shall assume the body to have rotated through an angle θ in the positive sense about an axis passing through a fixed point O in the direction and sense of a constant unit vector $\mathbf{u}$. Let P_1 and P_2 be the initial and terminal positions of any point P of the body and let M be the common foot of the perpendiculars let fall from P_1 and P_2 upon the axis. Let N be the midpoint of $P_1 P_2$. Then from the figure we have,

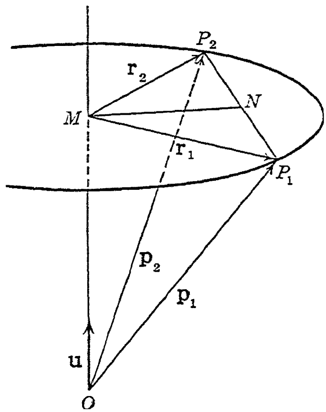

FIG. 46

$$(\cos \theta/2)\,|NP_2| = \pm\,(\sin \theta/2)\,|MN|,$$
$$|MN|\,NP_2 = \pm\,|NP_2|\,\mathbf{u} \times MN,$$

and consequently,

$$(\cos \theta/2)\, NP_2 = (\sin \theta/2)\, \mathbf{u} \times MN.$$

If we call $MP_1 = \mathbf{r}_1$, $MP_2 = \mathbf{r}_2$, $\cos \theta/2 = m$, $(\sin \theta/2)\,\mathbf{u} = \mathbf{n}$, the above equation becomes,

$$m\,(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{n} \times (\mathbf{r}_2 + \mathbf{r}_1), \qquad m^2 + \mathbf{n}^2 = 1.$$

If we call $OP_1 = \mathbf{p}_1$ and $OP_2 = \mathbf{p}_2$ then since,

$$\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{r}_2 - \mathbf{r}_1, \qquad \mathbf{n} \times (\mathbf{p}_2 + \mathbf{p}_1) = \mathbf{n} \times (\mathbf{r}_2 + \mathbf{r}_1),$$

it follows that,

$$(1) \qquad m\,(\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{n} \times (\mathbf{p}_2 + \mathbf{p}_1), \qquad m^2 + \mathbf{n}^2 = 1.$$

This is known as *Rodrigues' Formula.* For $m \neq 0$ the formula may be written in the convenient form,

$$(2) \qquad \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{w} \times (\mathbf{p}_2 + \mathbf{p}_1), \qquad \mathbf{w} = (\tan \theta/2)\,\mathbf{u}.$$

To solve equation (1) for $\mathbf{p}_2$ in terms of $\mathbf{p}_1$, m, $\mathbf{n}$ we easily obtain

from (1) the equations,

$$m^3 (\mathbf{p}_2 - \mathbf{p}_1) = m^2 \mathbf{n} \times (\mathbf{p}_2 + \mathbf{p}_1),$$
$$m^2 \mathbf{n} \times (\mathbf{p}_2 - \mathbf{p}_1) = m\, \mathbf{n}\cdot(\mathbf{p}_2 + \mathbf{p}_1)\, \mathbf{n} - m\, \mathbf{n}^2 (\mathbf{p}_2 + \mathbf{p}_1),$$
$$m\, \mathbf{n}\cdot(\mathbf{p}_2 - \mathbf{p}_1)\, \mathbf{n} = 0,$$

and on adding member for member we find,

$$m (\mathbf{p}_2 + \mathbf{p}_1) = 2m \{m^2 \mathbf{p}_1 + (\mathbf{n}\cdot\mathbf{p}_1)\, \mathbf{n} + m\, \mathbf{n}\times\mathbf{p}_1\}.$$

For $m \neq 0$ we have therefore,

$$(3) \quad \mathbf{p}_2 + \mathbf{p}_1 = 2 \{m^2 \mathbf{p}_1 + (\mathbf{n}\cdot\mathbf{p}_1)\, \mathbf{n} + m\, \mathbf{n}\times\mathbf{p}_1\}, \quad m^2 + \mathbf{n}^2 = 1,$$

or the equivalent,

$$(4) \qquad \mathbf{p}_2 + \mathbf{p}_1 = \frac{2}{1 + \mathbf{w}^2} \{\mathbf{p}_1 + (\mathbf{w}\cdot\mathbf{p}_1)\, \mathbf{w} + \mathbf{w}\times\mathbf{p}_1\},$$

which are immediately solvable for $\mathbf{p}_2$. For $m = 0$ equation (1) can not be directly solved for $\mathbf{p}_2$ but equation (3) must still hold if we make the additional assumption that $\mathbf{p}_2$ varies continuously with $\mathbf{p}_1$, m, $\mathbf{n}$. The equation then becomes,

$$(5) \qquad \mathbf{p}_2 + \mathbf{p}_1 = 2 (\mathbf{u}\cdot\mathbf{p}_1)\, \mathbf{u},$$

which is the form taken by Rodrigues' formula for $\theta = 180°$.

If P and Q be any two points fixed in the rigid body and if $\mathbf{r}_1$ and $\mathbf{r}_2$ be the initial and terminal values of the vector PQ, then all of the above relations obtained for $\mathbf{p}_1$ and $\mathbf{p}_2$ apply likewise to $\mathbf{r}_1$ and $\mathbf{r}_2$. To see this we have merely to write out the equation to be considered for the vectors OP and OQ and subtract the results member for member. Since Rodrigues' formula thus holds for all vectors fixed in the rigid body regardless of their position relative to the axis of rotation, it follows that the formula will still hold if the axis be shifted during the rotation provided it continues parallel to its original position.

EXERCISES

1. The sphere $x^2 + y^2 + z^2 = 81$ is rotated in the positive sense about the axis running through its center in the direction and sense of the vector (2, 2, 1) at the rate of one revolution in four seconds. Where will the point P of the sphere initially at $P_0 \equiv (3, -6, 6)$ be at the end of one second? Where and when will P cross the plane $3x - 2y - z - 6 = 0$?

Ans. $(6, -3, -6)$, $(6/5, 3, -42/5)$, $t = 1.590$
$(6/5, -123/25, 186/25)$, $t = 3.819$

2. The plane $x + 4y - z + 8 = 0$ is rotated through an angle of 60° in the positive sense about the axis running through the point $(1, -2, 1)$ in the direction and sense of the vector $(5, -1, 1)$. What is its equation in its terminal position? What are the terminal positions of the points $A_1 \equiv (-1, -1, 3)$, $B_1 \equiv (6, -3, 2)$ of this plane?

Ans. $y + z + 1 = 0$, $A_2 \equiv (-28/27, -10/3, 7/3)$
$B_2 \equiv (6, -3, 2)$

3. We may write equation (3) of this article,

$$\mathbf{p}_2 = -\mathbf{p}_1 + 2\{m^2\mathbf{p}_1 + (\mathbf{n}\cdot\mathbf{p}_1)\mathbf{n} + m\,\mathbf{n}\times\mathbf{p}_1\}$$

in the symbolic form,

$$\mathbf{p}_2 \quad e^{\theta\mathbf{u}}\,\mathbf{p}_1,$$

where $e^{\theta\mathbf{u}}$ indicates the operator which acting on $\mathbf{p}_1$ will produce $\mathbf{p}_2$. Show analytically that $e^{\theta\mathbf{u}}$ has the following properties.

(a) $e^{\theta\mathbf{u}}\,e^{\varphi\mathbf{u}}\,\mathbf{p} = e^{(\theta+\varphi)\mathbf{u}}\,\mathbf{p}$,
(b) $e^{0}\,\mathbf{p} = e^{2\pi\mathbf{u}}\,\mathbf{p} = \mathbf{p}$,
(c) $\dfrac{d}{d\theta}e^{\theta\mathbf{u}}\,\mathbf{p} = \mathbf{u}\times e^{\theta\mathbf{u}}\,\mathbf{p}$.

Compare with the familiar properties of the function of θ, $e^{\theta i} = \cos\theta + i\sin\theta$ where $i^2 = -1$.

4. Employ equation (c) of Problem 3 to find the initial vector velocity of the point P of Problem 1. *Ans.* $\mathbf{p}' = (9.42, -4.71, -9.42)$

45. Euler's Theorem; Poisson's Formula.

By the use of Rodrigues' formula we may now readily prove Euler's famous theorem on the displacements of a rigid body with one fixed point.

Euler's Theorem (1775). *Any displacement of a rigid body which leaves one of its points fixed may be produced by a rotation of the body through an angle of 180° or less about an axis passing through that point.*

Let the position of the body be determined by the position of the fixed point O and two others of its points P and Q such that O, P, Q are not collinear. Let the initial positions of these points be O, P_1, Q_1 and their terminal positions O, P_2, Q_2 and let us call,

$$OP_1 = \mathbf{p}_1, \qquad OQ_1 = \mathbf{q}_1, \qquad OP_2 = \mathbf{p}_2, \qquad OQ_2 = \mathbf{q}_2.$$

If then we can find a vector $\mathbf{w}$ satisfying the equations,

$$(1)\quad \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{w}\times(\mathbf{p}_2 + \mathbf{p}_1), \qquad \mathbf{q}_2 - \mathbf{q}_1 = \mathbf{w}\times(\mathbf{q}_2 + \mathbf{q}_1),$$

or a unit vector $\mathbf{u}$ satisfying the equations,

$$(2) \qquad \mathbf{p}_2 + \mathbf{p}_1 = 2\,(\mathbf{u}\cdot\mathbf{p}_1)\,\mathbf{u}, \qquad \mathbf{q}_2 + \mathbf{q}_1 = 2\,(\mathbf{u}\cdot\mathbf{q}_1)\,\mathbf{u},$$

then by Rodrigues' formula, § 44, (2) or § 44, (5), we shall have found a vector $\mathbf{w}$ or a unit vector $\mathbf{u}$ characterizing a rotation capable of producing the given displacement, and our theorem will be proven.

For brevity let us call,

$$\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{b}_1, \quad \mathbf{q}_2 - \mathbf{q}_1 = \mathbf{b}_2, \quad \mathbf{p}_2 + \mathbf{p}_1 = \mathbf{a}_1, \quad \mathbf{q}_2 + \mathbf{q}_1 = \mathbf{a}_2,$$

and write equations (1) as,

$$(1) \qquad \mathbf{b}_1 = \mathbf{w} \times \mathbf{a}_1, \qquad \mathbf{b}_2 = \mathbf{w} \times \mathbf{a}_2.$$

Since the points O, P, Q are not collinear we have,

$$(a) \qquad \mathbf{p}_1 \times \mathbf{q}_1 \neq 0,$$

while from properties of the rigid body proven in § 43 we may write,

$$\begin{aligned}
&(b) \quad \mathbf{a}_1\cdot\mathbf{b}_1 = \mathbf{p}_2^2 - \mathbf{p}_1^2 = 0, \qquad \mathbf{a}_2\cdot\mathbf{b}_2 = \mathbf{q}_2^2 - \mathbf{q}_1^2 = 0,\\
&(c) \quad \mathbf{a}_1\cdot\mathbf{b}_1 + \mathbf{a}_2\cdot\mathbf{b}_2 = 2\,(\mathbf{p}_2\cdot\mathbf{q}_2 - \mathbf{p}_1\cdot\mathbf{q}_1) = 0,\\
&(d) \quad (\mathbf{a}_2\times\mathbf{a}_1)\cdot(\mathbf{b}_2\times\mathbf{b}_1) = (\mathbf{a}_2\cdot\mathbf{b}_1)^2 = (\mathbf{a}_1\cdot\mathbf{b}_2)^2,\\
&(e) \quad (\mathbf{a}_2\times\mathbf{a}_1)\times(\mathbf{b}_2\times\mathbf{b}_1) = (\mathbf{a}_2\cdot\mathbf{b}_1)\,\mathbf{b}_2\times\mathbf{a}_1 + (\mathbf{a}_1\cdot\mathbf{b}_2)\,\mathbf{b}_1\times\mathbf{a}_2.
\end{aligned}$$

We first consider the case,

I. $\mathbf{a}_2 \times \mathbf{a}_1 = 0$. We shall here establish the existence of a unit vector $\mathbf{u}$ satisfying equations (2), the rotation being through 180° for this case only. There are essentially two possibilities to be discussed here,

$$\begin{aligned}
&\text{I, 1.} \quad \mathbf{a}_2 \times \mathbf{a}_1 = 0, \qquad \mathbf{p}_2 + \mathbf{p}_1 = \mathbf{q}_2 + \mathbf{q}_1 = 0,\\
&\text{I, 2.} \quad \mathbf{a}_2 \times \mathbf{a}_1 = 0, \qquad \mathbf{p}_2 + \mathbf{p}_1 \neq 0.
\end{aligned}$$

For Case I, 1 we choose for $\mathbf{u}$ the unit vector,

$$(3) \qquad \mathbf{u} = \frac{\mathbf{p}_1 \times \mathbf{q}_1}{|\mathbf{p}_1 \times \mathbf{q}_1|},$$

and have at once,

$$\begin{aligned}
\mathbf{p}_2 + \mathbf{p}_1 = 0 &= \frac{2\,[\mathbf{p}_1\,\mathbf{q}_1\,\mathbf{p}_1]}{(\mathbf{p}_1 \times \mathbf{q}_1)^2}\,\mathbf{p}_1 \times \mathbf{q}_1 = 2\,(\mathbf{u}\cdot\mathbf{p}_1)\,\mathbf{u},\\
+ \mathbf{q}_1 = 0 &\quad \frac{2\,[\mathbf{p}_1\,\mathbf{q}_1\,\mathbf{q}_1]}{(\mathbf{p}_1 \times \mathbf{q}_1)^2}\,\mathbf{p}_1 \times \mathbf{q}_1 = 2\,(\mathbf{u}\cdot\mathbf{q}_1)\,\mathbf{u},
\end{aligned}$$

thus verifying equations (2). For Case I, 2 we choose for $\mathbf{u}$ the unit vector,

$$\text{(4)} \qquad \mathbf{u} = \frac{\mathbf{p}_2 + \mathbf{p}_1}{|\mathbf{p}_2 + \mathbf{p}_1|},$$

and remembering that $\mathbf{p}_2^2 = \mathbf{p}_1^2$ we may write,

$$\mathbf{p}_2 + \mathbf{p}_1 = \frac{\mathbf{p}_2^2 + \mathbf{p}_1^2 + 2\,\mathbf{p}_2\cdot\mathbf{p}_1}{(\mathbf{p}_2 + \mathbf{p}_1)^2}(\mathbf{p}_2 + \mathbf{p}_1)$$
$$= \frac{2\,(\mathbf{p}_2 + \mathbf{p}_1)\cdot\mathbf{p}_1}{(\mathbf{p}_2 + \mathbf{p}_1)^2}(\mathbf{p}_2 + \mathbf{p}_1) = 2\,(\mathbf{u}\cdot\mathbf{p}_1)\,\mathbf{u},$$

thus verifying the first of equations (2). By hypothesis

$$0 = \mathbf{a}_2 \times \mathbf{a}_1 = (\mathbf{q}_2 + \mathbf{q}_1) \times (\mathbf{p}_2 + \mathbf{p}_1),$$

while by equation (d)

$$0 = \mathbf{a}_1\cdot\mathbf{b}_2 = (\mathbf{p}_2 + \mathbf{p}_1)\cdot(\mathbf{q}_2 - \mathbf{q}_1),$$

so that we may write,

$$0 = (\mathbf{p}_2 + \mathbf{p}_1) \times \{(\mathbf{q}_2 + \mathbf{q}_1) \times (\mathbf{p}_2 + \mathbf{p}_1)\}$$
$$= (\mathbf{p}_2 + \mathbf{p}_1)^2\,(\mathbf{q}_2 + \mathbf{q}_1) - (\mathbf{p}_2 + \mathbf{p}_1)\cdot(\mathbf{q}_2 + \mathbf{q}_1)\,(\mathbf{p}_2 + \mathbf{p}_1).$$

From these we have at once,

$$\mathbf{q}_2 + \mathbf{q}_1 = \frac{(\mathbf{p}_2 + \mathbf{p}_1)\cdot(\mathbf{q}_2 + \mathbf{q}_1)}{(\mathbf{p}_2 + \mathbf{p}_1)^2}(\mathbf{p}_2 + \mathbf{p}_1)$$
$$= \frac{2\,(\mathbf{p}_2 + \mathbf{p}_1)\cdot\mathbf{q}_1}{(\mathbf{p}_2 + \mathbf{p}_1)^2}(\mathbf{p}_2 + \mathbf{p}_1) = 2\,(\mathbf{u}\cdot\mathbf{q}_1)\,\mathbf{u},$$

which verifies the second of equations (2).

We next consider the Case,

II. $\mathbf{a}_2 \times \mathbf{a}_1 \neq 0$, in which we shall show that there exists a vector $\mathbf{w}$ satisfying equations (1). There are here two possibilities to be considered,

$$\begin{array}{lll} \text{II, 1.} & \mathbf{a}_2 \times \mathbf{a}_1 \neq 0, & \mathbf{b}_2 \times \mathbf{b}_1 = 0, \\ \text{II, 2.} & \mathbf{a}_2 \times \mathbf{a}_1 \neq 0, & \mathbf{b}_2 \times \mathbf{b}_1 \neq 0. \end{array}$$

For Case II, 1 we choose for $\mathbf{w}$ the vector,

$$\text{(5)} \qquad \mathbf{w} = \frac{[\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{b}_2]\,\mathbf{a}_1 + [\mathbf{a}_2\,\mathbf{a}_1\,\mathbf{b}_1]\,\mathbf{a}_2}{(\mathbf{a}_2 \times \mathbf{a}_1)^2}.$$

To verify the first of equations (1) we observe that under our

hypothesis $\mathbf{b}_2 \times \mathbf{b}_1 = 0$ we have by equation (d)

$$\mathbf{a}_1 \cdot \mathbf{b}_2 = \mathbf{a}_2 \cdot \mathbf{b}_1 = 0$$

and hence

$$\mathbf{b}_1 \times (\mathbf{a}_2 \times \mathbf{a}_1) = (\mathbf{a}_1 \cdot \mathbf{b}_1)\,\mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{b}_1)\,\mathbf{a}_1 = 0.$$

Consequently we may write,

$$\begin{aligned} 0 = (\mathbf{a}_2 \times \mathbf{a}_1) \times \{\mathbf{b}_1 \times (\mathbf{a}_2 \times \mathbf{a}_1)\} \\ = (\mathbf{a}_2 \times \mathbf{a}_1)^2\,\mathbf{b}_1 - [\mathbf{a}_2\,\mathbf{a}_1\,\mathbf{b}_1]\,\mathbf{a}_2 \times \mathbf{a}_1, \end{aligned}$$

from which we have at once,

$$\mathbf{b}_1 = \frac{[\mathbf{a}_2\,\mathbf{a}_1\,\mathbf{b}_1]\,\mathbf{a}_2 \times \mathbf{a}_1}{(\mathbf{a}_2 \times \mathbf{a}_1)^2} = \mathbf{w} \times \mathbf{a}_1,$$

thus verifying the first of equations (1). The verification of the second equation follows at once by a mere interchange of subscripts. In discussing the general Case, II, 2, we first observe that neither $\mathbf{a}_2 \cdot \mathbf{b}_1$ nor $\mathbf{a}_1 \cdot \mathbf{b}_2$ can be zero for by equations (d) and (e) we would then have $\mathbf{a}_2 \times \mathbf{a}_1 = 0$ or $\mathbf{b}_2 \times \mathbf{b}_1 = 0$, contrary to hypothesis. We choose for $\mathbf{w}$ the vector,

$$(6) \qquad \mathbf{w} \quad \frac{\mathbf{b}_2 \times \mathbf{b}_1}{\mathbf{a}_1 \cdot \mathbf{b}_2} = \frac{\mathbf{b}_1 \times \mathbf{b}_2}{\mathbf{a}_2 \cdot \mathbf{b}_1},$$

and equations (1) then become,

$$\mathbf{w} \times \mathbf{a}_1 = \frac{(\mathbf{b}_2 \times \mathbf{b}_1) \times \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{b}_2} = \frac{(\mathbf{a}_1 \cdot \mathbf{b}_2)\,\mathbf{b}_1 - (\mathbf{a}_1 \cdot \mathbf{b}_1)\,\mathbf{b}_2}{\mathbf{a}_1 \cdot \mathbf{b}_2} = \mathbf{b}_1,$$

$$\mathbf{w} \times \mathbf{a}_2 = \frac{(\mathbf{b}_1 \times \mathbf{b}_2) \times \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{b}_1} = \frac{(\mathbf{a}_2 \cdot \mathbf{b}_1)\,\mathbf{b}_2 - (\mathbf{a}_2 \cdot \mathbf{b}_2)\,\mathbf{b}_1}{\mathbf{a}_2 \cdot \mathbf{b}_1} = \mathbf{b}_2,$$

and are thus verified. Since the displacement of the body is completely determined by that of the points O, P, Q, it is clear that the vector $\mathbf{w}$ or $\mathbf{u}$ here given will satisfy equations (1) or (2), not only for the points O, P, Q, but for every other point of the rigid body. This completes the proof of Euler's theorem. The vector $\mathbf{w}$ is unique and $\mathbf{u}$ is unique except in sign.

By the use of Euler's theorem we may readily establish an important result known as *Poisson's Formula* concerned with the velocities of the points of a rigid body which moves with one point fixed. Let a rigid body move with one point O remaining fixed and let $OP = \mathbf{p}$ be the radius vector of any point P of the

body at the instant t and let $\mathbf{p} + \Delta\mathbf{p}$ be its radius vector at a subsequent instant $t + \Delta t$. Now it was shown in the proof of Euler's theorem that there exists a uniquely determined vector $\mathbf{w}$ such that Rodrigues' formula $\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{w} \times (\mathbf{p}_2 + \mathbf{p}_1)$, which in our case can be written,

$$\frac{\Delta\mathbf{p}}{\Delta t} = \frac{2\,\mathbf{w}}{\Delta t} \times \left(\mathbf{p} + \frac{\Delta\mathbf{p}}{2}\right),$$

holds for every point P of the body, it being here assumed that the time interval Δt has been taken so small as to avoid the case of a $180°$ rotation. Furthermore the above formulas for $\mathbf{w}$ show that $2\,\mathbf{w}/\Delta t$ is a continuous function of the values of $\Delta\mathbf{p}/\Delta t$ and $\mathbf{p} + \Delta\mathbf{p}/2$ for two points of the body and consequently if $\Delta\mathbf{p}/\Delta t$ and $\mathbf{p} + \Delta\mathbf{p}/2$ for these two points should approach limits, then $2\,\mathbf{w}/\Delta t$ would approach a limit also and these limits would satisfy the above equation. Thus if we allow Δt to approach zero we have on the assumption that $\Delta\mathbf{p}/\Delta t$ approaches a limit $\mathbf{p}'$, the relation,

$$\text{(7)} \qquad \mathbf{p}' = \omega \times \mathbf{p}, \qquad \text{where} \qquad \omega = \lim_{\Delta t \to 0} \frac{2\,\mathbf{w}}{\Delta t},$$

holding for the same value of ω for every point of the body at the instant considered. This equation is known as *Poisson's Formula* (1831) and gives us the velocity of every point of the body when the value of ω at the instant and the radius vector $\mathbf{p}$ of the point are known.

To get an explicit form for ω we let P and Q be any two points of the rigid body not collinear with O and let their radius vectors be $\mathbf{p}$ and $\mathbf{q}$ at the instant t and $\mathbf{p} + \Delta\mathbf{p}$, $\mathbf{q} + \Delta\mathbf{q}$ at the instant $t + \Delta t$. If these values be substituted into equations (5) and (6) we readily find on proceeding to the limit, allowing Δt to approach zero,

I. $\mathbf{p}' \times \mathbf{q}' = 0$

$$\text{(8)} \qquad \omega = \lim_{\Delta t \to 0} \frac{2\,\mathbf{w}}{\Delta t} = \frac{[\mathbf{p}\,\mathbf{q}\,\mathbf{q}']\,\mathbf{p} + [\mathbf{q}\,\mathbf{p}\,\mathbf{p}']\,\mathbf{q}}{(\mathbf{p} \times \mathbf{q})^2};$$

II. $\mathbf{p}' \times \mathbf{q}' \neq 0$

$$\text{(9)} \qquad \omega = \lim_{\Delta t \to 0} \frac{2\,\mathbf{w}}{\Delta t} = \frac{\mathbf{q}' \times \mathbf{p}'}{\mathbf{p} \cdot \mathbf{q}'} = \frac{\mathbf{p}' \times \mathbf{q}'}{\mathbf{q} \cdot \mathbf{p}'}.$$

To understand the mechanical significance of the vector $\boldsymbol{\omega}$ we consider the case of a rigid body rotating with a constant angular velocity θ' in the positive sense about a fixed axis passing through the point O in the direction and sense of a constant unit vector $\mathbf{u}$. Then in the time Δt the body will rotate through an angle $\theta' \Delta t$ and in the notation of Rodrigues' formula this rotation will be characterized by the vector,

$$\mathbf{w} = (\tan \theta' \Delta t/2) \mathbf{u}.$$

In this case the vector $\boldsymbol{\omega}$ of Poisson's formula will have the value,

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{2 \mathbf{w}}{\Delta t} = \mathbf{u} \lim_{\Delta t \to 0} \frac{2 \tan \theta' \Delta t/2}{\Delta t} = \theta' \mathbf{u}.$$

Thus in this case $\boldsymbol{\omega}$ has the direction and sense of the axis of rotation and a length equal to the angular velocity. Consequently Poisson's formula (7) which gives us the velocities of the points of a rigid body moving with one fixed point, enables us to state the theorem,

The velocities of the points of a rigid body with one fixed point are the same at any instant as they would be if the body were rotating in the positive sense about a fixed axis passing through the fixed point in the direction and sense of $\boldsymbol{\omega}$ *and with an angular velocity equal to the length of* $\boldsymbol{\omega}$.

As a consequence of this theorem we call $\boldsymbol{\omega}$ the *vector angular velocity* of the body and the axis through the fixed point in the direction and sense of $\boldsymbol{\omega}$ is known as the *instantaneous axis* of the body.

In reference to the last theorem it should perhaps be pointed out that it is only in respect to its velocities that the body behaves at each instant as if rotating about a fixed axis. The accelerations of the points of the body will in general be quite different than they would be under this rotation.

It was pointed out at the close of § 44 that Rodrigues' formula applies not only to the vector $\mathbf{p}$ running from the fixed point O to any point P of the rigid body, but is equally applicable to any vector $\mathbf{r}$ attached to the rotating rigid body. The same argument is immediately applicable to Poisson's formula (7) and shows us that we have,

$$\mathbf{r}' = \boldsymbol{\omega} \times \mathbf{r}, \tag{10}$$

where $\mathbf{r}$ is any vector attached to the rotating body. We can not, however in this case interpret $\mathbf{r}'$ as the velocity of the terminus of $\mathbf{r}$.

EXERCISES

1. A cube is given a displacement leaving two opposite vertices fixed and carrying each of the remaining vertices to a position originally occupied by another. Determine the vector $\mathbf{w}$ characterizing the equivalent rotation and thus show that the rotation is through 120°.

2. If the equivalent rotation in Problem 1 be carried out at a uniform rate in 1 second, find the scalar velocities of the moving vertices.
Ans. 1.7101

3. A rectangular parallelopiped of dimensions 2, 1, 1 is given a displacement leaving two opposite vertices fixed and carrying two other vertices into the positions originally occupied by two others. Determine the vector $\mathbf{w}$ characterizing the equivalent rotation and find the angle of this rotation. *Ans.* 101° 32′ 13″

4. If the equivalent rotation in Problem 3 be carried out at a uniform rate in 1 second, find the scalar velocities of the various moving vertices. *Ans.* 1.6177, 2.0463

5. The regular tetrahedron, $O \equiv (0, 0, 0)$, $P \equiv (1, 4, -1)$, $Q \equiv (0, 3, 3)$, $R \equiv (-3, 3, 0)$ is given a displacement leaving O fixed and carrying P to $(3, 3, 0)$ and Q to $(4, -1, -1)$. Determine the vector $\mathbf{w}$ characterizing the equivalent rotation and thus find the terminal position of R and the angle θ of the rotation.
Ans. $(3, 0, 3)$, $\theta = 123° 44′ 56″$

6. In Case II, 1 of the proof of Euler's theorem show that,

$$[\mathbf{p}_1\, \mathbf{q}_1\, \mathbf{w}] = [\mathbf{p}_2\, \mathbf{q}_2\, \mathbf{w}] = 0.$$

Hence show that in this case $\mathbf{w}$ may be written,

$$\mathbf{w} = \frac{\mathbf{q}_1 \cdot (\mathbf{q}_2 - \mathbf{q}_1)}{[\mathbf{p}_1\, \mathbf{q}_2\, \mathbf{q}_1]}\,\mathbf{p}_1 + \frac{\mathbf{p}_1 \cdot (\mathbf{p}_2 - \mathbf{p}_1)}{[\mathbf{q}_1\, \mathbf{p}_2\, \mathbf{p}_1]}\,\mathbf{q}_1.$$

7. If a rigid body with one fixed point moves so that the instantaneous axis remains fixed in the body, show that the axis also remains fixed in space, the motion being consequently a rotation about a fixed axis.

46. Rotation of the Reference Frame.

In developing Rodrigues' formula we considered the displacement of a particle attached to a rigid body which was given a certain rotation relative to the coördinate axes or *reference frame*, as it is often called. Without perhaps being conscious of it, we thought of ourselves while observing this displacement as being fixed relative to the reference frame because we said it is the reference frame that is standing still while the rigid body is moving. Clearly we might equally well have adopted another point

of view and thought of ourselves as attached to the rigid body so that it would seem to be standing still while the reference frame rotated.

To make this new aspect of the situation clear we had best think of two reference frames, a so-called fixed reference frame S_1 which was formerly the rigid body to which the particle P was attached and a displaced reference frame S_2 which takes the place of the old reference frame. Then if we know the coördinates of a point relative to S_1, we may seek to determine its coördinates relative to S_2 provided we have given in some way the displacement which would carry the reference frame from the position of S_1 into that of S_2. An important case of this is that in which S_1 and S_2 have a common origin. Then by Euler's theorem there must exist some rotation about an axis through the common origin which would carry S_1 into S_2 and we may employ the notation of Rodrigues' formula and characterize this rotation by the vector $\mathbf{w}$, or in case of a rotation through 180° by the unit vector $\mathbf{u}$, expressing $\mathbf{w}$ or $\mathbf{u}$ by its coördinates relative to S_1.

Thus if $\mathbf{p}_1$ be the value of the radius vector of any point P relative to the reference frame S_1 and if $\mathbf{p}_2$ be the value of the same radius vector relative to S_2 and if a rotation about the common origin characterized by a vector whose value relative to S_1 is $\mathbf{w}$ or $\mathbf{u}$ will carry S_1 into S_2, then $\mathbf{p}_1$ and $\mathbf{p}_2$ will in effect be related by Rodrigues' formula, § 44, (2) or § 44, (5). We must however make one slight modification due to the fact that, as explained above, we have changed our point of view as to which system is moving and which is standing still. Since a rotation of amount θ of S_1 relative to S_2 produces exactly the same relative displacement as a rotation about the same axis of amount $-\theta$ of S_2 relative to S_1, it is apparent that for our present purpose we must replace $\mathbf{w}$ by $-\mathbf{w}$ and $\mathbf{u}$ by $-\mathbf{u}$ in these equations. Thus we have for the transformation of coördinates due to a rotation of the reference frame about the origin,

$$(1) \qquad \mathbf{p}_1 - \mathbf{p}_2 = \mathbf{w} \times (\mathbf{p}_1 + \mathbf{p}_2),$$

$$(2) \qquad \mathbf{p}_1 + \mathbf{p}_2 = 2\,(\mathbf{u}\cdot\mathbf{p}_1)\,\mathbf{u},$$

where equation (1) may be solved for $\mathbf{p}_2$ by the equivalent form,

$$(3) \qquad \mathbf{p}_1 + \mathbf{p}_2 = \frac{2}{1+\mathbf{w}^2}\,\{\mathbf{p}_1 + (\mathbf{w}\cdot\mathbf{p}_1)\,\mathbf{w} - \mathbf{w}\times\mathbf{p}_1\}.$$

If the rotation of the reference frame is determined by giving the values with respect to both frames of the radius vectors of two points not collinear with the origin, then the formulas (3), (4), (5), (6) of the preceding section give the values of $\mathbf{w}$ or $\mathbf{u}$ for the rotation, if a change in the signs of $\mathbf{w}$ and $\mathbf{u}$ be introduced. Thus for the general case where $(\mathbf{p}_2 + \mathbf{p}_1) \times (\mathbf{q}_2 + \mathbf{q}_1) \neq 0$, $(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{q}_2 - \mathbf{q}_1) \neq 0$ we have,

$$(4) \qquad \mathbf{w} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{q}_2 - \mathbf{q}_1)}{(\mathbf{p}_2 + \mathbf{p}_1) \cdot (\mathbf{q}_2 - \mathbf{q}_1)} = \frac{(\mathbf{q}_2 - \mathbf{q}_1) \times (\mathbf{p}_2 - \mathbf{p}_1)}{(\mathbf{q}_2 + \mathbf{q}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1)}.$$

If we have the value $\mathbf{p}_1$ of the radius vector $\mathbf{p}$ relative to S_1 given for several points and we wish to determine the value $\mathbf{p}_2$ of $\mathbf{p}$ relative to S_2 for all of them for the same value of $\mathbf{w}$ or $\mathbf{u}$, we may conveniently proceed as follows. We first find the values of the three coördinate vectors $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ of the system S_1 relative to the system S_2 by formula (2) or (3), enabling us to write,

$$\begin{aligned} \mathbf{i}_1 &= \lambda_{11}\, \mathbf{i}_2 + \lambda_{12}\, \mathbf{j}_2 + \lambda_{13}\, \mathbf{k}_2, \\ \mathbf{j}_1 &= \lambda_{21}\, \mathbf{i}_2 + \lambda_{22}\, \mathbf{j}_2 + \lambda_{23}\, \mathbf{k}_2, \\ \mathbf{k}_1 &= \lambda_{31}\, \mathbf{i}_2 + \lambda_{32}\, \mathbf{j}_2 + \lambda_{33}\, \mathbf{k}_2, \end{aligned}$$

where $\mathbf{i}_2$, $\mathbf{j}_2$, $\mathbf{k}_2$ are the three coördinate vectors of system S_2. Then if $\mathbf{p} = x_1\, \mathbf{i}_1 + y_1\, \mathbf{j}_1 + z_1\, \mathbf{k}_1$ we have at once $\mathbf{p} = x_2\, \mathbf{i}_2 + y_2\, \mathbf{j}_2 + z_2\, \mathbf{k}_2$ where,

$$(5) \qquad \begin{aligned} x_2 &= \lambda_{11}\, x_1 + \lambda_{21}\, y_1 + \lambda_{31}\, z_1, \\ y_2 &= \lambda_{12}\, x_1 + \lambda_{22}\, y_1 + \lambda_{32}\, z_1, \\ z_2 &= \lambda_{13}\, x_1 + \lambda_{23}\, y_1 + \lambda_{33}\, z_1. \end{aligned}$$

This puts the transformation of coördinates into a form explicit in the coördinates and is very convenient in application. It will be clear that since $\mathbf{i}_1 \cdot \mathbf{j}_2 = \lambda_{12}$ it follows that λ_{12} is the cosine of the angle between the X-axis of S_1 and the Y-axis of S_2, with similar meanings for the other λ_{ij}.

Conversely, if the coördinate transformation (5) is given, we may easily show from equation (4) that for the general case,

$$(6) \qquad \mathbf{w} \equiv \left(\frac{\lambda_{13} + \lambda_{31}}{\lambda_{21} - \lambda_{12}},\ \frac{\lambda_{21} + \lambda_{12}}{\lambda_{32} - \lambda_{23}},\ \frac{\lambda_{32} + \lambda_{23}}{\lambda_{13} - \lambda_{31}} \right)$$

or the equivalent,

$$(6') \qquad \mathbf{w} \equiv \left(\frac{\lambda_{21} + \lambda_{12}}{\lambda_{13} - \lambda_{31}},\ \frac{\lambda_{32} + \lambda_{23}}{\lambda_{21} - \lambda_{12}},\ \frac{\lambda_{13} + \lambda_{31}}{\lambda_{32} - \lambda_{23}} \right).$$

Remembering that $|\mathbf{w}| = \tan \theta/2$, we have from this after some reduction,

$$\cos \theta = \tfrac{1}{2}(\lambda_{11} + \lambda_{22} + \lambda_{33} - 1). \tag{7}$$

for the determination of the angle of rotation θ.

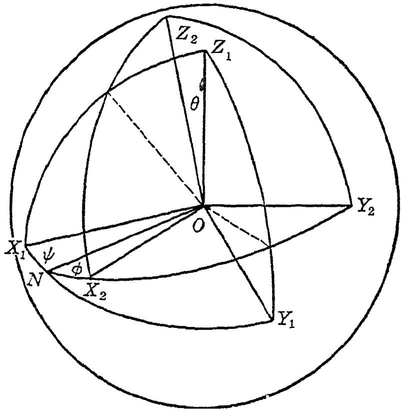

FIG. 47

The relative positions of two congruent rectangular reference frames S_1 and S_2 having a common origin O is often determined by *Euler's Angles* ψ, φ, θ which are defined as follows. Let X_1, Y_1, Z_1 be points at a unit's distance from O on the positive axes of S_1 and let X_2, Y_2, Z_2 be the same for S_2. Let N be a point on the line of intersection of the planes X_1OY_1 and X_2OY_2 at a unit's distance from O and on the side of O such that ON, OZ_1, OZ_2 have a positive sense of rotation. Then call the angles,

$$\begin{aligned} \psi &= X_1ON \text{ in the positive sense about } OZ_1, \\ \varphi &= NOX_2 \text{ in the positive sense about } OZ_2, \\ \theta &= Z_1OZ_2 \text{ in the positive sense about } ON. \end{aligned}$$

It will be observed that ψ and φ may vary from 0 to 2π but that N has been so chosen that θ varies only from 0 to π. An examination of certain spherical triangles involved in the figure will show that the cosines of the angles between the axes of S_1 and those of S_2 are expressed in terms of ψ, φ, θ as follows:

		X_1	Y_1	Z_1
	X_2	$\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta$	$\sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta$	$\sin\varphi\sin\theta$
(8)	Y_2	$-\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta$	$-\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta$	$\cos\varphi\sin\theta$
	Z_2	$\sin\psi\sin\theta$	$-\cos\psi\sin\theta$	$\cos\theta$

These cosines are of course also the coefficients λ_{ij} in the equations (5) of the transformation of coördinates.

By the aid of equation (6) we find after some trigonometric reduction the following coördinates for $\mathbf{w}$ in terms of ψ, φ, θ,

$$(9)\qquad \mathbf{w} \equiv \left(\frac{\cos\frac{\psi-\varphi}{2}\tan\frac{\theta}{2}}{\cos\frac{\psi+\varphi}{2}},\ \frac{\sin\frac{\psi-\varphi}{2}\tan\frac{\theta}{2}}{\cos\frac{\psi+\varphi}{2}},\ \frac{\sin\frac{\psi+\varphi}{2}}{\cos\frac{\psi}{2}} \right)$$

while the angle of rotation Θ, called θ in equation (7), is given by,

$$(10)\qquad \cos\frac{\Theta}{2} = \cos\frac{\psi+\varphi}{2}\cos\frac{\theta}{2}.$$

EXERCISES

1 Two coördinate systems S_1 and S_2 have the same sense of rotation and a common origin. The coördinates of a point P relative to the two systems are respectively $(3, 6, -3)$ and $(5, -5, 2)$ and those of Q are $(0, 3, -6)$ and $(0, -6, 3)$. About what axis and through what angle must S_1 be rotated to bring it into the position of S_2? Fill out the table of direction cosines for the transformation. *Ans.* $\mathbf{w} = (3, 1, 1)$, $\theta = 146° 26' 34''$

2. By a rotation of the axes the coördinates of a fixed point P are changed from (x_1, y_1, z_1) to (x_2, y_2, z_2) according to the following transformation. About what axis and through what angle were the coördinate axes rotated? Determine the new coördinates of the points P and Q whose original coördinates are $P_1 = (5, 0, 0)$, $Q_1 = (0, 4, -3)$.

$$\begin{aligned} 25x_2 &= -15x_1 + 16y_1 - 12z_1 \\ 25y_2 &= 20x_1 + 12y_1 - 9z_1 \\ 25z_2 &= - 15y_1 - 20z_1 \end{aligned}$$

Ans. $\mathbf{w} = (3, 6, -2)$, $\theta = 163° 44' 23''$

3. Find the least angle θ which the reference frames S_1 and S_2 may make with each other in order that the equations,

$$4x_1 + y_1 - z_1 - 3 = 0, \qquad x_2 - y_2 - 1 = 0$$

may be the equations of the same plane before and after a rotation of the reference frame. Find the value of $\mathbf{w}$ relative to S_1 and the equations of transformation. *Ans.* $\theta = 60°$, $\mathbf{w} = (1/9, 1/9, 5/9)$

4. Derive formulas (6), (6′), and (7) of this article.

5. Show that the coefficients λ_{ij} of the transformation (5) satisfy,

(*a*) six equations of the type, $\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1$,
(*b*) six equations of the type, $\lambda_{21}\lambda_{31} + \lambda_{22}\lambda_{32} + \lambda_{23}\lambda_{33} = 0$,
(*c*) three equations of the type, $\lambda_{21}^2 - \lambda_{12}^2 = \lambda_{13}^2 - \lambda_{31}^2$,
(*d*) nine equations of the type, $\lambda_{12} = \lambda_{31}\lambda_{23} - \lambda_{21}\lambda_{33}$,
(*e*) the determinant, $[\lambda_{ij}] = 1$.

47. Composition of Rotations and Angular Velocities.

If a rigid body with a fixed point O be given a rotation characterized in the notation of Rodrigues' formula by the vector $\mathbf{w}_1$ followed by a rotation characterized by $\mathbf{w}_2$, then by Euler's theorem there must exist a rotation carrying the body from its initial to its final position. Assuming that none of these rotations is through 180°, let us seek to determine the vector $\mathbf{w}_3$ characterizing the combined rotation. If we represent by $\mathbf{p}_1$ the radius vector OP of any point P of the rigid body in its initial position, by $\mathbf{p}_2$ its radius vector after the rotation $\mathbf{w}_1$ and by $\mathbf{p}_3$ its final radius vector, then by Rodrigues' formula we have,

$$(1) \qquad \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{w}_1 \times (\mathbf{p}_2 + \mathbf{p}_1),$$

$$(2) \qquad \mathbf{p}_3 - \mathbf{p}_2 = \mathbf{w}_2 \times (\mathbf{p}_3 + \mathbf{p}_2).$$

On multiplying the first of these equations through by $\mathbf{w}_1\cdot$ and the second by $\mathbf{w}_2\cdot$ and then multiplying the resulting scalars respectively by the vectors $\mathbf{w}_2$ and $\mathbf{w}_1$, we find,

$$0 = (\mathbf{w}_1\cdot\mathbf{p}_2 - \mathbf{w}_1\cdot\mathbf{p}_1)\,\mathbf{w}_2,$$
$$0 = (\mathbf{w}_2\cdot\mathbf{p}_3 - \mathbf{w}_2\cdot\mathbf{p}_2)\,\mathbf{w}_1.$$

Multiplying (1) by $\mathbf{w}_2 \times$ and (2) by $-\mathbf{w}_1 \times$ gives us,

$$\begin{aligned} \mathbf{w}_2 \times \mathbf{p}_2 - \mathbf{w}_2 \times \mathbf{p}_1 &= \mathbf{w}_2\cdot(\mathbf{p}_2 + \mathbf{p}_1)\,\mathbf{w}_1 - \mathbf{w}_2\cdot\mathbf{w}_1\,(\mathbf{p}_2 + \mathbf{p}_1), \\ \mathbf{w}_1 \times \mathbf{p}_1 - \mathbf{w}_1 \times \mathbf{p}_3 &= -\,\mathbf{w}_1\cdot(\mathbf{p}_3 + \mathbf{p}_2)\,\mathbf{w}_2 + \mathbf{w}_2\cdot\mathbf{w}_1\,(\mathbf{p}_3 + \mathbf{p}_2). \end{aligned}$$

We now add all six of the above equations member for member, obtaining,

$$(\mathbf{p}_3 - \mathbf{p}_1)(1 - \mathbf{w}_2\cdot\mathbf{w}_1) = (\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_2 \times \mathbf{w}_1) \times (\mathbf{p}_3 + \mathbf{p}_1).$$

If we call,

$$(3) \qquad \mathbf{w}_3 = \frac{\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_2 \times \mathbf{w}_1}{1 - \mathbf{w}_2\cdot\mathbf{w}_1}$$

this last equation becomes,

$$(4) \qquad \mathbf{p}_3 - \mathbf{p}_1 = \mathbf{w}_3 \times (\mathbf{p}_3 + \mathbf{p}_1),$$

which being Rodrigues' formula for the combined rotation shows us that this rotation is characterized by $\mathbf{w}_3$ as given above.

If now the rotations $\mathbf{w}_1$ and $\mathbf{w}_2$ take place in two time intervals both of length Δt then, as observed in § 45, the limits,

$$\lim_{\Delta t \to 0} \frac{2\,\mathbf{w}_1}{\Delta t} = \boldsymbol{\omega}_1, \qquad \lim_{\Delta t \to 0} \frac{2\,\mathbf{w}_2}{\Delta t} = \boldsymbol{\omega}_2,$$

will exist if we merely assume that the points of the body possess velocities at each instant. Consequently we may write,

$$(5)\qquad \omega_3 = \lim_{\Delta t \to 0} \frac{2\,\mathbf{w}_3}{\Delta t} = \lim_{\Delta t \to 0} \frac{\dfrac{2\,\mathbf{w}_1}{\Delta t} + \dfrac{2\,\mathbf{w}_2}{\Delta t} + \dfrac{\Delta t}{2}\dfrac{2\,\mathbf{w}_2}{\Delta t} \times \dfrac{2\,\mathbf{w}_1}{\Delta t}}{1 - \dfrac{(\Delta t)^2}{4}\dfrac{2\,\mathbf{w}_2}{\Delta t} \cdot \dfrac{2\,\mathbf{w}_1}{\Delta t}}$$
$$= \omega_1 + \omega_2.$$

Since ω_3 is thus independent of the order of the two infinitesimal rotations $\mathbf{w}_1$ and $\mathbf{w}_2$, it appears that they might be carried out simultaneously without affecting ω_3. We thus have the important,

Law of Composition of Vector Angular Velocities. If a rigid body with one fixed point be given simultaneous angular velocities ω_1 and ω_2, the resulting angular velocity is the vector sum, $\omega_3 = \omega_1 + \omega_2$.

This is a brief way of stating the fact that if a rigid body with one fixed point have a vector angular velocity ω_1 relative to a reference frame S_1 and if S_1 have a vector angular velocity ω_2 relative to another reference frame S_2, then the vector angular velocity of the body relative to S_2 is $\omega_3 = \omega_1 + \omega_2$.

This composition of angular velocities is well illustrated by the motion of a spinning top as it starts to run down. The top has a large angular velocity about a vertical axis and when it tips slightly to one side the force of gravity tends to give it a small angular velocity about a horizontal axis perpendicular to the vertical plane in which it is inclined. But the top does not fall over as soon as it takes this slight tip for the two vector angular velocities mentioned above combine by vector addition to form a vector angular velocity slightly inclined to the vertical and rotating about it. We say the top wobbles.

As an application of formula (3) we may find the derivative $\mathbf{w}'$ of the vector $\mathbf{w}$ when the rigid body with one fixed point has an angular velocity ω. We here let $\mathbf{w}$ characterize the rotation which would carry the body from its position at a fixed instant t_0 to its position at any instant t. The vector $\mathbf{w}$ is then a function of t. Let Δt be a small increment of time from the instant t and represent by $\mathbf{w} + \Delta\mathbf{w}$ the value of $\mathbf{w}$ at the instant $t + \Delta t$. Let us characterize by $\delta\mathbf{w}$ the rotation which would carry the body from its position at the instant t to its position at the instant $t + \Delta t$. Then the rotation $\mathbf{w} + \Delta\mathbf{w}$ is equivalent to the rotation $\mathbf{w}$ fol-

lowed by the rotation $\delta \mathrm{w}$ and by formula (3) we have,

$$\mathrm{w} + \Delta \mathrm{w} = \frac{\mathrm{w} + \delta \mathrm{w} + \delta \mathrm{w} \times \mathrm{w}}{1 - \delta \mathrm{w} \cdot \mathrm{w}},$$

and on subtracting w from both members we find,

$$\Delta \mathrm{w} = \frac{\delta \mathrm{w} + (\delta \mathrm{w} \cdot \mathrm{w})\, \mathrm{w} + \delta \mathrm{w} \times \mathrm{w}}{1 - \delta \mathrm{w} \cdot \mathrm{w}}.$$

Now by the definition of ω (§ 45, (7)) for the instant t,

$$\lim_{\Delta t \to 0} \frac{\delta \mathrm{w}}{\Delta t} = \frac{\omega}{2},$$

and hence if we divide the above expression for $\Delta \mathrm{w}$ by Δt and proceed to the limit allowing Δt to approach zero, we find the important relation,

$$\mathrm{w}' = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{w}}{\Delta t} = \tfrac{1}{2} \{\omega + (\omega \cdot \mathrm{w})\, \mathrm{w} + \omega \times \mathrm{w}\}. \tag{6}$$

EXERCISES

1. If a rigid body with one fixed point is given a rotation characterized by w_1 followed by a rotation through 180° characterized by the unit vector u_2, show that in the general case the resulting rotation will be characterized by,

$$\mathrm{w}_3 = \frac{\mathrm{u}_2 + \mathrm{u}_2 \times \mathrm{w}_1}{- \mathrm{u}_2 \cdot \mathrm{w}_1}.$$

2. If a rigid body with one fixed point is given a rotation through 180° characterized by the unit vector u_1 followed by a rotation through 180° characterized by the unit vector u_2, show that in the general case the resulting rotation will be characterized by,

$$\mathrm{w}_3 = \frac{\mathrm{u}_1 \times \mathrm{u}_2}{\mathrm{u}_1 \cdot \mathrm{u}_2}.$$

3. From equation (3) of this article derive the equation,

$$(1 - \mathrm{w}_2 \cdot \mathrm{w}_1)^2 (1 + \mathrm{w}_3^2) = (1 + \mathrm{w}_2^2)(1 + \mathrm{w}_1^2)$$

and thus prove,

Rodrigues' Theorem. Let A, B, C be a fixed spherical triangle on a sphere with center O. If a rigid body is rotated successively about the lines OA, OB, OC through the respective angles $2A$, $2B$, $2C$, it will return to its initial position.

4. Show that equation (3) of this article is equivalent to each of the equations,

$$\mathbf{w}_1 = \frac{\mathbf{w}_3 - \mathbf{w}_2 + \mathbf{w}_3 \times \mathbf{w}_2}{1 + \mathbf{w}_3 \cdot \mathbf{w}_2}, \qquad \mathbf{w}_2 = \frac{\mathbf{w}_3 - \mathbf{w}_1 + \mathbf{w}_1 \times \mathbf{w}_3}{1 + \mathbf{w}_1 \cdot \mathbf{w}_3}$$

Present both an analytic and a geometric argument.

5. If $\mathbf{w}$ characterizes the rotation carrying a rigid body with one fixed point from some given position to its position at the instant t, show that the angular velocity at the instant t is given by the formula,

$$\omega = \frac{2}{1 + \mathbf{w}^2} (\mathbf{w}' + \mathbf{w} \times \mathbf{w}').$$

6. The sphere $x^2 + y^2 + z^2 = 1$ has its center fixed and at each instant t has a displacement from a certain fixed position characterized by $\mathbf{w} \equiv (\cos t, \sin t, 0)$. Show that the angular velocity is of constant length and that the point which is at $(1, 0, 0)$ at the instant $t = 0$ moves on the intersection of the sphere with the cylinder, $x^2 + y^2 - x = 0$ and that its projection on the XY-plane traces out a circle with unit scalar velocity.

7. A rigid body is successively rotated in the positive sense about the X, Y and Z-axes through the respective angles 2α, 2β, 2γ. Calling,

$$\begin{aligned}
\kappa &= \cos\alpha \cos\beta \cos\gamma + \sin\alpha \sin\beta \sin\gamma \\
\lambda &= \sin\alpha \cos\beta \cos\gamma - \cos\alpha \sin\beta \sin\gamma \\
\mu &= \cos\alpha \sin\beta \cos\gamma + \sin\alpha \cos\beta \sin\gamma \\
\nu &= \cos\alpha \cos\beta \sin\gamma - \sin\alpha \sin\beta \cos\gamma,
\end{aligned}$$

show that the equivalent single rotation is characterized by the vector $\mathbf{w}$ where,

$$\kappa \mathbf{w} = \lambda \mathbf{i} + \mu \mathbf{j} + \nu \mathbf{k}.$$

Show that,

$$\kappa^2 + \lambda^2 + \mu^2 + \nu^2 = 1,$$

and that $\cos \theta/2 = \kappa$, where θ is the angle of the rotation.

8. Show that a rotation characterized by the vector $\mathbf{w}$ of a rigid body fixed at the origin can be resolved into successive rotations about the X, Y and Z-axes through the respective angles 2α, 2β, 2γ where,

$$\tan 2\alpha = \frac{2(\kappa\lambda + \mu\nu)}{-\lambda^2 - \mu^2 + \nu^2}, \qquad \sin 2\beta \quad \frac{2(\kappa\mu - \lambda\nu)}{\kappa^2 + \lambda^2 + \mu^2 + \nu^2},$$

$$\tan 2\gamma \quad \frac{2(\lambda\mu + \nu\kappa)}{\lambda^2 - \mu^2 - \nu^2 + \kappa^2},$$

and in which κ, λ, μ, ν are any four scalars, not all zero, such that,

$$\kappa \mathbf{w} = \lambda \mathbf{i} + \mu \mathbf{j} + \nu \mathbf{k}.$$

9. Resolve a rotation through 120° about the line $x = y = z$ into successive rotations about the X, Y, Z-axes.
Ans. 90° about the X-axis followed by 90° about the Z-axis or the reverse rotations in the reverse order.

10. Resolve a rotation through 90° about the Z-axis into two successive rotations of 180°, the first being about the X-axis.
Ans. Second is about the line, $x = y$, $z = 0$

11. If $\boldsymbol{\omega}$ is the vector angular velocity of the reference frame S_2 relative to the reference frame S_1, express the coördinates $(\omega_1, \omega_2, \omega_3)$ of $\boldsymbol{\omega}$ in S_1 and $(\omega_i, \omega_j, \omega_k)$ in S_2 in terms of Euler's angles ψ, φ, θ and their derivatives with respect to the time.
Ans.
$$\begin{aligned}
\omega_1 &= \sin\psi \sin\theta\ \varphi' + \cos\psi\ \theta', \\
\omega_2 &= -\cos\psi \sin\theta\ \varphi' + \sin\psi\ \theta', \\
\omega_3 &= \cos\theta\ \varphi' + \psi', \\
\omega_i &= \sin\varphi \sin\theta\ \psi' + \cos\varphi\ \theta', \\
\omega_j &= \cos\varphi \sin\theta\ \psi' - \sin\varphi\ \theta', \\
\omega_k &= \cos\theta\ \psi' + \varphi'
\end{aligned}$$

48. Chasles' Theorem; Mozzi's Theorem.

If all the points of a rigid body are displaced the same distance in the same direction and sense we say that the displacement of the body is a *translation.* If we represent the displacement by a vector $\mathbf{k}$ having this direction, sense and length then we may write,

$$(1) \qquad \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{k},$$

where $\mathbf{p}_2$ and $\mathbf{p}_1$ are the values of the radius vector OP from any fixed point O to any point P of the body before and after the translation. By Euler's theorem any displacement of a rigid body leaving one point fixed may be produced by a rotation of the body about some axis through that point and so we may properly call any displacement of a body which leaves one point fixed a *rotation.*

It is now easy to see that any displacement of a rigid body may be produced by a properly chosen translation followed by a properly chosen rotation, and in fact this can usually be done in an infinite number of ways. For, suppose a given displacement to carry a point A of the body from the position A_1 to the position A_2, the radius vectors of these positions from a fixed point O being $\mathbf{a}_1$ and $\mathbf{a}_2$. If we take the body in its initial position and perform upon it a translation for which $\mathbf{k} = \mathbf{a}_2 - \mathbf{a}_1$, we shall carry A into its final position A_2. Then the displacement carry-

ing the body from this new position to the final position of the given displacement will be a displacement with A_2 for a fixed point and will thus be a rotation. Our statement is thus proven. Since the point A used above might be any point of the body it is clear that a given displacement may usually be resolved into a translation followed by a rotation in infinitely many ways.

Furthermore we may show that it is possible to choose the above point A so that the displacement $\mathbf{k}$ will be parallel to the axis of the succeeding rotation. This fact constitutes,

Chasles' Theorem (1830). *Any displacement of a rigid body may be brought about by a translation in a certain direction combined with a rotation about an axis running in that direction, it being understood that either of these may be missing.*

Let A and P be any two points of the rigid body with radius vectors $OA = \mathbf{a}$ and $OP = \mathbf{p}$ from some fixed point O. Then $AP = \mathbf{p} - \mathbf{a}$ is a vector fixed in the rigid body and we may therefore apply to it exactly the same analytic argument applied in § 45 to the vector $\mathbf{p}$ to show that there always exists a vector $\mathbf{w}$ such that for all choices of A and P,

$$(2)\quad (\mathbf{p}_2 - \mathbf{a}_2) - (\mathbf{p}_1 - \mathbf{a}_1) = \mathbf{w} \times \{(\mathbf{p}_2 - \mathbf{a}_2) + (\mathbf{p}_1 - \mathbf{a}_1)\}$$

or a unit vector $\mathbf{u}$ such that,

$$(3)\qquad (\mathbf{p}_2 - \mathbf{p}_1) + (\mathbf{p}_1 - \mathbf{a}_1) = 2\,\mathbf{u}\cdot(\mathbf{p}_1 - \mathbf{a}_1)\,\mathbf{u},$$

where $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{p}_1$, $\mathbf{p}_2$ are the values of $\mathbf{a}$ and $\mathbf{p}$ before and after the given displacement. The explicit forms for $\mathbf{w}$ or $\mathbf{u}$ are of course obtained by replacing in equations (3), (4), (5), (6) of § 45 $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$ respectively by $\mathbf{p}_1 - \mathbf{a}_1$, $\mathbf{p}_2 - \mathbf{a}_2$, $\mathbf{q}_1 - \mathbf{a}_1$, $\mathbf{q}_2 - \mathbf{a}_2$, where A, P, Q are any three non-collinear points of the body.

In particular, if the above vector $\mathbf{w}$ has the value zero, then if A be chosen and fixed in the body we have,

$$\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{a}_2 - \mathbf{a}_1 = \mathbf{k}$$

for every choice of P. On comparison with equation (1) we see that the given displacement is in this case a translation and Chasles' theorem is proven for this case, the rotation reducing to zero. If $\mathbf{w}$ exists but is not zero, we pick a particular point P so that $\mathbf{p}_1$ and $\mathbf{p}_2$ are known quantities and then choose for A that

point of the body whose initial position A_1 is given by,

$$\text{(4)} \qquad \mathbf{a}_1 = \frac{1}{2}\left\{ \mathbf{p}_2 + \mathbf{p}_1 + \frac{\mathbf{w}\times(\mathbf{p}_2 - \mathbf{p}_1)}{\mathbf{w}^2} \right\}.$$

Following the method used in deriving equation (4) from equation (2) in § 44, we obtain from equation (2) of this article,

$$(\mathbf{p}_2 - \mathbf{a}_2) + (\mathbf{p}_1 - \mathbf{a}_1)$$
$$= \frac{2}{1+\mathbf{w}^2}\{\mathbf{p}_1 - \mathbf{a}_1 + \mathbf{w}\cdot(\mathbf{p}_1 - \mathbf{a}_1)\,\mathbf{w} + \mathbf{w}\times(\mathbf{p}_1 - \mathbf{a}_1)\},$$

which for our special choice of $\mathbf{a}_1$ yields,

$$\mathbf{a}_2 - \mathbf{a}_1 = \frac{\mathbf{w}\cdot(\mathbf{p}_2 - \mathbf{p}_1)}{\mathbf{w}^2}\,\mathbf{w} = \frac{\mathbf{w}\cdot(\mathbf{a}_2 - \mathbf{a}_1)}{\mathbf{w}^2}\,\mathbf{w} = k\,\mathbf{w}.$$

The point A being chosen, the scalar k is a constant independent of the subsequent choice of P and we may write equation (2) in the form,

$$\text{(5)} \qquad \mathbf{p}_2 - \mathbf{p}_1 = k\,\mathbf{w} + \mathbf{w}\times\{(\mathbf{p}_2 + \mathbf{p}_1) - (\mathbf{a}_2 + \mathbf{a}_1)\},$$

holding with this choice of A for every point P of the rigid body. If we write this in the form,

$$\text{(5')} \qquad (\mathbf{p}_2 - \mathbf{a}_0) - (\mathbf{p}_1 + k\,\mathbf{w} - \mathbf{a}_0)$$
$$= \mathbf{w}\times\{(\mathbf{p}_2 - \mathbf{a}_0) + (\mathbf{p}_1 + k\,\mathbf{w} - \mathbf{a}_0)\},$$

where $\mathbf{a}_0 = (\mathbf{a}_2 + \mathbf{a}_1)/2$ and compare with the equation of translation (1) and with Rodrigues' equation of rotation, § 44, (2), we see that we have a statement that the given displacement may be effected by a translation $k\,\mathbf{w}$ followed by a rotation $\mathbf{w}$ about an axis passing through A_0, where $OA_0 = \mathbf{a}_0$. Or again, if we write this equation in the form,

$$\text{(5'')} \qquad (\mathbf{p}_2 - \mathbf{a}_0 - k\,\mathbf{w}) - (\mathbf{p}_1 - \mathbf{a}_0)$$
$$= \mathbf{w}\times\{(\mathbf{p}_2 - \mathbf{a}_0 - k\,\mathbf{w}) + (\mathbf{p}_1 - \mathbf{a}_0)\},$$

we have a statement that the given displacement may be effected by the above rotation followed by the above translation.

If the vector $\mathbf{w}$ satisfying equation (2) does not exist, then there exists a unit vector $\mathbf{u}$ satisfying equation (3). We pick a particular point P so that $\mathbf{p}_1$ and $\mathbf{p}_2$ are known quantities and then choose for A that point of the body whose initial position A_1 is given by $\mathbf{a}_1 = (\mathbf{p}_2 + \mathbf{p}_1)/2$. With this choice of A equation (3)

yields,

$$\mathbf{a}_2 - \mathbf{a}_1 = \mathbf{u}\cdot(\mathbf{p}_2 - \mathbf{p}_1)\,\mathbf{u} = \mathbf{u}\cdot(\mathbf{a}_2 - \mathbf{a}_1)\,\mathbf{u} = k\,\mathbf{u}.$$

The point A being chosen, the scalar k is a constant independent of the subsequent choice of P and we may write equation (3) in the form,

$$(\mathbf{p}_2 - \mathbf{a}_0) + (\mathbf{p}_1 + k\,\mathbf{u} - \mathbf{a}_0) = 2\{\mathbf{u}\cdot(\mathbf{p}_1 + k\,\mathbf{u} - \mathbf{a}_0)\}\,\mathbf{u},$$

where $\mathbf{a}_0 = (\mathbf{a}_2 + \mathbf{a}_1)/2$ and which holds for every point P of the rigid body. If we compare this with the equation of translation (1) and with Rodrigues' equation of rotation § 44, (5), we see that we have a statement that the given displacement may be effected by a translation $k\,\mathbf{u}$ followed by a 180° rotation $\mathbf{u}$ about an axis passing through A_0, where $OA_0 = \mathbf{a}_0$. If we write this equation in the form,

$$(\mathbf{p}_2 - \mathbf{a}_0 - k\,\mathbf{u}) + (\mathbf{p}_1 - \mathbf{a}_0) = 2\,\{\mathbf{u}\cdot(\mathbf{p}_1 - \mathbf{a}_0)\}\,\mathbf{u},$$

we have a statement that the given displacement may be effected by the above rotation followed by the above translation. This completes the proof of Chasles' theorem.

A displacement of the above type, i.e. a rotation about a certain axis combined with a translation along that axis, is known as a *twist*. Chasles' theorem states in effect that every displacement of a rigid body may be brought about by a twist, in which however either the translation or the rotation may reduce to zero. A twist is also sometimes known as a *screw* as it is exactly the displacement of a machine screw turned into a fixed threaded hole.

The same limiting process which led us from Rodrigues' formula for finite rotations of a rigid body to Poisson's formula for the continuous motion of a body with one fixed point now enables us to derive from Chasles' theorem on displacements the analogous theorem of Mozzi on any continuous motion of a body. Let A and P be any two points of a rigid body which at the instant t have radius vectors $OA = \mathbf{a}$, $OP = \mathbf{p}$ from a fixed point O and which at the instant $t + \Delta t$ have radius vectors $\mathbf{a} + \Delta\mathbf{a}$ and $\mathbf{p} + \Delta\mathbf{p}$. With this notation equation (2) states that there exists a vector $\mathbf{w}$ such that for all choices of A and P we have,

$$\frac{\Delta\mathbf{p}}{\Delta t} - \frac{\Delta\mathbf{a}}{\Delta t} = \frac{2\mathbf{w}}{\Delta t} \times \left(\mathbf{p} - \mathbf{a} + \frac{\Delta\mathbf{p}}{2} - \frac{\Delta\mathbf{a}}{2}\right),$$

it being assumed that the time interval Δt has been taken so small as to avoid the case of 180° rotation. If in this equation we proceed to the limit allowing Δt to approach zero, we see as in § 45 that on the assumption of the existence of the limits,

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{p}}{\Delta t} = \mathbf{p}', \qquad \lim_{\Delta t \to 0} \frac{\Delta \mathbf{a}}{\Delta t} = \mathbf{a}',$$

we have the following

Fundamental Equation of the Kinematics of a Rigid Body,

$$(6) \qquad \mathbf{p}' = \mathbf{a}' + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a}), \qquad \text{where} \qquad \boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{2\,\mathbf{w}}{\Delta t}.$$

The explicit form for $\boldsymbol{\omega}$ is obtained by replacing in equation § 45, (8) or (9) $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q}$, $\mathbf{q}'$ respectively by $\mathbf{p} - \mathbf{a}$, $\mathbf{p}' - \mathbf{a}'$, $\mathbf{q} - \mathbf{a}$, $\mathbf{q}' - \mathbf{a}'$, where A, P, Q are any three non-collinear points of the body.

For $\boldsymbol{\omega} = 0$ we have $\mathbf{p}' = \mathbf{a}'$ and all points of the body have the same velocity at the instant considered. For $\boldsymbol{\omega} \neq 0$ we may transform equation (6) as follows. Make some particular choice of P so that $\mathbf{p}$ and $\mathbf{p}'$ are known quantities and then choose the point A so that,

$$OA = \mathbf{a} = \mathbf{p} + \frac{\boldsymbol{\omega} \times \mathbf{p}'}{\omega^2}.$$

Equation (6) gives for this choice of A,

$$\mathbf{a}' = \frac{\boldsymbol{\omega} \cdot \mathbf{p}'}{\omega^2}\,\boldsymbol{\omega} = \frac{\boldsymbol{\omega} \cdot \mathbf{a}'}{\omega^2}\,\boldsymbol{\omega} = k\,\boldsymbol{\omega}.$$

A being now fixed, the scalar k is unaffected by the subsequent choice of P and equation (6) takes the form,

$$(7) \qquad \mathbf{p}' = k\,\boldsymbol{\omega} + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a}),$$

holding for every point P of the body.

From equations (6) and (7) we may readily derive,

Mozzi's Theorem (1763). *The velocities of the points of a rigid body at any instant are what they would be if the body were rotating about a certain fixed axis and simultaneously had a motion of*

translation along this axis, it being understood that either or both of these may be missing.

By a *motion of translation* of a rigid body is meant a motion such that every resulting displacement is a translation, i.e. a motion in which at each instant all the points of the body have the same velocity. The proof of the theorem involves three cases.

I. $\boldsymbol{\omega} = 0$. In this case by equation (6) $\mathbf{p}' = \mathbf{a}'$ and hence all points of the body have the velocity $\mathbf{a}'$ at the instant considered. This velocity could be imparted to all points of the body by a properly chosen motion of translation and Mozzi's theorem follows for this case, the rotation being missing.

II. $\boldsymbol{\omega} \neq 0$, $\boldsymbol{\omega} \cdot \mathbf{a}' = 0$. In this case $k = 0$ and equation (7) reduces to $\mathbf{p}' = \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a})$. On comparison with equation § 45, (7), we see that these velocities could be imparted to the points of the body by rotating it about an axis through A with a vector angular velocity $\boldsymbol{\omega}$. Mozzi's theorem is thus proven for this case, the motion of translation being missing. Since in this case we have $\boldsymbol{\omega} \cdot \mathbf{p}' = 0$, we may conclude that if the velocity of one point of a rigid body is perpendicular to the vector angular velocity, then the velocities of all the points of the body are also.

III. $\boldsymbol{\omega} \cdot \mathbf{a}' \neq 0$. In this general case we see from equation (7) that if we were to give the body a motion of translation which would give all the points the same velocity $k\,\boldsymbol{\omega}$ and were to combine this with a rotation about an axis through A with vector angular velocity $\boldsymbol{\omega}$, then the points would have their actual velocities. This completes the proof of Mozzi's theorem. The axis through A in the direction and sense of $\boldsymbol{\omega}$ is known as the *instantaneous axis* or *axis of Mozzi.*

We may define a *helicoidal motion* of a rigid body as a rotation of the body about a fixed axis combined with a motion of translation along that axis, the velocity of translation being constantly proportional to the velocity of rotation. In such a motion all the points of the body describe coaxial helices. The theorem of Mozzi may be expressed by the statement,

At each instant in the motion of a rigid body there is a tangent helicoidal motion.

That is, there exists a helicoidal motion which would impart to all points of the body the velocities, but usually not the accelerations, which they actually have at the instant.

EXERCISES

1. For a moving unit cube $A, B, C, D, A_1, B_1, C_1, D_1$ the velocity of the point A is the vector AB, the velocity of C_1 lies along $C_1 C$ and the velocity of D_1 is in the plane $A_1 BCD_1$. Find the instantaneous axis and the velocities of rotation and translation.

   Ans. Pure motion of rotation about $A_1 D_1$

2. The sphere $x^2 + y^2 + z^2 = 9$ moves as a rigid body, the points P, Q, R on its surface having positions and velocities at a certain moment as follows:

$$\begin{array}{lll} \mathbf{p} = (2, -2, -1) & \mathbf{q} = (2, 1, -2) & \mathbf{r} = (0, 0, 3) \\ \mathbf{p}' = (0, 4, -8) & \mathbf{q}' = (-5, 6, -2) & \mathbf{r}' = (6, -6, 0) \end{array}$$

   Determine the instantaneous axis and the vector angular velocity of the sphere and the velocity of translation along the axis. If the sphere continues to rotate about this axis with an angular velocity increasing uniformly $2\pi - 6$ units per second, what will be the position and velocity of the point P one second after the given moment?

   Ans. Rotation about axis through origin with $\boldsymbol{\omega} = (2, 2, 1)$
   $\mathbf{p}_2 = (-22/9, 14/9, 7/9)$

3. Show that the lines, $(\mathbf{p} - \mathbf{p}_0) \times \mathbf{m} = 0$ of a rigid body which are at a given instant perpendicular to the velocities of their points satisfy the equation,

$$(\mathbf{a}' - \boldsymbol{\omega} \times \mathbf{a}) \cdot \mathbf{m} + \boldsymbol{\omega} \cdot \mathbf{n} = 0 \qquad \text{where} \qquad \mathbf{n} = \mathbf{p}_0 \times \mathbf{m}.$$

   The lines satisfying an equation which is linear and homogeneous in the six coördinates of the two vectors $\mathbf{m}$, $\mathbf{n}$ are said to form a *linear complex.* Show that if the rigid body has at the instant a pure motion of translation, the above complex consists of all lines perpendicular to the direction of the motion, while if the body has a pure motion of rotation the complex consists of all lines coplanar with the instantaneous axis. Show that the lines intersecting the axis at right angles always belong to the complex.

4. Show that the planes, each of which is perpendicular to the velocity of its point of intersection with a given line of a rigid body, have a line in common or are parallel. Where is this common line if the body has a pure motion of rotation?

5. Show that each plane of a rigid body has a point whose velocity is perpendicular to the plane, provided that $\boldsymbol{\omega} \cdot \mathbf{u} \neq 0$, where $\mathbf{u}$ is a unit vector perpendicular to the plane.

6. Show that each plane of a rigid body has a line of points whose velocities are parallel to the plane, provided $\boldsymbol{\omega} \times \mathbf{u} \neq 0$, where $\mathbf{u}$ is a unit vector perpendicular to the plane.

7. A rigid body slides along a curve, $\mathbf{p} = \mathbf{p}(t)$ rigidly attached to the trihedral $\mathbf{t}, \mathbf{n}, \mathbf{b}$ at the point on the curve (§ 32). Show that the vector angular velocity of the body is,

$$\omega = \frac{s'}{\rho}\mathbf{b} - \frac{s'}{\tau}\mathbf{t},$$

and that the instantaneous axis intersects the principal normal at right angles at a distance,

$$\frac{\rho}{1 + \rho^2/\tau^2}$$

from the point on the curve.

8. Show that any displacement of a rigid body can be brought about by two successive 180° rotations about properly chosen axes.

9. Show that if any three positions S_1, S_2, S_3 of a rigid body be given, then there always exists a fourth position S such that the body may be brought from S to each of the three given positions by a 180° rotation.

10. Prove *Halphen's Theorem* (1882). Let l_1, l_2, l_3 be any three lines and r_1 the common perpendicular of lines l_2 and l_3, and r_2 the common perpendicular of l_3 and l_1, and r_3 the common perpendicular of l_1 and l_2. Then if a rigid body be given a twist T_1 about l_1 consisting of a translation along l_1 of twice the distance from r_2 to r_3 and a rotation about l_1 through twice the angle from r_2 to r_3; followed by corresponding twists T_2 about l_2 and T_3 about l_3, the body will return to its original position. (See Problems 8 and 9.)

11. Prove *Poncelet's Construction for the Instantaneous Axis*. From any point O draw the vectors OP_1', OP_2', OP_3' equal respectively to the velocities $\mathbf{p}_1'$, $\mathbf{p}_2'$, $\mathbf{p}_3'$ of three points P_1, P_2, P_3 of the rigid body. Let M_1 and M_2 be the feet of the perpendiculars let fall from P_1 and P_2 on the plane P_1', P_2', P_3' and let M_1N and M_2N be drawn in this plane perpendicular respectively to $\mathbf{p}_1'$ and $\mathbf{p}_2'$. Then the instantaneous axis is the perpendicular to this plane at N.

49. Accelerations in the Rigid Body.

If A and P are any two points of a rigid body with radius vectors $\mathbf{a}$ and $\mathbf{p}$ from some fixed point O, and if we call $\mathbf{p} - \mathbf{a} = \mathbf{r}$, equation (6) of the last article becomes,

$$\text{(1)} \qquad \mathbf{p}' = \mathbf{a}' + \omega \times \mathbf{r}.$$

Differentiating this with respect to the time gives us for the acceleration $\mathbf{p}''$ of P,

$$\mathbf{p}'' = \mathbf{a}'' + \omega' \times \mathbf{r} + \omega \times \mathbf{r}',$$

or, since by equation (1) $\mathbf{r}' = \mathbf{p}' - \mathbf{a}' = \omega \times \mathbf{r}$, we have,

$$(2) \quad \mathbf{p}'' = \mathbf{a}'' + \omega' \times \mathbf{r} + \omega \times (\omega \times \mathbf{r})$$
$$= \mathbf{a}'' + \omega' \times \mathbf{r} + (\omega \cdot \mathbf{r})\,\omega - \omega^2\,\mathbf{r}.$$

This formula is somewhat complicated to visualize in this general form but it yields results of interest in certain special cases.

I. $\omega = 0$. In this case it appears from equation (1) that all points of the body have the same velocity $\mathbf{a}'$ and we may say that as far as velocities are concerned the body is momentarily in a motion of translation. However equation (2) then becomes,

$$\mathbf{p}'' = \mathbf{a}'' + \omega' \times \mathbf{r},$$

and it is evident that only when $\omega' = 0$ will the different points of the body all have the same acceleration.

I, *a*. $\omega = \omega' = 0$. Here equations (1) and (2) show that all points of the body have the same velocity and the same acceleration. This of course occurs in the case of a permanent motion of translation but it may also occur in other cases among which is that in which the body, while performing a helicoidal motion, slows down to instantaneous rest and immediately resumes motion about the same axis in the same sense as formerly. If ω and ω' exist at the instant of rest, they will both be zero and all points of the body will have zero acceleration.

II. $\omega \neq 0$. In this case there exists Mozzi's instantaneous axis of rotation and translation passing through a properly chosen point A in the direction and sense of ω. The term $\omega \times (\omega \times \mathbf{r})$ occurring as one component of $\mathbf{p}''$ in equation (2) is known as the *centripetal acceleration* $\mathbf{j}_0$ since it runs along the perpendicular from P to the instantaneous axis. In fact if we represent by $\mathbf{d}$ the vector running from P perpendicularly to the instantaneous axis, then we see by reference to § 22, Prob. 6 that $\mathbf{d} = \omega \times (\omega \times \mathbf{r})/\omega^2$ and consequently,

$$\mathbf{j}_0 = \omega \times (\omega \times \mathbf{r}) = \omega^2\,\mathbf{d}.$$

II, *a*. $\omega \neq 0$, $\mathbf{a}'' = 0$. This case arises when the body has a point A which is at rest or moving with uniform rectilinear motion at the instant considered. The acceleration of each point of the body then reduces to the centripetal acceleration together with the term $\omega' \times \mathbf{r}$ perpendicular to the plane of AP and ω'.

II, *b*. $\boldsymbol{\omega} \neq 0$, $\boldsymbol{\omega} \times \boldsymbol{\omega}' = 0$. In this case and in this case only there exists a helicoidal motion which would give to all points of the body the accelerations, but generally not the velocities, which they actually possess at the instant considered. For in a permanent helicoidal motion the velocity $\mathbf{p}'$ of each point would consist of the velocity $\theta' \mathbf{u} \times \mathbf{r}$ due to rotation with a variable angular velocity θ' about an axis through a fixed point A in the direction and sense of the constant unit vector $\mathbf{u}$, combined with a proportional velocity of translation $k\theta' \mathbf{u}$ in the direction of $\mathbf{u}$, k being a constant. Every helicoidal motion would thus have the velocities of the points of the body given by the formula,

$$\mathbf{p}' = k\theta' \mathbf{u} + \theta' \mathbf{u} \times \mathbf{r},$$

and consequently the acceleration given by,

$$\mathbf{p}'' = k\theta'' \mathbf{u} + \theta'' \mathbf{u} \times \mathbf{r} + \theta' \mathbf{u} \times \mathbf{r}'.$$

In choosing a helicoidal motion we have $\mathbf{u}$, θ', θ'', k, A at our disposal and consequently for this formula to give the value of the actual acceleration,

$$\mathbf{p}'' = \mathbf{a}'' + \boldsymbol{\omega}' \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}'$$

for all points of the body at a given instant it will be necessary and sufficient that $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ be parallel and that we be able to find a point A with acceleration $\mathbf{a}''$ parallel to $\boldsymbol{\omega}$. By hypothesis $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are parallel in this case and we shall show that we may find the point A. Let a particular point P of the body be chosen so that $\mathbf{p}$ and $\mathbf{p}''$ are known quantities. We then choose A so that,

$$AP = \mathbf{r} = -\frac{\omega^2 \mathbf{p}'' + \boldsymbol{\omega}' \times \mathbf{p}''}{\omega^4 + \omega'^2}.$$

On substituting this value for $\mathbf{r}$ in equation (2), we find after some reduction, bearing in mind that $\boldsymbol{\omega} \times \boldsymbol{\omega}' = 0$,

$$\mathbf{a}'' = \frac{\boldsymbol{\omega} \cdot \mathbf{p}''}{\omega^2} \boldsymbol{\omega}.$$

The point A so chosen thus has an acceleration parallel to $\boldsymbol{\omega}$ and the theorem is proven. This theorem on accelerations is the analog of Mozzi's theorem on velocities but, unlike Mozzi's theorem, it is applicable only under the condition $\boldsymbol{\omega} \times \boldsymbol{\omega}' = 0$.

The above case occurs whenever $\boldsymbol{\omega}$ is momentarily fixed in direction. In particular it occurs whenever $\boldsymbol{\omega}$ is permanently fixed in direction. Two cases of the latter situation are permanent helicoidal motion and plane motion which is not a translation. A rigid body has a *plane motion* when a plane fixed in the body moves in a plane fixed in space. We shall consider such motion in detail in § 52.

EXERCISES

1. If a rigid body moves with one point O fixed, show that the tangential component of the acceleration of each point P is p times the measure of $\boldsymbol{\omega}'$ on the normal at P to the cone traced by p, where $p = OP$ and $\boldsymbol{\omega}$ is the vector angular velocity.

2. If a rigid body moves with one point O fixed, discuss those lines through O on which the acceleration of the points lies along the line. Setting $\mathbf{p}'' = \lambda\,\mathbf{p}$, show that λ must satisfy the cubic,

$$\lambda^3 + 2\omega^2\lambda^2 + (\omega^4 + \omega'^2)\,\lambda + (\boldsymbol{\omega}\times\boldsymbol{\omega}')^2 = 0$$

and that for each root of this cubic the corresponding line is along

$$\mathbf{p} = (\omega^2 + \lambda)^2\,\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\omega}')\,\boldsymbol{\omega}' - (\omega^2 + \lambda)\,\boldsymbol{\omega}\times\boldsymbol{\omega}'.$$

Show that the plane through each such line parallel to $\boldsymbol{\omega}$ is perpendicular to the plane through this line parallel to $\boldsymbol{\omega}'$. Discuss in particular the case $\boldsymbol{\omega}\cdot\boldsymbol{\omega}' = 0$, showing that the line through O parallel to $\boldsymbol{\omega}'$ satisfies the given condition.

3. Show that at any instant in the motion of a rigid body there is a point on each line of the body at which the acceleration is perpendicular to the line, provided $\boldsymbol{\omega}\times\mathbf{u} \neq 0$, where $\mathbf{u}$ is a unit vector parallel to the line.

4. Show that at any instant in the motion of a rigid body for which $\boldsymbol{\omega}\times\boldsymbol{\omega}' = 0$ all points on each line parallel to the instantaneous axis have the same acceleration.

5. Show that at every instant in the motion of a rigid body at which $\boldsymbol{\omega}\times\boldsymbol{\omega}' \neq 0$ there exists one and only one point of the body having zero acceleration.

50. Derivative of a Vector Relative to Two Reference Systems.

In our discussion of vector functions of a scalar (§ 24) we considered the changes in a variable vector as they were observed by a single observer, or as observed by several observers all employing the same reference system. We shall now imagine a vector $\mathbf{p}$ to vary as a function of the time t, but to be observed

by observers employing two different reference systems, S_1 and S_2 which may be in motion relative to each other. As an example of this we might think of the vector $\mathbf{p}$ as the velocity of a tennis ball with which two passengers are playing tennis on the deck of a steamship. To the players, busy with their game, the ship naturally forms the reference system and their estimates of the velocity of the ball at any instant are clearly quite different from the estimate of some observer who may be watching the game from the shore through a field glass. The observed value and variation of a vector quantity thus depend on the state of motion of the observer and his reference frame.

We shall however here make explicitly the important assumption that the value of a *scalar quantity* is independent of the motion of the observer and his reference frame relative to the measured quantity. For instance, we shall assume that the length of a pencil is the same for us when it rests on the table before us as it is when falling from the table to the floor. This assumption, which seems at first so thoroughly justified, is not made in *relativity mechanics*, and in fact one of the most fundamental differences between the classical mechanics here under discussion and the newer relativity mechanics lies in the fact that the one accepts and the other does not accept this basic assumption. A special case of this assumption is the hypothesis previously made that "Time is absolute" (§ 2). The length of an interval of time, like the length of the pencil mentioned above, is a scalar quantity and we are assuming that the observers, although employing different reference systems, are nevertheless using the same measurement of time in making their observations. Just how they are to succeed in doing this when, as in the above example, they are widely separated is a question presenting both practical and theoretical difficulties of a high order. The attempt to solve this question in a thorough fashion again leads us to relativity mechanics and we shall not here undertake to discuss it.

In discussing the variation of the vector $\mathbf{p}$ relative to the two reference systems S_1 and S_2 it will be convenient for us to think of each reference system as having its own measure of the time which we shall call t_1 and t_2 respectively. These are really identical with the true time t in accordance with our above postulate that time is absolute but at first we shall regard them

as functions of t employed in place of the time in their respective systems. Since the systems S_1 and S_2 may be in motion relative to each other, we may first set up a means of expressing this motion. To the observer employing the system S_1 the system S_2 moves like a rigid body and its position at any instant may be completely determined for him by giving the values of certain quantities as functions of his time t_1. For instance, if A_1 and A_2 are points fixed in S_1 and S_2 respectively and if we denote by $\mathbf{a}_2$ the vector $A_1 A_2$ and by $\boldsymbol{\omega}_2$ the vector angular velocity of S_2 relative to S_1, then $\mathbf{a}_2$ and $\boldsymbol{\omega}_2$ as functions of t_1 will determine the motion of S_2 relative to S_1, if the position of S_2 in S_1 be known for some value of t_1. We may indicate this by the single equation,

$$S_2 = S_2(t_1),$$

although the actual equations giving the position might be six in number if scalar functions of t_1 were employed.

We may regard the vector $\mathbf{p}$ as a function of t_1 as it would be observed by the observer employing system S_1 or as a function of t_2 in S_2. Thus we would have,

$$\mathbf{p} = \mathbf{p}_1(t_1) \qquad \text{or} \qquad \mathbf{p} = \mathbf{p}_2(t_2),$$

but to equate these two expressions for $\mathbf{p}$ we must refer them both to the same reference system, as S_1. The result of referring $\mathbf{p}_2(t_2)$ to the system S_1 will of course depend not only on $\mathbf{p}_2(t_2)$ but on the position of S_2 in S_1. We may indicate this fact by the equation,

$$\mathbf{p}_1(t_1) = \mathbf{f}\{S_2(t_1),\ \mathbf{p}_2(t_2)\},$$

where it is understood as before that the complete expression of the position of S_2 as a function of t_1 may require as many as six scalar equations and the expression of $\mathbf{p}_2$ as a function of t_2 may require three scalar equations. In any case we may reduce the right member of this equation to a single vector function of t_1 and t_2 and write,

$$\text{(1)} \qquad \mathbf{p} = \mathbf{F}(t_1, t_2),$$

in which now $\mathbf{p}$ is referred to S_1.

If in the right member of this equation we should at some instant t fix t_1 at the value which it then had and allow only t_2 to vary from then on, we would in effect be fixing the system S_2

in its position relative to S_1 and the resulting values of $\mathbf{F}$ would express the values of $\mathbf{p}$ as observed in system S_2 and then referred to S_1. An observer employing the system S_1 might naturally call this the *relative value* of $\mathbf{p}$; i.e., the value relative to S_2. If on the other hand we were at the instant t to fix t_2 at the value which it then had and were to allow only t_1 to vary from then on, we would in effect be attaching the vector $\mathbf{p}$ to the system S_2 and an observer employing system S_1 might naturally call the resulting value of $\mathbf{F}$ the *drag* or *body value* of $\mathbf{p}$ from the instant t since it would be the value which $\mathbf{p}$ would have if dragged along with the body S_2 from the instant t.

If we now find the derivative of $\mathbf{p}$ with respect to t by differentiating equation (1), we have,

$$\mathbf{p}' = \frac{\partial \mathbf{F}}{\partial t_1}\frac{dt_1}{dt} + \frac{\partial \mathbf{F}}{\partial t_2}\frac{dt_2}{dt},$$

or since t_1 and t_2 are in fact identical with t,

$$\mathbf{p}' = \frac{\partial \mathbf{F}}{\partial t_1} + \frac{\partial \mathbf{F}}{\partial t_2}. \tag{2}$$

Again differentiating with respect to t and afterwards setting $t_1 = t_2 = t$ gives us,

$$\mathbf{p}'' = \frac{\partial^2 \mathbf{F}}{\partial t_1^2} + \frac{\partial^2 \mathbf{F}}{\partial t_1\,\partial t_2} + \frac{\partial^2 \mathbf{F}}{\partial t_2\,\partial t_1} + \frac{\partial^2 \mathbf{F}}{\partial t_2^2}. \tag{3}$$

Equations (2) and (3), like equation (1), are relative to system S_1.

The process of partial differentiation thus furnishes us with an analysis of $\mathbf{p}'$ and $\mathbf{p}''$ into various terms. It only remains to examine these terms with a view to their mechanical interpretation. In equation (2) this is very easy. The term $\partial\mathbf{F}/\partial t_1$ is of course what the value of $\mathbf{p}'$ would be if t_2 were held constant at the instant under consideration. It is thus the derivative of the drag value of the vector at the instant and is called the *drag derivative*. Similarly $\partial\mathbf{F}/\partial t_2$ is the value which $\mathbf{p}'$ would have if t_1 were held constant at the instant and is thus the derivative of the relative value of $\mathbf{p}$ and is known as the *relative derivative*. We thus have the important theorem,

The true derivative of a vector is the sum of the drag derivative and the relative derivative.

Similarly in equation (3) it is evident that $\partial^2\mathbf{F}/\partial t_1^2$, being what the value of $\mathbf{p}''$ would be if t_2 were fixed at the instant, is the *second drag derivative* of $\mathbf{p}$, while $\partial^2\mathbf{F}/\partial t_2^2$ is the *second relative derivative* of $\mathbf{p}$. Evidently $\partial^2\mathbf{F}/\partial t_2\,\partial t_1$ is the relative derivative of the drag derivative of $\mathbf{p}$ and $\partial^2\mathbf{F}/\partial t_1\,\partial t_2$ is the drag derivative of the relative derivative of $\mathbf{p}$.

The names true derivative, drag derivative and relative derivative here employed are natural to a person using the reference system S_1. They have come down to us from a time when it was generally supposed that it was possible to determine a reference system actually fixed in space and such a system was regarded as the basic one to which a moving system should be referred. We now regard the concept of being fixed in space as completely illusory so that, although the above names are still employed, they have lost a large part of their descriptive quality. The reference system S_2 might thus equally well have been adopted as the one to which we would ultimately refer.

The drag derivative $\partial\mathbf{F}/\partial t_1$, being the derivative of a vector rigidly attached to the system S_2, is given by the formula, § 45, (10),

$$\frac{\partial\mathbf{F}}{\partial t_1} = \boldsymbol{\omega}_2 \times \mathbf{p}, \tag{4}$$

where $\boldsymbol{\omega}_2$ is the vector angular velocity of S_2 relative to S_1. If we represent this drag derivative by $\mathbf{p}_1'$ and the relative derivative by $\mathbf{p}_2'$, we may write equation (2) as,

$$\mathbf{p}' = \mathbf{p}_1' + \mathbf{p}_2' = \mathbf{p}_2' + \boldsymbol{\omega}_2 \times \mathbf{p}. \tag{2$'$}$$

Similarly we have,

$$\frac{\partial^2\mathbf{F}}{\partial t_1\,\partial t_2} = \frac{\partial\mathbf{p}_2'}{\partial t_1} = \boldsymbol{\omega}_2 \times \mathbf{p}_2',$$

while if we differentiate equation (4) partially with respect to t_2 and recall that $\boldsymbol{\omega}_2$ is a function of t_1 only, we find,

$$\frac{\partial^2\mathbf{F}}{\partial t_2\,\partial t_1} = \boldsymbol{\omega}_2 \times \mathbf{p}_2'.$$

It thus appears that the order of the partial differentiation of $\mathbf{F}$ with respect to the two variables t_1 and t_2 is immaterial. This is of course a natural extension of the corresponding fact for scalar functions of two variables.

If we indicate by corresponding subscripts partial differentiation with respect to t_1 and t_2, we may now write equation (3) in the form,

$$(3') \qquad \mathbf{p}'' = \mathbf{p}''_{11} + \mathbf{p}''_{22} + 2\,\omega_2 \times \mathbf{p}'_2.$$

The term $2\,\omega_2 \times \mathbf{p}'_2$ is called the *complementary second derivative* of $\mathbf{p}$ and we may express this equation by the theorem,

The true second derivative of a vector is the sum of the second drag derivative, the second relative derivative, and the complementary second derivative.

Since $\mathbf{p}''_{11}$ is the second derivative of a vector rigidly attached to the system S_2 its value may be computed by formula, § 49, (2), to be,

$$(5) \qquad \mathbf{p}''_{11} = \omega'_2 \times \mathbf{p} + (\omega_2 \cdot \mathbf{p})\,\omega_2 - \omega_2^2\,\mathbf{p}.$$

If at any moment the system S_2 has a zero angular velocity relative to S_1, it follows from equation (4) that the drag derivative of a vector is then zero and its true and relative derivatives are consequently equal. If the angular velocity remains zero throughout an interval of time, as when the system S_2 has a motion of translation relative to S_1, then by equations (3′) and (5) the complementary second derivative and the drag second derivative of a vector are both zero and its true and relative second derivatives are consequently equal.

EXERCISES

1. Let a vector $\mathbf{p}$ vary as a function of the scalar t relative to two reference systems, S_1 and S_2. Let $\mathbf{i}_2$, $\mathbf{j}_2$, $\mathbf{k}_2$ be the coördinate vectors of the system S_2 expressed in S_1 and let x_2, y_2, z_2 be the coördinates of $\mathbf{p}$ in S_2. Obtain in terms of these quantities and their derivatives expressions for the true, drag, and relative derivatives of $\mathbf{p}$ in S_1 and expressions for the various second derivatives. Derive equations (2′) and (3′) of this article.

2. Prove that if a reference system S_2 has a permanent motion of translation relative to a system S_1, then the relative derivatives of all orders of a vector are equal to the corresponding true derivatives.

51. Relative Motion; Theorem of Coriolis.

We have seen in the preceding article that the derivative of a vector function of the time will generally be different in different reference systems and we have a formula giving us a relation

between these derivatives,

§ 50, (2′) $$\mathbf{p}' = \mathbf{p}_1' + \mathbf{p}_2'.$$

Since the velocity of a moving point relative to any reference system is defined as the derivative with respect to the time and in that system of a vector running to the moving point from a point fixed in the system, it appears that the above equation should give information as to the relation between the velocities of a point in two different systems. Let us assume that we have two reference systems S_1 and S_2 which may be in motion relative to each other, and a point P moving freely relative to both systems. As in the preceding article, we shall find it convenient to regard each of the two systems S_1 and S_2 as having its own measure of time, t_1 and t_2 respectively. These may be thought of as functions of the true time t, although we ultimately identify both of them with t. Then the position of the point P in the system S_1 at each instant may be determined by giving the vector A_1P as a function of t_1 and the position of P in S_2 by giving A_2P as a function of t_2, where A_1 and A_2 are points fixed in the systems S_1 and S_2 respectively. Let us agree to pick as A_1 and A_2 points fixed in their respective systems but coinciding with each other at the instant t to be considered, and let us indicate by P_1 and P_2 two points similarly attached to S_1 and S_2 but coinciding at this instant with each other at the position then occupied by the moving point P.

We now apply the above formula to the vector A_1P obtaining,

(1) $$(A_1P)' = (A_1P)_1' + (A_1P)_2',$$

where the significance of the various derivatives will be clear from the preceding article. Since the first term of the second member is the partial derivative of A_1P with respect to t_1, we are in this term regarding t_2 as a constant and hence regarding A_1P as rigidly attached to the system S_2. We may equally well express this by replacing A_1 by A_2 and P by P_2 and if we do this, we may then express the differentiation as with respect to the true time t, since A_2 and P_2 are not functions of t_2. Thus since $(A_1P)_1' = (A_2P_2)'$ equation (1) may be written,

(2) $$(A_1P)' = (A_2P_2)' + (A_1P)_2'.$$

As a particular case we may let the point P move so as to con-

stantly coincide with the point A_2. Equation (2) then becomes,

$$(A_1 A_2)' = (A_2 A_2)' + (A_1 A_2)'_2,$$

and as the first term of the second member is evidently zero we may write this as,

$$0 = (A_1 A_2)' + (A_2 A_1)'_2,$$

which, being added term for term to equation (2) yields,

$$(3) \qquad (A_1 P)' = (A_1 P_2)' + (A_2 P)'_2.$$

The first member is the velocity of the point P in the system S_1. From the point of view of an observer employing this system we call this the *true velocity* $\mathbf{v}$ of P. The first term of the second member is the true velocity of P_2, i.e. the velocity of a point coinciding with P at the instant considered, but rigidly attached to the system S_2. This is called the *drag velocity* $\mathbf{v}_1$ of P since it is the velocity which P would have in S_1 if dragged along with S_2. The last term of the second member is the velocity of P in the system S_2 and is thus called the *relative velocity* $\mathbf{v}_2$ by the observer employing S_1. Equation (3),

$$(3) \qquad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2,$$

states the important fact known as the,

Law of Composition of Velocities. The true velocity of a point is the sum of its drag velocity and its relative velocity.

Thus if a man be walking on the deck of a moving ship his velocity at any instant relative to the shore is the vector sum of his velocity relative to the ship and the velocity relative to the shore of that point of the deck at which he is at the instant considered.

The law of composition of velocities follows naturally from the definition of the velocity of a point as a vector quantity, and in fact is susceptible of a number of simple mechanical and analytic proofs. A more striking fact is that a similar simple relation does *not* hold in the case of the accelerations. If we apply equation § 50, (3′),

$$\mathbf{p}'' = \mathbf{p}''_{11} + \mathbf{p}''_{22} + 2\,\mathbf{p}''_{21}$$

to the vector $A_1 P$ we have,

$$(4) \qquad (A_1 P)'' = (A_1 P)''_{11} + (A_1 P)''_{22} + 2\,(A_1 P)''_{21}.$$

Since the first term of the second member is the second partial derivative of $A_1 P$ with respect to t_1, we are in this term regarding t_2 as a constant and hence regarding $A_1 P$ as rigidly attached to the system S_2. We may equally well express this by replacing A_1 by A_2 and P by P_2 and if we do this, we may then express the differentiation as with respect to the true time t, since A_2 and P_2 are not functions of t_2. Thus since $(A_1 P)''_{11} = (A_2 P_2)''$ equation (4) may be written,

$$(5) \qquad (A_1 P)'' = (A_2 P_2)'' + (A_1 P)''_{22} + 2\,(A_1 P)''_{21}.$$

As a particular case we may let the point P so move as to coincide constantly with A_2 and equation (5) then reads,

$$(A_1 A_2)'' = (A_2 A_2)'' + (A_1 A_2)''_{22} + 2\,(A_1 A_2)''_{21}.$$

Since the first term of the second member is evidently zero we may write this as,

$$0 = (A_1 A_2)'' + (A_2 A_1)''_{22} + 2\,(A_2 A_1)''_{21},$$

which being added term for term to equation (5) yields,

$$(6) \qquad (A_1 P)'' = (A_1 P_2)'' + (A_2 P)''_{22} + 2\,(A_2 P)''_{21}.$$

The first member is the acceleration $\mathbf{j}$ of the point P in the system S_1 and employing this reference system we call it the *true acceleration* of P. The first term of the second member is the true acceleration of P_2, i.e. the acceleration of a point coinciding with P at the instant considered but rigidly attached to the system S_2. We call this the *drag acceleration* $\mathbf{j}_1$ of P since it is the acceleration which P would have if dragged along with S_2. The second term of the second member is the acceleration of P in the system S_2 and is called the *relative acceleration* $\mathbf{j}_2$. But now in this case we have the third term of the second member which we may indicate by $\mathbf{j}_3$ and call the *complementary acceleration* or the *acceleration of Coriolis*. Equation (6) may be written,

$$(6) \qquad \mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3,$$

and states the important,

Theorem of Coriolis. The true acceleration of a point is the sum of its drag acceleration, relative acceleration and complementary acceleration.

The complementary acceleration $2(A_2 P)''_{21}$ is evidently twice the drag derivative of the relative velocity $(A_2 P)'_2$ and consequently may be written (§ 50, (4)),

$$(7) \qquad \mathbf{j}_3 = 2\,\boldsymbol{\omega}_2 \times \mathbf{v}_2.$$

It is thus perpendicular to the relative velocity of P and to the vector angular velocity of S_2 in S_1 and vanishes whenever either of these vectors is zero or when they are parallel to each other.

If we think of the point P whose motion we have been considering as having associated with it a mass m so that it becomes a particle, then we may by Newton's second law, $m\,\mathbf{j} = \lambda\,\mathbf{f}$, associate each of the above accelerations with the force capable of producing it. In classical mechanics we assume that the mass m of a particle is a constant independent of the reference frame employed, while λ is an absolute constant, so that by equation (6) we have,

$$(8) \qquad \mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3,$$

where $\mathbf{f} = m\,\mathbf{j}/\lambda$, $\mathbf{f}_1 = m\,\mathbf{j}_1/\lambda$, $\mathbf{f}_2 = m\,\mathbf{j}_2/\lambda$, $\mathbf{f}_3 = m\,\mathbf{j}_3/\lambda$. An observer A_1 employing reference system S_1 might naturally call these vectors $\mathbf{f}$, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ respectively the *true force, drag force, relative force* and *complementary force.* If we consider our particle at a given instant with a given mass, position and velocity, then the drag force $\mathbf{f}_1$ and the complementary force $\mathbf{f}_3$ are completely determined at the instant if we know the motion of S_2 relative to S_1. We shall call the quantity $-(\mathbf{f}_1 + \mathbf{f}_3)$ the *local force in* S_2 acting on the given particle. It will be observed that the local force in no way depends on the true or relative force at the moment. Equation (8) may now be written,

$$(9) \qquad \mathbf{f}_2 = \mathbf{f} + \mathbf{k},$$

in which $\mathbf{k}$ is the local force.

It is implied in Newton's second law (§ 36) that forces simultaneously applied to a particle are subject to vector addition and a reference frame for which this holds is said to be a *Newtonian reference frame.* Thus an observer A_1 employing reference system S_1, which he assumes to be Newtonian, and observing that two forces $\mathbf{f}$ and $\mathbf{g}$ are applied to a particle will expect the particle to move as if a single force $\mathbf{f} + \mathbf{g}$ had been applied. But like-

wise an observer A_2 employing system S_2, which he also assumes to be Newtonian, and observing that the two relative forces $\mathbf{f}_2$ and $\mathbf{g}_2$ are applied to the particle will expect the particle to move as if a single relative force $\mathbf{f}_2 + \mathbf{g}_2$ had been applied. These observers A_1 and A_2 can not in general both be right in these expectations for if the total true force is $\mathbf{f} + \mathbf{g}$, then by equation (9) the total relative force is $\mathbf{f} + \mathbf{g} + \mathbf{k} = \mathbf{f}_2 + \mathbf{g}_2 - \mathbf{k}$ and not $\mathbf{f}_2 + \mathbf{g}_2$; while if the total relative force is $\mathbf{f}_2 + \mathbf{g}_2$ then the true force will be $\mathbf{f}_2 + \mathbf{g}_2 - \mathbf{k} = \mathbf{f} + \mathbf{g} + \mathbf{k}$ and not $\mathbf{f} + \mathbf{g}$. Thus the presence of the local force $\mathbf{k}$ will reveal that the reference system employed is not Newtonian.

If the observer A_2 employing reference system S_2 finds that forces are not subject to vector addition in this frame he may nevertheless continue to regard S_2 as a Newtonian frame by explaining the presence of the local force by assuming some suitable "law of nature." Thus in experiments conducted in our laboratories we find a strong downward force constantly in operation. Instead of regarding this as the local force due to a constantly accelerated expansion of the earth we prefer to think of it as due to the presence of the earth itself and call it the force of gravity. We have here to make the somewhat unnatural assumption that this force is *exactly* proportional to the mass of the particle.

EXERCISES

1. A person traveling eastward at the rate of 3 miles per hour finds that the wind seems to blow directly from the north. On doubling his speed it appears to come from the northeast. What is the speed of the wind? *Ans.* 4.24

2. An airplane is to fly to a point 400 miles distant with a favoring 50 mile wind making an angle of 60° with its path. If the maximum speed of the plane in still air is 70 miles an hour, what is the shortest time in which it can make the trip? If the plane were kept at right angles to the path, how long would the trip take and what would be the speed of the plane relative to the air? *Ans.* $t = 5$, $t = 16$, $v = 25$

3. A man stands in the rain holding a tube open at the ends and 5 feet long. He finds that the raindrops fall directly through the tube if he inclines the top 4 feet to the north, but if he walks south at the rate of 7 feet per second he need only incline the top 3 feet to the north. What is the speed of the raindrops? *Ans.* 20

4. In driving an automobile a man observes that when going north at 20 miles an hour the raindrops stream straight down the front windows, but down the side windows at an angle of 30° with the vertical. On continuing in the same direction at 30 miles an hour the raindrops stream down the side windows at an angle of 45° with the vertical. Determine the speed of the raindrops.

Ans. 24.49

5. A man rows upstream on the Cam and sees a floating bottle when he is one mile above Cambridge. He then rows upstream for one hour and rows back to Cambridge, seeing the bottle again just as he arrives. What is the speed of the current?

Ans. 1/2 mile per hour

6. If the wind is blowing with a constant direction but varying speed and an airplane is to go from one designated point to another in the least possible time, show that its path should be a straight line. Does the statement hold if the direction of the wind varies?

7. Show that if a moving reference frame S_2 has a zero vector angular velocity $\boldsymbol{\omega}$ at an instant, then all the points in space have the same drag velocity at that instant. Show that if $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are both zero at an instant, then all the points in space have the same drag acceleration at the instant.

8. A circle is drawn in the XY-plane with center at $A \equiv (2, 0, 0)$ and radius 1. The circle rotates about the Y-axis with a uniform angular velocity $\omega = 1$ while a point P traces out the circle with a constant unit scalar velocity relative to the circle. Assume P to be initially on the X-axis and call the angle through which AP has turned in the rotating plane θ. Express in terms of θ the lengths of the true and drag velocities and of the complementary acceleration.

Ans. Complementary acceleration $2|\sin\theta|$

9. The velocity of light being 300,000 km. per second and the velocity of the earth in its orbit being 30 km. per second, how must a telescope be set to view a star at right angles to the earth's path at the instant? On what theory as to the nature of light is your conclusion based?

10. A rigid body moves so that a straight line fixed in the body passes constantly through a fixed point O. Show that the velocity of the point of the body at O is along the line.

11. A particle of mass m acted upon by a force $\mathbf{f}$ has its motion referred to a reference frame with fixed origin and constant vector angular velocity $\boldsymbol{\omega}$. Show that its equation of relative motion is,

$$\mathbf{q}'' + 2\,\boldsymbol{\omega} \times \mathbf{q}' = \boldsymbol{\omega} \times (\mathbf{q} \times \boldsymbol{\omega}) + \mathbf{f}/m,$$

where $\mathbf{q}$ is the radius vector of the particle relative to the moving reference frame. If $\mathbf{f}$ is of the form $\mathbf{f} = \varphi(\mathbf{q}^2)\,\mathbf{q}$, derive *Jacobi's*

integral of this equation,

$$\mathbf{q}'^2 = (\boldsymbol{\omega} \times \mathbf{q})^2 + \frac{1}{m} \int \varphi(\mathbf{q}^2)\, d(\mathbf{q}^2) + C.$$

Extend this to the case in which $\mathbf{f}$ is of the form,

$$\mathbf{f} = \sum \varphi_i\{(\mathbf{q} - \mathbf{q}_i)^2\}, \qquad (\mathbf{q}_i \text{ const.}).$$

12. Show that for experiments carried out in a laboratory on the surface of the earth the local force acting upon a particle of mass m for motion relative to the laboratory is effectively,

$$\mathbf{k} = -\, gm\, \mathbf{u} - 2m\, \boldsymbol{\omega} \times \mathbf{p}',$$

where $\mathbf{u}$ is the unit upward vector at the laboratory and $\boldsymbol{\omega}$ is the vector angular velocity of the earth.

13. Discuss the free motion of a particle for short distances near the surface of the earth, taking into account the rotation of the earth. If $\mathbf{u}$ is the unit upward vector in the neighborhood considered, then for motion relative to the ground there,

$$\mathbf{p}'' = -\, g\, \mathbf{u} - 2\, \boldsymbol{\omega} \times \mathbf{p}',$$

where $\boldsymbol{\omega}$ is the vector angular velocity of the earth. Integrating find,

$$\mathbf{p}' = -\, g\, \mathbf{u}\, t - 2\, \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{p}_0) + \mathbf{p}_0'.$$

Let θ be the angular distance of the neighborhood from the north pole and choose a coördinate system with the origin at the initial point of the motion, the XY-plane horizontal, the X-axis eastward, the Y-axis northward and the Z-axis upward. Then derive,

$$\begin{aligned} x' &= 2\omega \cos\theta\, y - 2\omega \sin\theta\, z + x_0', \\ y' &= -\, 2\omega \cos\theta\, x + y_0', \\ z' &= -\, gt + 2\omega \sin\theta\, x + z_0'. \end{aligned}$$

Since ω is very small drop all terms involving ω to higher powers than the first and obtain,

$$\begin{aligned} x &= \frac{g}{3}\, \omega \sin\theta\, t^3 + \omega\, (y_0' \cos\theta - z_0' \sin\theta)\, t^2 + x_0'\, t, \\ y &= -\, \omega \cos\theta\, x_0'\, t^2 + y_0'\, t, \\ z &= -\, \frac{g}{2}\, t^2 + \omega \sin\theta\, x_0'\, t^2 + z_0'\, t. \end{aligned}$$

Interpret these results.

14. A body is projected from the earth's surface with a scalar velocity v in a direction making an angle α with the horizontal and such that the projection of this direction on the horizontal lies at an angle β east of north. If θ is the angular distance of the neighbor-

hood from the north pole, determine at the time t after projection the horizontal distance ξ of the body in the vertical plane of projection, the horizontal deviation η to the right of this plane, and the vertical distance ζ, omitting in each case all powers of the earth's angular velocity ω above the first. Discuss the effect of the earth's rotation on the range over level ground and on a body falling from rest.

Ans.
$$\xi = v \cos \alpha \, t - \omega \sin \theta \sin \beta \left(v \sin \alpha \, t^2 - \frac{g}{3} t^3 \right),$$
$$\eta = \omega \cos \theta \, v \cos \alpha \, t^2 - \omega \sin \theta \cos \beta \left(v \sin \alpha \, t^2 - \frac{g}{3} t^3 \right),$$
$$\zeta = v \sin \alpha \, t - \frac{g}{2} t^2 + \omega \sin \theta \, v \cos \alpha \sin \beta \, t^3$$

15. What is the motion relative to the earth of a particle on a smooth horizontal plane at the earth's surface if account be taken of the earth's rotation?

Ans. Uniform motion on a circle tangent to the direction of projection and deviating to the right of it in the northern hemisphere

52. Plane Motion.

As a special case of the motion of a rigid body we may consider the motion of a plane figure in its plane. This is known as *plane motion.* It will be clear that any facts which we may deduce in this study will be at once applicable to the motion of any rigid body in which a plane section of the body continues to move in the same fixed plane. Such motions are very common and important in applied mechanics.

In § 48 in the proof of Mozzi's theorem we found that the state of motion of a rigid body at any instant falls into one of the three cases,

I. $\omega = 0$,	Motion of translation,
II. $\omega \neq 0$, $\omega \cdot \mathbf{a}' = 0$,	Motion of rotation,
III. $\omega \cdot \mathbf{a}' \neq 0$,	Helicoidal motion,

in all of which we have,

$$(1) \qquad \mathbf{p}' = \mathbf{a}' + \omega \times (\mathbf{p} - \mathbf{a}), \qquad OP = \mathbf{p}, \qquad OA = \mathbf{a},$$

where A and P are any two points of the rigid body and O any fixed point. In Case I all the points of the body have the same velocity at the instant considered. This case may arise in plane motion and presents no peculiarities other than that the veloci-

ties of all the points of the figure are not only equal but lie in the plane of the motion. When however we consider Cases II and III for plane motion we may easily show that if $\omega \neq 0$ then $\omega \cdot \mathbf{a}' = 0$, thus eliminating helicoidal motion. For as explained in the proof of Mozzi's theorem (§ 48) ω, when not zero, is given by the formula,

$$\omega = \frac{(\mathbf{p}' - \mathbf{a}') \times (\mathbf{q}' - \mathbf{a}')}{(\mathbf{q} - \mathbf{a}) \cdot (\mathbf{p}' - \mathbf{a}')},$$

A, P, Q being any three non-collinear points of the plane figure. Since $\mathbf{p}'$, $\mathbf{q}'$, $\mathbf{a}'$ lie in the plane of the motion, ω must be perpendicular to that plane and hence $\omega \cdot \mathbf{a}' = 0$. In plane motion the body is thus always in a state of translation or rotation. In what follows we shall be concerned almost wholly with the case of rotation since the case of translation presents little of interest, as noted above.

In plane motion at each instant at which the plane figure is in a state of rotation there exists a point C of the figure which has zero velocity.

To locate this point C we choose any particular point A of the figure so that $\mathbf{a}$ and $\mathbf{a}'$ are known quantities and then determine C by the formula,

$$AC = \mathbf{c} - \mathbf{a} = \frac{\omega \times \mathbf{a}'}{\omega^2}. \tag{2}$$

We first note that the point C so chosen is a point of the plane figure, for ω being perpendicular to the plane, $\mathbf{c} - \mathbf{a}$ must lie in the plane. Furthermore C, as a point of the rigid plane figure, has zero velocity, for by equation (1) its velocity $\mathbf{c}'$ is,

$$\mathbf{c}' = \mathbf{a}' + \omega \times (\mathbf{c} - \mathbf{a}) = \mathbf{a}' + \frac{\omega \times (\omega \times \mathbf{a}')}{\omega^2} = \frac{\omega \cdot \mathbf{a}'}{\omega^2}\,\omega,$$

which vanishes due to the fact that $\omega \cdot \mathbf{a}' = 0$. This point C is known as the *instantaneous center* or *center of velocities* and is evidently simply the point at which the instantaneous axis of the theorem of Mozzi pierces the plane of the motion.

If now A be once more any point of the rigid plane figure, we have,

$$\mathbf{p}' = \mathbf{a}' + \omega \times (\mathbf{p} - \mathbf{a}), \qquad 0 = \mathbf{a}' + \omega \times (\mathbf{c} - \mathbf{a}),$$

and consequently by subtraction,

$$(3) \qquad \mathbf{p}' = \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{c}).$$

If we compare this with equation § 45, (7), $\mathbf{p}' = \boldsymbol{\omega} \times \mathbf{p}$ and recall the significance of $\boldsymbol{\omega}$ in that equation we have at once the theorem,

The velocities of the points of the rigid plane figure are what they would be if the figure were rotating in the positive sense relative to $\boldsymbol{\omega}$ *about* C *as a fixed point with an angular velocity* $\theta' = |\boldsymbol{\omega}|$.

The instantaneous center will in general vary from moment to moment both in the fixed plane and in the moving figure and will thus trace out in the fixed plane a curve which we shall call the *space centrode* S_1 and a curve in the moving figure which we shall call the *body centrode* S_2. We now recall the formula, $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ of the preceding article connecting the true velocity $\mathbf{v}$, the drag velocity $\mathbf{v}_1$ and the relative velocity $\mathbf{v}_2$ of any point. By definition of C its drag velocity is constantly zero so that for C we have $\mathbf{v} = \mathbf{v}_2 = \mathbf{w}$ in which now $\mathbf{w}$ is both the true and relative velocity of C. The vector $\mathbf{w}$ is thus both the velocity with which C traces out the space centrode S_1 and the velocity with which it traces out the body centrode S_2. Since these velocities have the same direction it follows that the body and space centrodes are tangent to each other at C. The two vectors have also the same length and sense and consequently the point of contact C of the two centrodes travels along them at the same speed and in the same sense. These facts concerning the two centrodes may be described by the following statement.

The body centrode S_2 *rolls without slipping on the space centrode* S_1, *their point of contact being the instantaneous center* C.

Rodrigues' formula (§ 44) shows after a slight change in notation that if a rigid body is rotated through an angle θ in the positive sense of rotation relative to a unit vector $\mathbf{u}$ about an axis remaining parallel to $\mathbf{u}$, and if $\mathbf{p}_0$ and $\mathbf{p}$ are the initial and final values of a vector fixed in the rigid body, then,

$$\mathbf{p} = \cos\theta\, \mathbf{p}_0 + (1 - \cos\theta)\, (\mathbf{u}\cdot\mathbf{p}_0)\, \mathbf{u} + \sin\theta\, \mathbf{u} \times \mathbf{p}_0.$$

For brevity we may write this equation in the symbolic form,

$$\mathbf{p} = e^{\theta\mathbf{u}}\, \mathbf{p}_0,$$

the operator $e^{\theta\mathbf{u}}$ applied to a vector indicating rotation of this vector through the angle θ in the positive sense about $\mathbf{u}$. If

now $\mathbf{p}_0$, $\mathbf{p}$, and θ vary as functions of t while $\mathbf{u}$ remains constant, we have on differentiating this equation and reducing by means of it,

$$(4) \qquad \mathbf{p}' = \theta' \mathbf{u} \times \mathbf{p} + e^{\theta \mathbf{u}} \mathbf{p}_0'.$$

The reader will recognize that $\mathbf{p}'$, $\theta' \mathbf{u} \times \mathbf{p}$, $e^{\theta \mathbf{u}} \mathbf{p}_0'$ are respectively the true derivative, the drag derivative and the relative derivative of $\mathbf{p}$. Evidently this equation may be applied to vectors rigidly attached to a figure in plane motion, since the instantaneous axis remains perpendicular to the plane of the motion and hence parallel to a constant unit vector $\mathbf{u}$.

Let us now consider an instant t_0 in the plane motion of a rigid plane figure at which the instantaneous center C occupies a position C_0. Let us indicate by C_1 the position occupied by C at a subsequent instant $t_0 + \Delta t$ and by C_2 the position which C would have occupied at the latter instant if it had moved with its actual speed along the position occupied by the body centrode S_2 at the instant t_0. Let $\mathbf{t}_1$ be the unit tangent vector to S_1 at C_1 and let $\mathbf{t}_2$ be the unit tangent vector to S_2 at C_2, both drawn in the sense of C's motion. Let $\Delta\theta$ be the positive angle through which the rigid plane figure turns in the time interval Δt in the positive sense of rotation relative to a unit vector $\mathbf{u}$ drawn perpendicular to the plane in the proper sense. The motion of the plane figure being such as to keep the body centrode constantly tangent to the space centrode at C, it appears that a rotation $\Delta\theta$ would carry the vector $\mathbf{t}_2$ into $\mathbf{t}_1$, so that we may write,

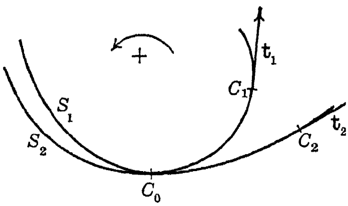

FIG. 48

$$\mathbf{t}_1 = e^{\Delta\theta \mathbf{u}} \mathbf{t}_2.$$

Differentiating with respect to the time by means of formula (4), we have at the instant $t_0 + \Delta t$,

$$\mathbf{t}_1' = \theta' \mathbf{u} \times \mathbf{t}_1 + e^{\Delta\theta \mathbf{u}} \mathbf{t}_2'.$$

We now recall that from the first of the Frenet formulas, § 32, (8), we have,

$$\frac{d\mathbf{t}_1}{ds} = \frac{1}{\rho_1} \mathbf{n}_1, \qquad \frac{d\mathbf{t}_2}{ds} = \frac{1}{\rho_2} \mathbf{n}_2,$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit normal vectors to S_1 and S_2 at C_1 and C_2 on the concave side of these curves and ρ_1 and ρ_2 are their radii of curvature, while s is the common length of arc along S_1 and S_2. Since $ds/dt = |\mathbf{w}| = w$ these may be written,

$$\mathbf{t}_1' = \frac{w}{\rho_1}\mathbf{n}_1, \qquad \mathbf{t}_2' = \frac{w}{\rho_2}\mathbf{n}_2,$$

and our equation takes the form,

$$\frac{w}{\rho_1}\mathbf{n}_1 = \theta'\,\mathbf{u}\times\mathbf{t}_1 + \frac{w}{\rho_2}e^{\Delta\theta\,\mathbf{u}}\,\mathbf{n}_2. \tag{5}$$

Since $\Delta\theta$ has been taken as positive, θ' will also be positive if the time interval Δt is sufficiently small. Let us agree to regard ρ_1 as positive when a 90° rotation in the positive sense relative to $\mathbf{u}$ carries $\mathbf{t}_1$ into $\mathbf{n}_1$, with the corresponding agreement for ρ_2. Then the three vectors forming the terms of equation (5) all have the same direction and sense and we may therefore equate the lengths of the two members obtaining,

$$\frac{w}{\rho_1} = \theta' + \frac{w}{\rho_2}.$$

Since $\boldsymbol{\omega} = \theta'\,\mathbf{u}$ it follows that $\omega = |\boldsymbol{\omega}| = \theta'$ and we have the interesting,

(6) *Formula of Euler-Savary,* $$\frac{1}{\rho_1} - \frac{1}{\rho_2} = \frac{\omega}{w}.$$

At each instant of rotation there exists a point M of the moving plane figure whose velocity is equal to the true and relative velocity $\mathbf{w}$ *of the instantaneous center C.*

For if we choose the point M so that,

$$CM = \mathbf{m} - \mathbf{c} = \frac{\mathbf{w}\times\boldsymbol{\omega}}{\boldsymbol{\omega}^2},$$

then the velocity $\mathbf{m}'$ of M is given by formula (3) as,

$$\mathbf{m}' = \boldsymbol{\omega}\times(\mathbf{m} - \mathbf{c}) = \frac{\boldsymbol{\omega}\times(\mathbf{w}\times\boldsymbol{\omega})}{\boldsymbol{\omega}^2} = \mathbf{w}.$$

From the way in which M was chosen it is evident that the vector CM makes an angle of 90° with the vector $\mathbf{w}$ and that the sense of rotation from $\mathbf{w}$ to CM is negative relative to $\boldsymbol{\omega}$, while the length of CM is w/ω.

We may now put the formula of Euler-Savary in a form involving this point M. If we designate by N_1 and N_2 the centers of curvature of the centroids S_1 and S_2 for their point of contact C, then C, M, N_1, N_2 are all points along the normal to the centroids at C. If we adopt either sense along this normal as positive and indicate by CM, CN_1, CN_2 the measures of these vectors on the normal, we have by formula (6),

(7) *Formula of Euler-Savary,*
$$\frac{1}{CN_2} - \frac{1}{CN_1} = \frac{1}{CM}\cdot$$

The reader will find it instructive to draw figures illustrating the various possibilities as to relative size and sense of these vectors.

Whenever ω and ω' *are not both zero there exists a point D of the plane figure which has zero acceleration.*

To locate this point D we first choose any particular point A of the moving plane figure so that $\mathbf{a}$ and $\mathbf{a}''$ are known quantities. We then choose D so that,

$$AD = \mathbf{d} - \mathbf{a} = \frac{\omega^2\,\mathbf{a}'' + \omega' \times \mathbf{a}''}{\omega^4 + \omega'^2} \tag{8}$$

It is evident that D is a point of the rigid plane figure for $\mathbf{a}''$ and $\omega' \times \mathbf{a}''$ both lie in this plane. The acceleration of any point P of the figure is given by formula, § 49, (2), which for this case reduces to,

$$\mathbf{p}'' = \mathbf{a}'' + \omega' \times (\mathbf{p} - \mathbf{a}) - \omega^2\,(\mathbf{p} - \mathbf{a}). \tag{9}$$

This gives for the acceleration $\mathbf{d}''$ of the point D,

$$\mathbf{d}'' = \mathbf{a}'' + \frac{(\omega' \cdot \mathbf{a}'')\,\omega' - \omega'^2\,\mathbf{a}'' - \omega^4\,\mathbf{a}''}{\omega^4 + \omega'^2}$$

which vanishes due to the fact that $\omega' \cdot \mathbf{a}'' = 0$. If now A be once more any point of the plane figure, equation (9) gives for the point D,

$$0 = \mathbf{a}'' + \omega' \times (\mathbf{d} - \mathbf{a}) - \omega^2\,(\mathbf{d} - \mathbf{a}),$$

and on subtracting from equation (9) we find,

$$\mathbf{p}'' = \omega' \times (\mathbf{p} - \mathbf{d}) - \omega^2\,(\mathbf{p} - \mathbf{d}). \tag{10}$$

If we compare this with what equation (9) would be if A were a fixed point and $\mathbf{a}''$ consequently zero we see that,

The accelerations of the points of the moving plane figure are what they would be if the figure were rotating with its actual angular velocity and acceleration about the point D as a fixed point.

The point D is consequently known as the *center of accelerations.* Equation (10) analyzes the acceleration $\mathbf{p}''$ into the component $-\omega^2(\mathbf{p}-\mathbf{d})$ along DP and the component $\boldsymbol{\omega}' \times (\mathbf{p}-\mathbf{d})$ perpendicular to DP. The ratio of the lengths of these components and their orientation relative to DP are clearly the same at any given instant for all points P of the figure. It follows that $\mathbf{p}''$ makes everywhere the same obtuse angle $\arctan - |\boldsymbol{\omega}'|/\omega^2$ with DP in the positive sense of rotation relative to $\boldsymbol{\omega}$. The length of the acceleration is clearly proportional to the distance DP.

We get another analysis of the acceleration $\mathbf{p}''$ into two components as follows. We differentiate both members of equation (3) with respect to the time and in the result replace $\mathbf{p}'$ by $\boldsymbol{\omega} \times (\mathbf{p}-\mathbf{c})$ and $\mathbf{c}'$ by $\boldsymbol{\omega} \times (\mathbf{m}-\mathbf{c})$, the latter being allowed by the fact that the true velocity of C is the body velocity of M. Thus we have,

$$\mathbf{p}'' = \boldsymbol{\omega}' \times (\mathbf{p}-\mathbf{c}) - \omega^2(\mathbf{p}-\mathbf{m}), \tag{11}$$

which by aid of equation (3) yields,

$$\mathbf{p}' \times \mathbf{p}'' = \{\omega^2(\mathbf{p}-\mathbf{c})\cdot(\mathbf{p}-\mathbf{m})\}\,\boldsymbol{\omega} \tag{12}$$

and

$$\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}') = -\{\omega^4(\mathbf{p}-\mathbf{c})\cdot(\mathbf{p}-\mathbf{m})\}\,(\mathbf{p}-\mathbf{c}). \tag{13}$$

It is clear from inspection of the figure that the angle between the vectors $\mathbf{p}-\mathbf{c}$ and $\mathbf{p}-\mathbf{m}$ will be acute or obtuse according as P lies outside or inside the circle having C and M at the extremities of a diameter. Consequently $(\mathbf{p}-\mathbf{c})\cdot(\mathbf{p}-\mathbf{m})$ is negative for interior points and positive for exterior points. But by equation § 32, (4) the unit normal vector to the path of P has the direction and sense of $\mathbf{p}' \times (\mathbf{p}'' \times \mathbf{p}')$ and since for a plane curve this normal always extends from P towards the concave side of the curve, we conclude from equation (13) that at any instant,

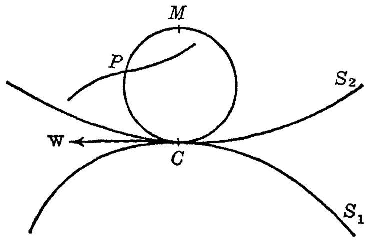

FIG. 49

The path of any point P of the rigid plane figure is concave or convex towards C at the point P according as P lies outside or inside the circle having C and M at the extremities of a diameter.

A point of inflection of a curve is (§ 32) a point at which $\mathbf{p}' \times \mathbf{p}'' = 0$, $\mathbf{p}' \neq 0$. From equations (3) and (12) it appears that these conditions will be satisfied for $\omega \neq 0$ only at all points of the above circle except C. Thus,

The circle with CM as diameter is the locus of the points of the rigid plane figure which are at the points of inflection of their paths, the point C being omitted.

This circle is consequently known as the *circle of inflections.* Evidently in particular the center of accelerations D is on this circle. It may also be shown that the path of any point on the body centrode has a cusp at the point where this point becomes the instantaneous center.

EXERCISES

1. Discuss the motion of a plane figure in its plane when a circle fixed in the moving figure rolls without slipping on a line fixed in the plane. Locate the point M and the circle of inflections. Find the paths traced by points of the moving figure, especially points on the moving circle.
2. Discuss the motion of a plane figure in its plane when two points P and Q of the moving figure slide respectively along two lines k and l fixed in the plane. Describe the body and space centrodes and the circle of inflections. Show that the points of the moving figure describe ellipses and find what these reduce to for points on the body centrode. (*Elliptical lathe of Leonardo da Vinci.*)
3. Discuss the motion of a rigid plane figure in its plane when a point P of the figure slides along a fixed line k and a line l of the figure through P passes constantly through a fixed point A. Locate the instantaneous center C and the circle of inflections. Describe the body centrode. Find the paths traced by points of the moving figure, especially points on the body centrode.
4. Discuss the motion of a rigid plane figure in its plane when a right angle PQR of the moving figure moves so that the point P at a fixed distance a from Q slides on the side AB of the fixed right angle ABC and the side QR of the moving angle passes constantly through the fixed point C which is at a distance a from B. Locate the instantaneous center C, and the circle of inflections. Describe the space and body centrodes.
5. A plane quadrilateral $PQRS$ is formed of four rigid rods freely pivoted at the vertices. If this moves in its plane, show that the

instantaneous centers of the four rods form the vertices of a quadrilateral which is circumscribed about the quadrilateral $PQRS$. If one of the four rods is fixed, show that the instantaneous center of the opposite rod is at the point of intersection of the other two rods produced.

6. If P and Q are any two points of a plane figure in plane motion with the respective accelerations $\mathbf{p}''$ and $\mathbf{q}''$ and R is the intersection of the lines $P\,\mathbf{p}''$ and $Q\,\mathbf{q}''$, show that P, Q, R, D lie on a circle, D being the center of accelerations.

7. Show that in plane motion the curvature of the path of the point P at P is for $\boldsymbol{\omega} \neq 0$,

$$\frac{1}{\rho} = \frac{(\mathbf{p} - \mathbf{c})\cdot(\mathbf{p} - \mathbf{m})}{|\mathbf{p} - \mathbf{m}|^3}.$$

Thus show that the instantaneous center C is the center of curvature of the paths of the points of the common tangent of the space and body centrode at these points.

8. Show that in plane motion the body acceleration of the instantaneous center is $\mathbf{w} \times \boldsymbol{\omega}$.

9. Show that at any instant in plane motion the locus of the points of the moving figure at which the acceleration is perpendicular to the velocity is a circle with the equation,

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega}'\,(\mathbf{p} - \mathbf{c})^2 - \omega^2\,\mathbf{w}\cdot(\mathbf{p} - \mathbf{c}) \quad 0.$$

Where does this circle cut the circle of inflections? Show that the two circles are orthogonal.

10. If E is the intersection, other than C, of the line through C in the direction of $\mathbf{w}$ with the circle of Problem 9, show that the acceleration $\mathbf{p}''$ of each point P of the moving figure may be resolved into the two components,

$$\mathbf{p}'' = \boldsymbol{\omega}' \times (\mathbf{p} - \mathbf{e}) - \omega^2\,(\mathbf{p} - \mathbf{c}).$$

The point E is known as the *center of geometric accelerations*. Show that it is collinear with D and M.

11. If the plane figures A, B, C are moving in the same plane, show that the instantaneous centers of B relative to A, of C relative to B and of A relative to C are collinear.

CHAPTER VI

SLIDING VECTORS

53. Definitions.

Each of the vectors which we have thus far employed in the study of mechanics embodies in itself the three concepts of a certain direction, a certain sense along that direction and a certain length. We said that two vectors were equal when and only when they were the same in these three respects. In the topics to be discussed in this and the following three chapters we shall often find it convenient to combine other ideas into a single concept. Thus if we take a certain straight line, a certain sense along this line and a certain length we form a new concept which we shall call a *sliding vector*. A sliding vector can be visualized as an arrow free to slide on the fixed line along which it lies but not, like the vectors previously considered, free to move throughout space. We shall say that two sliding vectors are equal only if they lie along the same line, point in the same sense and have the same length. In contradistinction to these new sliding vectors the vectors previously considered may be called *free vectors*. Occasionally we may also consider *attached vectors*. Here the ideas of a definite point in space, a definite direction, a definite sense along that direction and a length are all combined in the one concept. We may visualize an attached vector as an arrow with its initial point at a certain fixed point in space. For two attached vectors to be equal requires that they have the same initial point, the same direction, sense and length.

As an example of a quantity completely characterized by a free vector we might mention the translation of a rigid body, the force of gravity in a small region of the earth's surface, the velocity of light from a distant star. In each case the elements of length, direction and sense are involved, but in no case is the vector associated with any particular point or line. Examples of sliding vectors have previously occurred without being mentioned as such. Thus the vector angular velocity ω of a rigid body at any instant could well be associated with the line about which the body is then

rotating and the whole rotation thus characterized by a single sliding vector. For this reason a sliding vector is sometimes known as a *rotor*. The same thing is true of the vector $\mathbf{w}$ of Rodrigues' formula for a displacement of rotation of a rigid body, for if $\mathbf{w}$ be associated with the axis about which the body could be rotated to produce the displacement, the resulting sliding vector would completely characterize the displacement. It is also evident that a force acting on a rigid body is capable of being completely characterized by a sliding vector because in the case of a rigid body the exact point of application of the force is immaterial provided the amount, direction, sense and line of action are determined. As examples of quantities requiring an attached vector for their complete characterization we may mention the velocity of the particles of water in a conduit, the force of gravity throughout the solar system, a force applied to a deformable body. In each of these cases the ideas of length, direction and sense must be associated with some definite point before the concept is complete.

The operations of addition and subtraction of two vectors, multiplication of a vector by a scalar, and dot and cross-multiplication of two vectors are performed on sliding and attached vectors exactly as if they were free vectors, and the result, when a vector, is thought of as a free vector unless otherwise specified.

54. Moment with Respect to a Point.

A sliding vector is evidently completely determined by the value $\mathbf{l}$ of the vector as a free vector and the radius vector $\mathbf{a}$ from the origin O to any point A of the line λ along which the sliding vector lies. It is clear that the vector $\mathbf{a}$ is not uniquely determined by the sliding vector since any scalar multiple of $\mathbf{l}$ could be added to $\mathbf{a}$ without change in the position of the line λ. With the sliding vector given by $\mathbf{a}$ and $\mathbf{l}$ we may define a new vector $\mathbf{m}_O$ called the *moment of the sliding vector with respect to O* by the formula,

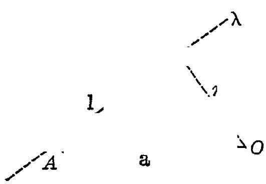

FIG. 50

$$\text{(1)} \qquad \mathbf{m}_O = \mathbf{a} \times \mathbf{l}.$$

This vector $\mathbf{m}_O$ depends only on the given sliding vector $\mathbf{l}$, λ and the point O because the particular choice of the point A on

the line λ does not affect the value of $\mathbf{m}_O$. For if another point B on λ had been chosen we would have,

$$\mathbf{b} \times \mathbf{l} = (\mathbf{a} + \mathbf{b} - \mathbf{a}) \times \mathbf{l} = \mathbf{a} \times \mathbf{l} + (\mathbf{b} - \mathbf{a}) \times \mathbf{l} = \mathbf{a} \times \mathbf{l}.$$

It is obvious that the moment $\mathbf{m}_O$ of $\mathbf{l}$, λ is perpendicular to the plane of O and λ and we may easily show that,

The length of $\mathbf{m}_O$ *is the product of the length of* $\mathbf{l}$ *and the distance from* O *to* λ.

For by formula § 22, (10) the distance r from any point Q to the line λ is,

$$r = \frac{|(\mathbf{q} - \mathbf{a}) \times \mathbf{l}|}{l}, \qquad \mathbf{q} = OQ, \qquad l = |\mathbf{l}|$$

and for Q at O this becomes,

$$(2) \qquad r = \frac{|\mathbf{a} \times \mathbf{l}|}{l} = \frac{|\mathbf{m}_O}{l} \qquad \text{or} \qquad |\mathbf{m}_O| = lr.$$

If we have given $\mathbf{l}$ and the moment $\mathbf{m}_O$ with respect to the point O for a certain sliding vector, then we may determine the moment $\mathbf{m}_P$ of this sliding vector with respect to another point P as follows,

$$(3) \quad \mathbf{m}_P = PA \times \mathbf{l} = (\mathbf{a} - \mathbf{p}) \times \mathbf{l} = \mathbf{a} \times \mathbf{l} - \mathbf{p} \times \mathbf{l}$$
$$= \mathbf{m}_O - \mathbf{p} \times \mathbf{l}.$$

We may put this result into words by the statement,

The moment of a sliding vector with respect to a point P is equal to its moment with respect to another point O diminished by what its moment with respect to O would be if it passed through P.

55. Coördinates of a Sliding Vector.

It is clear that the sliding vector $\mathbf{l}$ on the line λ may be completely characterized by giving the vector $\mathbf{l}$ as a free vector and the radius vector $\mathbf{a} = OA$ of some point A of the line λ. Here $\mathbf{a}$ is properly speaking an attached vector since its initial point O is likewise assumed to be given. From some points of view, however, it is more satisfactory to characterize the sliding vector $\mathbf{l}$, λ by giving the vector $\mathbf{l}$ as a free vector together with the moment $\mathbf{m}$ of $\mathbf{l}$, λ with respect to O. Here $\mathbf{m}$, like $\mathbf{a}$, is in effect an attached vector since O is assumed to be given.

We may regard $\mathbf{l}$ and $\mathbf{m}$ where $\mathbf{l}\cdot\mathbf{m} = 0$ as the two *vector coördinates* of the sliding vector $\mathbf{l}$, λ. We have seen that $\mathbf{m}$ is uniquely determined by $\mathbf{l}$, λ and we may easily show that the line λ is uniquely determined by $\mathbf{l}$ and $\mathbf{m}$ where $\mathbf{l}\cdot\mathbf{m} = 0, \mathbf{l} \neq 0$. For it was shown in § 20 that a necessary and sufficient condition that the equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, $(\mathbf{a} \neq 0)$ possess a solution $\mathbf{x}$ is that $\mathbf{a}\cdot\mathbf{b} = 0$ and that one such solution is $\mathbf{x} = \mathbf{b} \times \mathbf{a}/\mathbf{a}^2$. Thus in our case there always exists a vector $\mathbf{a}$ such that $\mathbf{m} = \mathbf{a} \times \mathbf{l}$ and one such value of $\mathbf{a}$ is $\mathbf{l} \times \mathbf{m}/\mathbf{l}^2$. Furthermore if $\mathbf{b}$ is any other vector such that $\mathbf{m} = \mathbf{b} \times \mathbf{l}$, then by subtraction $(\mathbf{b} - \mathbf{a}) \times \mathbf{l} = 0$ and $\mathbf{b} - \mathbf{a}$ is parallel to $\mathbf{l}$. The line through A in the direction of $\mathbf{l}$ thus coincides with the line through B in the direction of $\mathbf{l}$ and is the only line on which $\mathbf{l}$ can lie to form a sliding vector having the given moment $\mathbf{m}$ with respect to O.

The two vector coördinates $\mathbf{l}$, $\mathbf{m}$ $(\mathbf{l}\cdot\mathbf{m} = 0)$ of the sliding vector are equivalent to the six scalar coördinates,

$$(l_1, l_2, l_3, m_1, m_2, m_3) \qquad \text{where} \qquad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$$

and $\mathbf{l} \equiv (l_1, l_2, l_3)$, $\mathbf{m} \equiv (m_1, m_2, m_3)$. Because of the above condition which these six scalar coördinates must always satisfy there are essentially five independent scalar coördinates for a sliding vector and we may say that the sliding vectors of space constitute a *five parameter family*.

As developed in § 22 the equation of the line λ is $(\mathbf{p} - \mathbf{a}) \times \mathbf{l} = 0$ and this may evidently be written $\mathbf{p} \times \mathbf{l} = \mathbf{m}$. The line is thus determined by the values of $\mathbf{l}$ and $\mathbf{m}$ and so the six scalars $(l_1, l_2, l_3, m_1, m_2, m_3)$, $l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$, also serve as a set of coördinates for the line, but in this case it is evident that they could all be multiplied through by any non-vanishing scalar without affecting the line. In other words, when $(l_1, l_2, l_3, m_1, m_2, m_3)$ are regarded as the coördinates of the line λ it is only their ratios that are of significance and they are said to be *homogeneous coördinates*. This leaves only four quantities that can be chosen independently and the lines of space thus form a *four parameter family*. These six coördinates connected by the above relation and regarded as homogeneous are known as *Plücker line coördinates*. The vector coördinates $\mathbf{l}$, $\mathbf{m}$ of any sliding vector lying on a given line are likewise the vector coördinates of the line but in the case of the line they must be regarded as homo-

geneous since multiplying them both by the same non-vanishing scalar would not affect the line.

EXERCISES

1. Find the distance r from the point B to the line with vector coördinates $\mathbf{l}$, $\mathbf{m}$.

   Ans. $r = |\mathbf{b} \times \mathbf{l} - \mathbf{m}|/|\mathbf{l}|$

2. Find the radius vector $\mathbf{q}$ of the foot of the perpendicular let fall from the point B upon the line with vector coördinates $\mathbf{l}$, $\mathbf{m}$.

   Ans. $\mathbf{q} = \mathbf{l} \times \mathbf{m}/\mathbf{l}^2 + (\mathbf{b} \cdot \mathbf{l})\, \mathbf{l}/\mathbf{l}^2$

3. Find the radius vector $\mathbf{q}$ of the point of intersection of the line with vector coördinates $\mathbf{l}$, $\mathbf{m}$ and the plane with equation $(\mathbf{p} - \mathbf{b}) \cdot \mathbf{n} = 0$.

   Ans. $\mathbf{q} = \mathbf{n} \times \mathbf{m}/\mathbf{l} \cdot \mathbf{n} + (\mathbf{b} \cdot \mathbf{n})\, \mathbf{l}/\mathbf{l} \cdot \mathbf{n}$

4. Show that the two sliding vectors with vector coördinates $\mathbf{l}_1$, $\mathbf{m}_1$ and $\mathbf{l}_2$, $\mathbf{m}_2$ will be coplanar if and only if,

$$\mathbf{l}_1 \cdot \mathbf{m}_2 + \mathbf{l}_2 \cdot \mathbf{m}_1 = 0.$$

5. Show that if the two lines with vector coördinates $\mathbf{l}_1$, $\mathbf{m}_1$ and $\mathbf{l}_2$, $\mathbf{m}_2$ are coplanar and not parallel, they will intersect at the point Q where,

$$\mathbf{q} = \mathbf{m}_2 \times \mathbf{m}_1/\mathbf{l}_1 \cdot \mathbf{m}_2 = \mathbf{m}_1 \times \mathbf{m}_2/\mathbf{l}_2 \cdot \mathbf{m}_1.$$

6. Find the vector coördinates $\mathbf{l}$, $\mathbf{m}$ of the sliding vector running along the common perpendicular from the line $\mathbf{l}_1$, $\mathbf{m}_1$ to the line $\mathbf{l}_2$, $\mathbf{m}_2$.

$$\textit{Ans.} \quad \mathbf{l} = \frac{\mathbf{l}_1 \cdot \mathbf{m}_2 + \mathbf{l}_2 \cdot \mathbf{m}_1}{(\mathbf{l}_2 \times \mathbf{l}_1)^2}\, \mathbf{l}_2 \times \mathbf{l}_1,$$
$$\mathbf{m} = \frac{(\mathbf{m}_1 \cdot \mathbf{l})\, \mathbf{l}_2 \times \mathbf{l} - (\mathbf{m}_2 \cdot \mathbf{l})\, \mathbf{l}_1 \times \mathbf{l}}{\mathbf{l}_1 \cdot \mathbf{m}_2 + \mathbf{l}_2 \cdot \mathbf{m}_1}$$

7. Find the vector coördinates $\mathbf{l}$, $\mathbf{m}$ of the line of intersection of the two planes with equations, $(\mathbf{p} - \mathbf{a}_1) \cdot \mathbf{m}_1 = 0$ and $(\mathbf{p} - \mathbf{a}_2) \cdot \mathbf{m}_2 = 0$.

   Ans. $\mathbf{l} = \mathbf{m}_1 \times \mathbf{m}_2$, $\mathbf{m} = (\mathbf{a}_2 \cdot \mathbf{m}_2)\, \mathbf{m}_1 - (\mathbf{a}_1 \cdot \mathbf{m}_1)\, \mathbf{m}_2$

56. Moment with Respect to an Axis.

The moment of a sliding vector with respect to a fixed axis is defined as the measure upon the axis of the moment of the sliding vector with respect to any point of the axis.

Thus if the sliding vector has a moment $\mathbf{m}_O$ with respect to O and if the axis Δ passes through O in the direction and sense of the unit vector $\mathbf{u}$ then the moment m_Δ of the sliding vector with respect to Δ is

$$m_\Delta = \mathbf{u} \cdot \mathbf{m}_O.$$

And if A is any point of the line λ then the moment of the sliding

vector $\mathbf{l}$, λ with respect to Δ is

$$m_\Delta = \mathbf{u}\cdot\mathbf{m}_O = \mathbf{u}\cdot\mathbf{a}\times\mathbf{l} = [\mathbf{u}\,\mathbf{a}\,\mathbf{l}], \qquad \mathbf{a} = OA.$$

For this definition to have a unique significance it is necessary that the value of m_Δ be independent of the particular point O chosen on the axis Δ. This is clearly the case, for if the point O were shifted to a different position on the fixed axis Δ the effect would be to add to the vector $\mathbf{a}$ some scalar multiple of $\mathbf{u}$, and since,

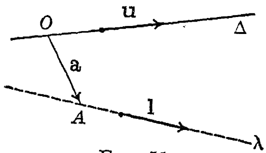

FIG. 51

$$[\mathbf{u}, \mathbf{a} + k\,\mathbf{u}, \mathbf{l}] = [\mathbf{u}, \mathbf{a}, \mathbf{l}],$$

this would not alter the value of m_Δ.

An important property of m_Δ is the following.

The moment of the sliding vector $\mathbf{l}$, λ *with respect to the axis* Δ *is numerically equal to the length of the projection of* $\mathbf{l}$ *on a plane perpendicular to* Δ *multiplied by the length of the common perpendicular of* λ *and* Δ.

Since the value of m_Δ is independent of the choice of O on Δ and of A on λ we may choose them as the feet of the common perpendicular of λ and Δ. If then $\mathbf{r} = OA$, we have,

$$m_\Delta = \mathbf{u}\cdot\mathbf{r}\times\mathbf{l} = \mathbf{l}\times\mathbf{u}\cdot\mathbf{r}.$$

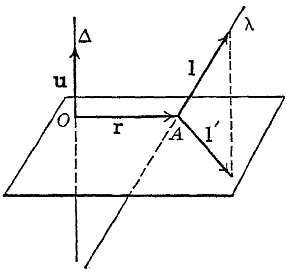

FIG. 52

But if we represent by $\mathbf{l}'$ the projection of $\mathbf{l}$ on a plane perpendicular to Δ, we have by the fundamental theorem of the vector product $\mathbf{l}\times\mathbf{u} = \mathbf{l}'\times\mathbf{u}$ and therefore have,

$$m_\Delta = \mathbf{l}'\times\mathbf{u}\cdot\mathbf{r} = [\mathbf{l}', \mathbf{u}, \mathbf{r}].$$

Since the vectors $\mathbf{l}'$, $\mathbf{u}$, $\mathbf{r}$ are mutually perpendicular the determinant $[\mathbf{l}', \mathbf{u}, \mathbf{r}]$ has a numerical value equal to the product of the lengths of the three vectors appearing in it and we have,

$$|m_\Delta| = l'\,r, \qquad l' = |\mathbf{l}'|, \qquad r = |\mathbf{r}|$$

as we wished to prove.

EXERCISES

1. The moment m_{12} of an axis Δ_1 with respect to an axis Δ_2 is defined as the moment with respect to Δ_2 of a unit sliding vector lying on Δ_1 and having the sense of Δ_1. Show that $m_{12} = m_{21}$. This is consequently called the *mutual moment* of the two axes.
2. Show that the mutual moment of two axes is the product of the distance between them and the sine of the angle from the one to the other in the positive sense of rotation relative to a vector running from the other to the one.
3. Show that two axes are coplanar if and only if their mutual moment is zero.
4. If three sliding vectors $(\mathbf{l}_1, \mathbf{m}_1)$, $(\mathbf{l}_2, \mathbf{m}_2)$, $(\mathbf{l}_3, \mathbf{m}_3)$ form a null system, i.e.,
$$\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 = 0,$$
show that the mutual moment of each pair of their axes is zero and that the three sliding vectors are thus coplanar.
5. If four sliding vectors $(\mathbf{l}_1, \mathbf{m}_1)$, $(\mathbf{l}_2, \mathbf{m}_2)$, $(\mathbf{l}_3, \mathbf{m}_3)$, $(\mathbf{l}_4, \mathbf{m}_4)$ form a null system, i.e.,
$$\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 + \mathbf{l}_4 = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \mathbf{m}_4 = 0,$$
find a relation which must hold between the mutual moments of their axes and express the ratios of the lengths of the vectors in terms of the mutual moments of the axes.

   Ans. One of the quantities, $\sqrt{m_{12}\, m_{34}}$, $\sqrt{m_{13}\, m_{42}}$, $\sqrt{m_{14}\, m_{23}}$ equals the sum of the other two
$$l_1^2 : l_2^2 : l_3^2 : l_4^2 = m_{23}\, m_{34}\, m_{42} : m_{34}\, m_{41}\, m_{13} : m_{41}\, m_{12}\, m_{24} : m_{12}\, m_{23}\, m_{31}$$

57. Systems of Sliding Vectors.

If a collection of any finite number of sliding vectors be regarded as a single object the whole collection is known as a *system of sliding vectors* and represented by the single letter Σ. We shall represent by n the number of vectors in the system Σ and assume that they have been numbered from 1 to n, calling the vectors $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \cdots, \mathbf{l}_n$ and their corresponding moments with respect to some point O, $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \cdots, \mathbf{m}_n$. The *sum* $\mathbf{S}$ *of the system* Σ is the sum of the vectors $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \cdots, \mathbf{l}_n$ and the *moment* $\mathbf{M}_O$ *of the system* Σ with respect to O is the sum of the vectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \cdots, \mathbf{m}_n$. Thus we have,
$$\mathbf{S} = \mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_n, \qquad \mathbf{M}_O = \mathbf{m}_1 + \mathbf{m}_2 + \cdots + \mathbf{m}_n.$$
The two vectors $\mathbf{S}$ and $\mathbf{M}_O$ are known as the *elements of reduction* of the system Σ.

By equation § 54, (3) the moment $\mathbf{m}_{iP}$ of the ith vector of the system with respect to any point P is given in terms of its moment $\mathbf{m}_{iO}$ with respect to O by the equation,

$$\mathbf{m}_{iP} = \mathbf{m}_{iO} - \mathbf{p} \times \mathbf{l}_i, \qquad \mathbf{p} = OP.$$

If we form the sum of such equations for all vectors of the system we have the formula,

$$\mathbf{M}_P = \mathbf{M}_O - \mathbf{p} \times \mathbf{S} \tag{1}$$

for the change of center of moments of the system from O to P. By definition the scalar $\mathbf{S}^2$ is independent of the choice of center of moments and if we form the dot-product of both members of equation (1) with $\mathbf{S}$ we see that $\mathbf{S} \cdot \mathbf{M}_P = \mathbf{S} \cdot \mathbf{M}_O$ and this scalar is therefore also independent of the choice of center of moments. The two scalars $\mathbf{S}^2$ and $\mathbf{S} \cdot \mathbf{M}$ are consequently known as the *invariants of the system* Σ.

As in the case of a single vector, the moment of the system Σ with respect to an axis Δ is defined as the measure upon Δ of the moment of Σ with respect to any point of Δ. Thus if the axis Δ passes through the point O in the direction and sense of a unit vector $\mathbf{u}$, the moment M_Δ of Σ with respect to Δ is,

$$M_\Delta = \mathbf{u} \cdot \mathbf{M}_O. \tag{2}$$

If P is any other point on the same axis Δ then for a proper choice of the scalar k we have $\mathbf{p} = OP = k\,\mathbf{u}$ and by equation (1) we find that,

$$\mathbf{u} \cdot \mathbf{M}_P = \mathbf{u} \cdot \mathbf{M}_O - \mathbf{u} \cdot k\,\mathbf{u} \times \mathbf{S} = \mathbf{u} \cdot \mathbf{M}_O. \tag{3}$$

It thus appears that M_Δ is independent of the particular point chosen on the axis Δ.

EXERCISES

1. Three sliding vectors have their lines of action on the three sides of a triangle. If the moment of this system with respect to a point not in its plane is zero, show that each of the three vectors must be zero.

2. Prove the following theorem due to Varignon by consideration of the system of sliding vectors involved.

   If perpendiculars are let fall from any point in the plane of a parallelogram upon two sides and a diagonal which are concurrent, then the product of the diagonal by its perpendicular is

equal to the sum or difference of the products of the sides by their perpendiculars according as the point is without or within the parallelogram.

58. Equivalence of Systems.

Two systems of sliding vectors are said to be equivalent if they have the same sum and the same moment with respect to any one point.

Thus the system Σ_1 having a sum $\mathbf{S}_1$ and moment $\mathbf{M}_1$ with respect to a point O is equivalent to the system Σ_2 having a sum $\mathbf{S}_2$ and moment $\mathbf{M}_2$ with respect to O whenever,

$$\mathbf{S}_1 = \mathbf{S}_2, \qquad \mathbf{M}_1 = \mathbf{M}_2$$

for some choice of the point O. From equation § 57, (1),

$$\mathbf{M}_P = \mathbf{M}_O - \mathbf{p} \times \mathbf{S}.$$

we have the theorem,

Two equivalent systems have the same moment with respect to every point of space.

For by the definition of equivalence the second member of the above equation will have the same value for both systems and consequently the first member must also.

The ordinary equals sign is used to express the equivalence of two systems. Thus the equivalence $\Sigma_1 = \Sigma_2$ is coextensive with the two equations, $\mathbf{S}_1 = \mathbf{S}_2$, $\mathbf{M}_1 = \mathbf{M}_2$. A system Σ_3 is said to be the *sum of two systems* Σ_1 and Σ_2 if it consists of all the sliding vectors in Σ_1 and all the sliding vectors in Σ_2. Thus the equivalence $\Sigma_3 = \Sigma_1 + \Sigma_2$ is coextensive with the two equations, $\mathbf{S}_3 = \mathbf{S}_1 + \mathbf{S}_2$, $\mathbf{M}_3 = \mathbf{M}_1 + \mathbf{M}_2$. A system Σ is said to be a *null system* if it is equivalent to a system consisting of no sliding vectors. Thus the equivalence $\Sigma = 0$ which expresses this fact is coextensive with the two equations, $\mathbf{S} = 0$, $\mathbf{M} = 0$. Two systems Σ_1 and Σ_2 are said to be *in equilibrium* if their sum is a null system, i.e. for $\Sigma_1 + \Sigma_2 = 0$ or $\mathbf{S}_1 + \mathbf{S}_2 = 0$, $\mathbf{M}_1 + \mathbf{M}_2 = 0$.

A system of sliding vectors whose lines of action, $\lambda_1, \lambda_2, \cdots, \lambda_n$ all pass through the same point A is called a *concurrent system.* We may now prove the,

Theorem of Varignon (Caen, 1654–Paris, 1722). *A concurrent system of sliding vectors is equivalent to a single sliding vector equal to its sum and passing through its point of concurrence.*

Let the concurrent system Σ consist of the vectors $\mathbf{l}_1, \mathbf{l}_2, \cdots, \mathbf{l}_n$ with lines of action concurrent at A. Then the moment of the system with respect to the point O is,

$$\mathbf{M} = \mathbf{a}\times\mathbf{l}_1 + \mathbf{a}\times\mathbf{l}_2 + \cdots + \mathbf{a}\times\mathbf{l}_n, \qquad \mathbf{a} = OA.$$

The sum of the system $\mathbf{S} = \mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_n$ as a sliding vector with line of action through A has a moment with respect to $\mathbf{O}$ given by $\mathbf{a} \times \mathbf{S}$. Evidently we have,

$$\mathbf{a} \times \mathbf{S} = \mathbf{a} \times (\mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_n)$$
$$= \mathbf{a}\times\mathbf{l}_1 + \mathbf{a}\times\mathbf{l}_2 + \qquad + \mathbf{a}\times\mathbf{l}_n = \mathbf{M}.$$

The system Σ is thus equivalent to the single sliding vector $\mathbf{S}$ with line of action through A since Σ and $\mathbf{S}$ have the same sum and the same moment with respect to O. The theorem as here proven is rather trivial but this is due to the fact that we have already proven the distributive law for vector multiplication.

Varignon's theorem shows that if a concurrent system be removed and replaced by a single vector equal to its sum and concurrent with it, neither the sum nor the moment with respect to any point will be affected. This replacement is one of the *elementary operations*. The reverse of this operation, i.e. replacing a single sliding vector by a system concurrent with it and of which it is the sum, is the other elementary operation. Since the first operation does not alter the sum nor the moment it follows that its reverse can not. These elementary operations could clearly be carried out on a concurrent system or single sliding vector constituting a part of some larger system without affecting the sum or moment of this larger system. We may therefore conclude that,

If a system Σ_1 can be transformed into a system Σ_2 by elementary operations on any of its vectors then, Σ_1 and Σ_2 are equivalent.

The converse of this theorem is also true.

If the systems Σ_1 and Σ_2 are equivalent, then it is possible to transform Σ_1 into Σ_2 by elementary operations on certain of its vectors.

In this case however the proof must be much more extended. We shall proceed by showing that if Σ_1 and Σ_2 are equivalent, then we may by elementary operations transform them both into the same system Σ_0, known as the reduced form. Then since the reverse of each elementary operation is an elementary operation it follows that the elementary operations carrying Σ_1 into Σ_0

followed by the reverses in reverse order of the elementary operations that would carry Σ_2 into Σ_0 will when all applied constitute a set of elementary operations carrying Σ_1 into Σ_2. We first prove,

Lemma I. Any system Σ of sliding vectors may be reduced by elementary operations to a system consisting of three sliding vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ *passing respectively through three arbitrarily chosen non-collinear points* B_1, B_2, B_3.

For consider any sliding vector $\mathbf{l}_i$, λ_i of the system Σ. If the line λ_i is not coplanar with the three points B_1, B_2, B_3 we may choose a point A_i on it which is not coplanar with B_1, B_2, B_3 and find three vectors $\mathbf{k}_{1i}$, $\mathbf{k}_{2i}$, $\mathbf{k}_{3i}$ having respectively the directions of $B_1 A_i$, $B_2 A_i$, $B_3 A_i$ and whose sum is $\mathbf{l}_i$. In effect this may be done by the formula § 21, (3),

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\,\mathbf{x} = [\mathbf{b}\,\mathbf{c}\,\mathbf{x}]\,\mathbf{a} + [\mathbf{c}\,\mathbf{a}\,\mathbf{x}]\,\mathbf{b} + [\mathbf{a}\,\mathbf{b}\,\mathbf{x}]\,\mathbf{c},$$

which resolves any vector $\mathbf{x}$ into components parallel to any three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. We may then by the second elementary operation replace the sliding vector $\mathbf{l}_i$, λ_i by the three sliding vectors $\mathbf{k}_{1i}$, $\mathbf{k}_{2i}$, $\mathbf{k}_{3i}$ lying respectively on the lines $B_1 A_i$, $B_2 A_i$, $B_3 A_i$ and hence concurrent with it. On the other hand, if the line λ_i should be coplanar with B_1, B_2, B_3, then if we choose any point A_i on it not collinear with some two of the Bs as B_1 and B_2 we may find two vectors $\mathbf{k}_{1i}$ and $\mathbf{k}_{2i}$ having respectively the directions of $B_1 A_i$ and $B_2 A_i$ and whose sum is $\mathbf{l}_i$. For in this case the vector $\mathbf{l}_i$ will be coplanar with A_i, B_1, B_2 and we have a formula, § 21, Prob. 5,

$$(\mathbf{a}\times\mathbf{b})^2\,\mathbf{x} = (\mathbf{b}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{x})\,\mathbf{a} + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{x})\,\mathbf{b},$$

resolving any vector $\mathbf{x}$ into components parallel to any two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ coplanar with it. We may then by the second elementary operation replace the sliding vector $\mathbf{l}_i$, λ_i by the two sliding vectors $\mathbf{k}_{1i}$, $\mathbf{k}_{2i}$ lying respectively on the lines $B_1 A_i$ and $B_2 A_i$ and hence concurrent with it.

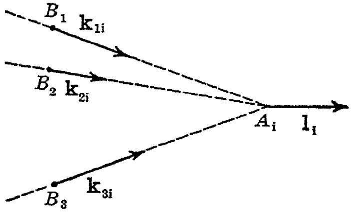

Fig. 53

When every vector of the system Σ has thus by the second elementary operation been replaced by vectors whose lines of

action pass through B_1 or B_2 or B_3, we may by the first elementary operation replace all the sliding vectors $\mathbf{k}_{1i}$ whose lines of action pass through B_1 by a single sliding vector $\mathbf{k}_1$ equal to their sum and concurrent with them at B_1. Similar replacements being performed at B_2 and B_3, the lemma is established. With the aid of this lemma we may readily establish,

Lemma II. Any system Σ of sliding vectors may be reduced by elementary operations to a system consisting of two sliding vectors $\mathbf{k}$ *and* $\mathbf{h}$ *with lines of action passing respectively through the points P and Q, where one of the points as P may be chosen arbitrarily and the other Q may be chosen as any point in the plane π through P perpendicular to the moment* $\mathbf{M}_P$ *of Σ with respect to P.*

To accomplish this we first employ the method of Lemma I and reduce Σ to three sliding vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ with the line of action of $\mathbf{k}_1$ passing through the given point P. Then let C be any point other than P common to the plane of P and $\mathbf{k}_2$ and the plane of P and $\mathbf{k}_3$. Since $\mathbf{k}_2$ is coplanar with P and C we may as before by the second elementary operation replace $\mathbf{k}_2$ by two sliding vectors, one with line of action through P and the other with line of action through C. The same having been done for $\mathbf{k}_3$ we then by the first elementary operation replace all the resulting sliding vectors with lines of action through P by a single sliding vector through P and all those through C by a single sliding vector $\mathbf{g}$ through C. The moment $\mathbf{M}_P$ of the system Σ with respect to P now is expressible simply as,

$$\mathbf{M}_P = PC \times \mathbf{g},$$

showing that the sliding vector $\mathbf{g}$ through C lies in the plane π through P perpendicular to $\mathbf{M}_P$. Since $\mathbf{g}$ is thus coplanar with the given points P and Q it may by the second elementary operation be replaced by two sliding vectors in this plane, one with line of action through P and the other $\mathbf{h}$ with line of action through Q. Finally the two sliding vectors now at P may by the first elementary operation be replaced by a single vector $\mathbf{k}$ at P, and the system Σ thus reduced to the form stated in the lemma. We next prove,

Lemma III. Any system of sliding vectors Σ may be reduced as in Lemma II with the further proviso that the sliding vector $\mathbf{h}$ *through Q is to be perpendicular to PQ. When so reduced the resulting sliding vectors* $\mathbf{k}$ *and* $\mathbf{h}$ *are uniquely determined by the sum* $\mathbf{S}$ *and the moment* $\mathbf{M}_P$ *of the given system and the chosen points P and Q.*

For after the given system has been reduced as in Lemma II we shall have $\mathbf{M}_P = PQ \times \mathbf{h}$ and therefore the sliding vector $\mathbf{h}$ lies in the plane π and can consequently by the second elementary operation be replaced by two vectors in that plane, one along PQ and one perpendicular to PQ at Q. The one along PQ may by the first elementary operation be combined with the vector $\mathbf{k}$ into a single sliding vector through P which we may again call $\mathbf{k}$, while the vector perpendicular to PQ at Q may now be called $\mathbf{h}$. This proves the first part of the lemma. If now the sum $\mathbf{S}$ and moment $\mathbf{M}_P$ of Σ be given together with the points P and Q, we may show that the vectors $\mathbf{k}$ and $\mathbf{h}$ are uniquely determined. For let $\mathbf{h}_1$ and $\mathbf{h}_2$ be any possible values of $\mathbf{h}$. Then we have,

$$\mathbf{M}_P \doteq PQ \times \mathbf{h}_1, \qquad \mathbf{M}_P = PQ \times \mathbf{h}_2,$$

and therefore by subtraction,

$$PQ \times (\mathbf{h}_2 - \mathbf{h}_1) = 0.$$

But since $\mathbf{h}_1$ and $\mathbf{h}_2$ are both perpendicular to PQ we have,

$$PQ \cdot (\mathbf{h}_2 - \mathbf{h}_1) = 0,$$

and since PQ is not zero it follows from these two equations that $\mathbf{h}_2 - \mathbf{h}_1$ is zero and $\mathbf{h}$ is consequently unique. And if $\mathbf{k}_2$ and $\mathbf{k}_1$ are any two possible values of $\mathbf{k}$ we have,

$$\mathbf{S} = \mathbf{k}_1 + \mathbf{h}, \qquad \mathbf{S} = \mathbf{k}_2 + \mathbf{h},$$

and by subtraction $\mathbf{k}_2 = \mathbf{k}_1$, and $\mathbf{k}$ is unique. The lemma is thus established.

It follows therefore that if any two equivalent systems Σ_1 and Σ_2 are reduced as in Lemma III with the same choice of P and Q, the resulting systems will be identical. We showed in the reasoning outlined at the beginning that if any two systems are reducible by elementary operations to the same system then each is reducible to the other. The theorem is therefore proven.

EXERCISES

1. If a system of two sliding vectors is equivalent to another system of two sliding vectors, show that the volume of the tetrahedron having these two sliding vectors for opposite edges is the same for both systems.

2. Show that a system of coplanar sliding vectors will be a null system if its sum and its moment with respect to any axis perpendicular to its plane are both zero.

3. If a system of sliding vectors is a null system, show that its projection on any plane or on any line will also be a null system.

4. Show that if the moment of a system of coplanar sliding vectors with respect to a point not in its plane is zero, then the system is a null system.

5. Show that if the moments $\mathbf{M}_A$, $\mathbf{M}_B$, $\mathbf{M}_C$ of a system of sliding vectors with respect to three non-collinear points A, B, C are all zero, then the system is a null system.

6. Show that any system of sliding vectors can be reduced by elementary operations to a system of six sliding vectors lying along the edges of a tetrahedron having any four given non-coplanar points as vertices. Show that these six sliding vectors are uniquely determined by the four given points and the sum $\mathbf{S}$ and moment $\mathbf{M}_O$ of the system and thus prove that equivalent systems are reducible to each other by elementary operations.

59. Categories of Reduction.

A system Σ of sliding vectors for which the sum $\mathbf{S}$ is zero is called a *couple*. Formula § 57, (1),

$$\mathbf{M}_P = \mathbf{M}_O - \mathbf{p} \times \mathbf{S},$$

shows at once that in this case $\mathbf{M}_P = \mathbf{M}_O$ and hence,

The moment $\mathbf{M}$ *of a couple is a constant free vector independent of the point with respect to which the moment is taken.*

A system consisting of two sliding vectors having the same direction and length but opposite senses is called an *elementary couple*. It is evident that in this case $\mathbf{S}$ is zero and an elementary couple is a couple. Let us choose two points A_1 and A_2 respectively on the two parallel lines of action λ_1 and λ_2 of the two sliding vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ of an elementary couple and at opposite ends of a common perpendicular of these two lines. If we call the vector $A_1A_2 = \mathbf{r}$ and take the moment of the system with respect to A_1 we evidently have,

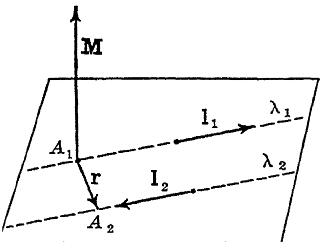

FIG. 54

$$\mathbf{M} = \mathbf{r} \times \mathbf{l}_2,$$

and $\mathbf{M}$ being perpendicular to $\mathbf{r}$ and $\mathbf{l}_2$ is perpendicular to the plane π in which the two vectors of the couple lie. Furthermore since $\mathbf{r}$ and $\mathbf{l}_2$ are perpendicular it follows that,

$$M = rl, \qquad M = |\mathbf{M}| \qquad r = |\mathbf{r}| \qquad l = |\mathbf{l}_1| = |\mathbf{l}_2|$$

It is evident from this that an elementary couple can always be constructed having any given moment $\mathbf{M}$. It is merely necessary to take a plane π perpendicular to $\mathbf{M}$ and in this plane to draw two parallel lines λ_1 and λ_2 at any convenient distance r apart. Then draw along each of these lines a sliding vector of length $l = M/r$, giving them opposite senses. From what has been shown above it is clear that if the sense of these vectors is properly chosen the moment of the elementary couple thus formed will be the given vector $\mathbf{M}$.

We may now classify all systems of sliding vectors into four *categories of reduction* and show that any system is reducible by the elementary operations to a system consisting of a single sliding vector and an elementary couple, either or both of which may be missing in certain cases. This classification is on the basis of the values of the elements of reduction, $\mathbf{S}$ and $\mathbf{M}_O$.

I. $\mathbf{S}\cdot\mathbf{M} \neq 0$. In this case the given system Σ is reducible by elementary operations to a system consisting of a single sliding vector and an elementary couple and it is not possible to dispense with either of them. For if we form a system consisting of a single sliding vector equal to the sum $\mathbf{S}$ of the given system and with its line of action through any point A and also an elementary couple with its moment $\mathbf{M}$ equal to the moment $\mathbf{M}_A$ of Σ with respect to this point A, then clearly the system thus formed has the same sum and the same moment with respect to A that the given system Σ has. Hence the given system can be reduced by elementary operations to the new system. Furthermore it is not possible to dispense with the single vector, for in that case the sum of the reduced system would be zero, contrary to the hypothesis $\mathbf{S} \cdot \mathbf{M} \neq 0$. And it is not possible to dispense with the couple, for in that case the moment of the reduced system with respect to any point on the line of action of the single vector would be zero, contrary to the hypothesis that the invariant $\mathbf{S}\cdot\mathbf{M} \neq 0$.

II. $\mathbf{S}\cdot\mathbf{M} = 0$, $\mathbf{S} \neq 0$. In this case the given system Σ may be reduced by elementary operations to a single sliding vector. We have here to construct a single sliding vector equal to the

sum $\mathbf{S}$ of the given system and with line of action passing through a point A so chosen that,

$$\mathbf{a} \times \mathbf{S} = \mathbf{M}_O, \qquad \mathbf{a} = OA,$$

where $\mathbf{M}_O$ is the moment of Σ with respect to O. We saw in § 20 that a necessary and sufficient condition for the existence of a vector $\mathbf{a}$ satisfying this equation for $\mathbf{S} \neq 0$ is that $\mathbf{S} \cdot \mathbf{M} = 0$ and that under this condition every value of $\mathbf{a}$ is given by,

$$\mathbf{a} = \frac{\mathbf{S} \times \mathbf{M}_O}{\mathbf{S}^2} + k\,\mathbf{S},$$

where k is an arbitrary scalar. Consequently under our hypotheses the vector $\mathbf{a}$ exists and the locus of the possible positions of the point A being a line parallel to $\mathbf{S}$, it appears that the sliding vector equivalent to the given system is unique.

III. $\mathbf{M} \neq 0, \mathbf{S} = 0$. In this case the given system is by definition a couple. We have seen that an elementary couple may always be formed having any given moment $\mathbf{M}$ and consequently our given couple may always be reduced by elementary operations to an elementary couple.

IV. $\mathbf{M} = 0, \mathbf{S} = 0$. In this case the given system is by definition a null system and evidently may be reduced by elementary operations so that no sliding vectors remain.

EXERCISES

1. Show that any system Σ of sliding vectors for which $\mathbf{S} \cdot \mathbf{M} \neq 0$ may be reduced by elementary operations to a single sliding vector and an elementary couple in which either a point on the line of action of the single sliding vector or the plane of the elementary couple may be chosen at random, provided that this plane is not chosen parallel to the sum $\mathbf{S}$ of the system.
2. Show that any system Σ of sliding vectors for which $\mathbf{S} \neq 0$ may be reduced by elementary operations to two sliding vectors perpendicular to each other.
3. Show that a system of coplanar sliding vectors may always be reduced by elementary operations to a single sliding vector or to an elementary couple.

60. Central Axis of a System.

Being given a system Σ of sliding vectors for which the sum $\mathbf{S}$ is not zero, let us seek to locate a point A such that the moment $\mathbf{M}_A$ of the system with respect to A is parallel to $\mathbf{S}$, i.e. $\mathbf{M}_A \times \mathbf{S}$

$= 0$. By equation § 57, (1) we have,

$$(1) \qquad \mathbf{M}_A = \mathbf{M}_O - \mathbf{a} \times \mathbf{S}, \qquad \mathbf{a} = OA,$$

so that the condition $\mathbf{M}_A \times \mathbf{S} = 0$ which we are imposing on A becomes $\mathbf{M}_O \times \mathbf{S} + \mathbf{S} \times (\mathbf{a} \times \mathbf{S}) = 0$ or,

$$(2) \qquad \mathbf{M}_O \times \mathbf{S} + \mathbf{S}^2\, \mathbf{a} - (\mathbf{a} \cdot \mathbf{S})\, \mathbf{S} = 0.$$

Clearly one solution of this equation in $\mathbf{a}$ is,

$$\mathbf{a} = \frac{\mathbf{S} \times \mathbf{M}_O}{\mathbf{S}^2}$$

and furthermore if $\mathbf{a}_1$ and $\mathbf{a}_2$ are any two solutions, then by substitution and subtraction we have,

$$\mathbf{S}^2\,(\mathbf{a}_2 - \mathbf{a}_1) = \{(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{S}\}\, \mathbf{S},$$

and $\mathbf{a}_2 - \mathbf{a}_1$ must be parallel to $\mathbf{S}$. It is thus clear that every solution of equation (2) is of the form,

$$(3) \qquad \mathbf{a} = \mathbf{S} \times \mathbf{M}_O + \lambda\, \mathbf{S},$$

and on substituting in the equation we find that formula (3) gives a solution of equation (2) for every choice of the scalar λ. The locus of the possible positions of A is thus a straight line parallel to $\mathbf{S}$ and hence to $\mathbf{M}_A$. If we assign the sense of $\mathbf{M}_A$ to this line, the axis thus formed is known as the *central axis* of the system Σ.

The significance of this name appears in the following theorem. *The moment of a given system Σ of sliding vectors with respect to any point P of space is the same as the velocity which that point would have if it were a point of a rigid body rotating about the central axis with an angular velocity* $\mathbf{S}$ *and simultaneously sliding along the central axis with a velocity* $\mathbf{M}_A$*, where A is any point of the central axis.*

To see this we first observe that since the moment $\mathbf{M}_A$ for any point A of the central axis is parallel to $\mathbf{S}$, we may write,

$$(4) \qquad \mathbf{M}_A = k\, \mathbf{S} \qquad \text{where} \qquad k \quad \frac{\mathbf{M} \cdot \mathbf{S}}{\mathbf{S}^2}$$

the scalar k being the ratio of two invariants and consequently an invariant independent of the point about which moments are taken. If we substitute this value of $\mathbf{M}_A$ in equation (1) and

subtract the resulting equation member for member from equation § 57, (1), we have the relation,

$$\mathbf{M}_P = k\,\mathbf{S} + \mathbf{S} \times (\mathbf{p} - \mathbf{a}). \tag{5}$$

On the other hand, formula § 48, (7),

$$\mathbf{p}' = k\,\boldsymbol{\omega} + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a}),$$

gives the velocity $\mathbf{p}'$ of any point P of a rigid body which is rotating with an angular velocity $\boldsymbol{\omega}$ about an axis through A and simultaneously sliding along this axis with a velocity $k\,\boldsymbol{\omega}$. On comparing these last two equations we see that the theorem is established.

It is not difficult to get a clear mental picture of the velocities of the various points of a rigid body which is thus rotating about and sliding along the axis of Mozzi and by the aid of the above theorem this picture now enables us to visualize the moments of a system of sliding vectors with respect to the various points of space. The following three theorems become in fact almost self evident.

The moment of the system Σ is the same with respect to all points of any line parallel to the central axis.

For if P and Q are any two points of such a line then $\mathbf{p} - \mathbf{q}$ is parallel to $\mathbf{S}$ and on evaluating $\mathbf{M}_P$ and $\mathbf{M}_Q$ by formula (5) and subtracting we find $\mathbf{M}_P - \mathbf{M}_Q = \mathbf{S} \times (\mathbf{p} - \mathbf{q}) = 0$ and the theorem follows.

The moment of the system Σ with respect to any point P is perpendicular to the vector $\mathbf{r}$ *running perpendicularly from the central axis to P.*

By the result of § 22, Prob. 6 this vector $\mathbf{r}$ has the value,

$$\mathbf{r} = \frac{\mathbf{S} \times \{(\mathbf{p} - \mathbf{a}) \times \mathbf{S}\}}{\mathbf{S}^2}$$

and if we form the dot-product of this with the value of $\mathbf{M}_P$ as given by equation (5) we have,

$$\mathbf{r} \cdot \mathbf{M}_P = \frac{k}{\mathbf{S}^2}\,[\mathbf{S}, \mathbf{S}, \{(\mathbf{p} - \mathbf{a}) \times \mathbf{S}\}]$$
$$- \frac{1}{\mathbf{S}^2}\,[\{(\mathbf{p} - \mathbf{a}) \times \mathbf{S}\}, \mathbf{S}, \{(\mathbf{p} - \mathbf{a}) \times \mathbf{S}\}] = 0,$$

and the theorem follows.

The length $|\mathbf{M}_P|$ *of the moment of a given system with respect to a given point P depends only on the distance r from P to the central axis of the system. It is shortest for points A on the central axis.*

By equation § 22, (10) the distance r from P to the central axis is,

$$r = |(\mathbf{p} - \mathbf{a}) \times \mathbf{S}|/|\mathbf{S}|$$

and consequently,

$$r^2 = \{(\mathbf{p} - \mathbf{a}) \times \mathbf{S}\}^2/\mathbf{S}^2.$$

If we square both members of equation (5) we have,

$$\mathbf{M}_P^2 = k^2\,\mathbf{S}^2 + \{(\mathbf{p} - \mathbf{a}) \times \mathbf{S}\}^2$$

and on introducing the above value of r^2 this becomes,

$$\mathbf{M}_P^2 = (k^2 + r^2)\,\mathbf{S}^2,$$

or,

$$|\mathbf{M}_P| = \sqrt{k^2 + r^2}\,|\mathbf{S}|,$$

from which our theorem is obvious.

In the case of a system for which $\mathbf{S}\cdot\mathbf{M} = 0$, $\mathbf{S} \neq 0$, we saw in the preceding article that the system is equivalent to a unique single sliding vector. The formula for the determination of a point A of this vector is identical with our formula (3) for the determination of a point A on the central axis. Thus the unique sliding vector equivalent to the system lies on the central axis.

EXERCISE

1. Describe a construction for the central axis of a system of sliding vectors when the sum $\mathbf{S}$ and moment $\mathbf{M}_O$ are given with $\mathbf{S} \neq 0$.

61. Lines of Zero Moment.

We saw in a theorem of the preceding article that the moment of a system of sliding vectors with respect to a point P is perpendicular to the line through P intersecting the central axis at right angles. We may put this theorem in another form by saying that the moment of the system with respect to any line which intersects the central axis at right angles is zero.

The question is thus raised as to the location of all the lines such that the moment of the system with respect to them is zero. Such a line is called a *line of zero moment*. Due to the invariance of the moments of the system under a sliding of the system along the central axis and a rotation of the system about the central axis we need ultimately only locate such a line by its distance r

from the central axis and the angle θ which it makes with the central axis. At first however we may determine the line by a unit vector $\mathbf{u}$ along it and the vector $\mathbf{r}$ running from the central axis to the line along their common perpendicular. If A is any point of the central axis and P is any point of the line under consideration, then if the line is to be a line of zero moment we must have $\mathbf{u}\cdot\mathbf{M}_P = 0$. For $\mathbf{S} \neq 0$ we may employ equation (5) of the preceding article to write this in the form,

$$(1)\qquad 0 = \mathbf{u}\cdot\mathbf{M}_P = \mathbf{u}\cdot(k\,\mathbf{S} + \mathbf{S}\times\mathbf{r}) = \mathbf{S}\cdot(k\,\mathbf{u} + \mathbf{r}\times\mathbf{u}).$$

Now $\mathbf{S}$ and $\mathbf{u}$ are two vectors perpendicular to $\mathbf{r}$ and we shall indicate by θ the angle from $\mathbf{S}$ to $\mathbf{u}$ in the positive sense of rotation relative to $\mathbf{r}$. Then $\mathbf{r}\times\mathbf{u}$ is also perpendicular to $\mathbf{r}$ and the angle from $\mathbf{S}$ to $\mathbf{r}\times\mathbf{u}$ in this sense is $\pi/2 + \theta$. Equation (1) thus takes the form,

$$(2)\qquad 0 = |\mathbf{S}|\,(k\cos\theta - r\sin\theta).$$

For systems in the first category of reduction discussed in article § 59 we have $\mathbf{S}\cdot\mathbf{M} \neq 0$ and since k and $|\mathbf{S}|$ are then not zero it is evident that our equation will be satisfied only for $r = \cos\theta = 0$ or for $\tan\theta = k/r$. Thus the lines of zero moment intersecting the central axis are perpendicular to it and as their distance r from the central axis increases the angle θ which they make with it gradually diminishes, approaching the limit zero as r becomes infinite.

For systems in the second category of reduction and hence reducible to a unique sliding vector we have $\mathbf{S}\cdot\mathbf{M} = 0$, $\mathbf{S} \neq 0$. Equation (2) still holds but k is here zero and the equation is evidently satisfied only if $r = 0$ or $\theta = 0$. Thus the lines of zero moment constitute all the lines intersecting the central axis and all the lines parallel to it, or in other words they are the lines which are individually coplanar with the central axis. For couples we have $\mathbf{S} = 0$ and equation (2) is not applicable, but since the moment $\mathbf{M}$ of the system is now an invariant, the same with respect to all points of space, it is evident that the lines perpendicular to $\mathbf{M}$ form all the lines of zero moment. For a null system it is of course clear that every line is a line of zero moment.

62. Systems of Parallel Sliding Vectors.

A system of sliding vectors which are all parallel possesses certain peculiar properties and has certain special applications. In

this case we may take a unit vector $\mathbf{u}$ parallel to the vectors of the system and express each vector $\mathbf{l}_i$ of the system in the form,

$$(1) \qquad \mathbf{l}_i = l_i \, \mathbf{u},$$

where l_i is a positive or negative scalar. Then the sum $\mathbf{S}$ of the system is given by,

$$\mathbf{S} = l_1 \mathbf{u} + l_2 \mathbf{u} + \cdots + l_n \mathbf{u} = (l_1 + l_2 + \cdots + l_n)\, \mathbf{u}$$

or

$$(2) \qquad \mathbf{S} = l_0 \, \mathbf{u} \qquad \text{where} \qquad l_0 = \sum_{i=1} l_i.$$

We shall now assume that the sum $\mathbf{S}$ is not zero and that consequently l_0 is not zero. If A_i is any point on the line of action λ_i of the sliding vector $\mathbf{l}_i$ then for the moment $\mathbf{M}_O$ of the system with respect to O we have,

$$\mathbf{M}_O = \sum_{i=1}^{n} (\mathbf{a}_i \times l_i \, \mathbf{u}) = \left(\sum_{i=1}^{n} l_i \, \mathbf{a}_i \right) \times u, \qquad \mathbf{a}_i = OA_i,$$

or

$$(3) \qquad \mathbf{M}_O = l_0 \, \mathbf{a}_0 \times \mathbf{u},$$

where

$$(4) \qquad \mathbf{a}_0 = \left(\sum_{i=1}^{n} l_i \, \mathbf{a}_i \right) / l_0.$$

It is evident from equations (2) and (3) that $\mathbf{S} \cdot \mathbf{M}_O = 0$ and since $\mathbf{S}$ is not zero the system must be reducible to a single sliding vector. In fact it is immediately apparent from equations (2) and (3) that we have,

$$(5) \qquad \mathbf{M}_O = \mathbf{a}_0 \times \mathbf{S},$$

so that,

The system Σ of parallel sliding vectors is equivalent to a single sliding vector $\mathbf{S}$ *with line of action through the point* A_0 *where*

$$OA_0 = \mathbf{a}_0.$$

The particular point A_0 given by formula (4) is of course also on the central axis of the system and since formula (4) in no way involves the unit vector $\mathbf{u}$ it follows that A_0 would still be on the central axis of the system if $\mathbf{u}$ were allowed to take on any vector

value. In fact this point A_0 is not uniquely determined by the given system of sliding vectors but depends on the particular points A_i chosen on their lines of action and on the corresponding scalars l_i. The point A_0 with the associated scalar l_0 is known as the *centroid* of the system consisting of the points A_i and their associated scalars l_i, the relations being as noted above,

$$(6) \qquad l_0 \, \mathbf{a}_0 = \sum_{i=1}^{n} l_i \, \mathbf{a}_i, \qquad l_0 = \sum_{i=1}^{n} l_i.$$

A fundamental property of the centroid is the following:

The centroid of a system of points and associated scalars is not altered if any sub-system chosen from the given system is replaced by its centroid.

The proof is very simple. Let the given system consist of the points $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots, B_m$ with the corresponding scalars $l_1, l_2, \cdots, l_n, k_1, k_2, \cdots, k_m$. Then by definition the centroid C_0, h_0 of the entire system is given by the pair of equations,

$$(7) \qquad h_0 \, \mathbf{c}_0 = \sum_{i=1}^{n} l_i \, \mathbf{a}_i + \sum_{i=1}^{m} k_i \, \mathbf{b}_i, \qquad h_0 = \sum_{i=1}^{n} l_i + \sum_{i=1}^{m} k_i,$$

while the centroid of the sub-system A_i, l_i is determined by the pair of equations (6). By substitution from equations (6) we may reduce equations (7) to the form,

$$(8) \qquad h_0 \, \mathbf{c}_0 = l_0 \, \mathbf{a}_0 + \sum_{i=1}^{m} k_i \, \mathbf{b}_i, \qquad h_0 = l_0 + \sum_{i=1}^{m} k_i.$$

But these are exactly the equations for determination of the centroid of a system consisting of the points $A_0, B_1, B_2, \cdots, B_m$ with the associated scalars $l_0, k_1, k_2, \cdots, k_m$. The theorem is therefore established.

EXERCISE

1. Prove the *Law of the Lever*. Let three parallel coplanar sliding vectors $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ pass respectively through the three points A_1, A_2, A_3 which lie in this order on a straight line. A necessary and sufficient condition that this system form a null system is that $\mathbf{l}_3$ and $\mathbf{l}_1$ have the same sense and $\mathbf{l}_2$ the opposite sense and that,

$$\frac{l_1}{A_2 A_3} = \frac{l_2}{A_3 A_1} = \frac{l_3}{A_1 A_2},$$

where $l_i = |\mathbf{l}_i|$.

CHAPTER VII

STATICS OF PARTICLES AND OF A RIGID BODY

63. Statics of a Particle.

If we consider a set of one or more particles each of which may be acted upon by one or more forces, then if all the particles remain at rest we say that the set of particles is in *equilibrium* under the action of the system of forces. The study of the systems of forces which produce equilibrium in sets of particles constitutes the branch of theoretical mechanics known as *statics*. Since no motion takes place in equilibrium the idea of time does not enter explicitly into the study of statics but time is tacitly involved since it appears in the definition of force which we have employed.

We shall first consider the statics of a single particle. In § 36 we saw that in the interpretation of Newton's first and second laws of motion we regard the forces acting on a particle as vectors and subject to vector addition. Thus if forces $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ act simultaneously on a particle, it is their vector sum $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \cdots + \mathbf{f}_n$ which is employed in the equation of motion $m\,\mathbf{j} = \lambda\,\mathbf{f}$. Since $\mathbf{j}$ is the time derivative of the velocity $\mathbf{v}$ of the particle it is clear that the velocity remains constant only when $\mathbf{f}$ is zero. Thus,

A single particle at rest is in equilibrium only if the total force acting on it is zero.

We shall see in § 66 that the force of gravity at the earth's surface acts on a material rigid body as if the body were a particle of the same mass called the *centroid* of the body. If the body has its matter uniformly distributed and possesses a point, line, or plane of symmetry, the centroid will be on that point, line, or plane. The fact that material rigid bodies behave in this respect like particles greatly extends the usefulness of the above theorem.

In addition to certain forces acting on a particle we may impose the condition that the particle remain on a given surface or curve. This may be approximated in a physical model by a

small sphere sliding between two slightly separated surfaces or by a bead sliding on a wire passing through it. Since this additional condition modifies the acceleration of the particle we regard it as introducing a force **k** called a *constraint* which is always just sufficient to maintain the particle on the given surface or curve. The other forces acting are in contradistinction called the *applied forces* **f**. If the surfaces or wire in the physical model were very smooth they would offer little resistance to the motion of the bead along them. Consequently when we speak of a particle as moving on a *smooth surface or curve* we imply that the constraint thus imposed is a force always normal to the surface or curve at the particle. We may now prove the theorem,

The necessary and sufficient condition that a particle constrained to a smooth surface or curve be in equilibrium under the action of an applied force **f** *is that* **f** *be normal to the surface or curve at the particle.*

For if the particle is in equilibrium the total force acting on it must be zero, i.e. $\mathbf{f} + \mathbf{k} = 0$, and **f**, like **k**, must be normal to the surface or curve. On the other hand if the particle is at rest but not in equilibrium, it must then move from the position of rest in the direction of the total force $\mathbf{f} + \mathbf{k}$ which can not then be zero. Thus if **t** is the unit tangent vector to the path of the particle at the point considered, we have $\mathbf{t}\cdot(\mathbf{f} + \mathbf{k}) \neq 0$ since **t** and $\mathbf{f} + \mathbf{k}$ are parallel non-vanishing vectors. But **k** is normal to the surface or curve while the path of the particle is along it and consequently we have $\mathbf{t}\cdot\mathbf{k} = 0$. But then $\mathbf{t}\cdot\mathbf{f} \neq 0$ and **f** is not normal to the surface or curve. It follows that if **f** is normal to the surface or curve, the particle must be in equilibrium for if it were not **f** could not be normal to the surface or curve. The condition is thus sufficient.

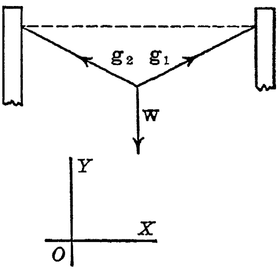

FIG. 55

We illustrate the application of these theorems by a few simple examples.

A wet sweater weighing 11 pounds is hung in the center of a 30 foot clothes line and causes it to sag $2\frac{3}{4}$ feet. What is the tension in the line?

Representing the weight of the sweater by **W** and the tension in the left and right halves of the line by $\mathbf{g}_1$ and $\mathbf{g}_2$ respectively,

we have for equilibrium,

$$\mathbf{g}_1 + \mathbf{g}_2 + \mathbf{W} = 0.$$

We represent by θ_1 and θ_2 the angles which $\mathbf{g}_1$ and $\mathbf{g}_2$ make with the X-axis as drawn and by g_1, g_2, W the lengths of the corresponding vectors. Measuring the above equation on the two coördinate axes gives us,

$$g_1 \cos \theta_1 + g_2 \cos \theta_2 = 0, \qquad g_1 \sin \theta_1 + g_2 \sin \theta_2 - W = 0,$$

while from the figure it is evident that,

$$-\tan \theta_1 = \tan \theta_2 = \frac{11}{60}.$$

Solving we have,

$$g_1 = g_2 = \frac{61}{22} W = 30.5$$

A right circular cylinder of weight W rests between two smooth planes which make angles of 30° and 60° with the horizontal and 30° with each other, the axis of the cylinder being horizontal. Find the amount of the force exerted by each plane on the cylinder.

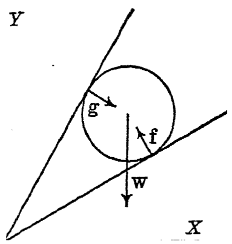

Fig. 56

There are three forces acting on the cylinder. The force of gravity $\mathbf{W}$ acts as if the cylinder were replaced by its centroid, which, since the cylinder is assumed to be homogeneous, is a particle of the same weight located at the midpoint of the axis. There are also the forces $\mathbf{f}$ and $\mathbf{g}$ exerted by the two inclined planes which, since the planes are smooth, will act at right angles to these planes. The condition for equilibrium is then,

$$\mathbf{f} + \mathbf{g} + \mathbf{W} = 0.$$

Measuring this equation on the two coördinate axes in the figure we find the two scalar equations,

$$-\frac{1}{2}f + \frac{\sqrt{3}}{2}g = 0, \qquad \frac{\sqrt{3}}{2}f - \frac{1}{2}g - W = 0,$$

where f, g, W are the lengths of the respective forces. These

yield,

$$f = \sqrt{3}\, W, \qquad g = W.$$

A step ladder has two equal sections of length 50 inches each hinged together at one end and joined at their other ends by a brace 28 inches long. If a 144 pound man stands on the top of the ladder and the ladder rests on a smooth floor, what will be the tension in the brace if we neglect the weight of the ladder?

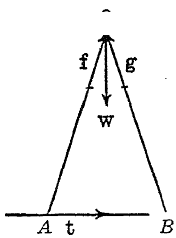

FIG. 57

Let us indicate by **f** and **g** the compression in the two sections of the ladder and by **W** the weight of the man while **t** is the tension in a certain sense along the brace. For equilibrium at the hinge O we must have,

$$\mathbf{f} + \mathbf{g} + \mathbf{W} = 0,$$

where **f** and **g** act upward along their sections. Measuring this equation on horizontal and vertical axes in the plane of the figure gives us,

$$\frac{7}{25} f - \frac{7}{25} g = 0, \qquad \frac{24}{25} f + \frac{24}{25} g - W = 0,$$

which yield,

$$f = g = \frac{25}{48} W.$$

But we must also have equilibrium at the bottom end A of a section of the ladder. Here we have the force $-\mathbf{f}$ acting along a section of the ladder, the tension **t** acting along the brace and the pressure **h** of the floor acting upward because the floor is smooth. Hence we have,

$$-\mathbf{f} + \mathbf{t} + \mathbf{h} = 0,$$

and measuring this equation on the coördinate axes gives,

$$\frac{7}{25} f + t = 0, \qquad -\frac{24}{25} f + h = 0.$$

Hence,

$$t = \frac{7}{48} W, \qquad h = \frac{1}{2} W.$$

Thus the tension in the brace is, $t = \frac{7}{48} W = 21$ pounds.

EXERCISES

1. A stone weighing 80 pounds hangs by a chain 15 feet long and is pulled 9 feet from the vertical by a horizontal rope attached to it. What are the tensions in the chain and the rope? *Ans.* 100, 60

2. A weight W is attached at a point C of a cord whose ends A and B are fixed. If the two parts of the cord make angles α and β with the horizontal, find the amounts a and b of the tensions in them.

$$Ans. \quad a = \frac{\sin \beta}{\sin (\alpha + ,} W, \qquad b = \frac{\sin \alpha}{\sin (\alpha + \beta)} W$$

A cord with equal weights W attached at its ends is hung over two smooth pegs A and B fixed in a vertical wall. Find the vector pressures on the pegs when AB makes an angle θ with the horizontal.
Ans. Pressures make angles α and β with horizontal and are of amounts a and b where,

$$= -45^\circ + \theta/2, \quad \beta = 225^\circ + \theta/2, \quad a = 2W \cos (45^\circ + \theta/2),$$
$$b = 2W \cos (45^\circ - \theta/2)$$

4. Let $AB = c$ be the vertical mast and $AC = b$ be the jib of a crane, the ends B and C being connected by a chain of length a. If a weight W be suspended from C, find the tension T in the chain and the thrust S in the jib. *Ans.* $S = (b/c)W$, $T = (a/c)W$

5. In the crane of Problem 4 if the jib be hinged at A and the weight be supported by a cable passing over a pulley at C and attached at B, in what position of the jib will there be equilibrium?
Ans. $a = c$

6. Three rods 13 feet, 20 feet, and 21 feet in length are hinged together to form a triangle which is set up in a vertical plane with the 21 foot side resting on a smooth horizontal floor. If a weight of 63 pounds is placed at the top, find the tension or compression in each rod. *Ans.* 52, 25, 20

7. Prove *Lami's Theorem.* If three forces acting on a particle are in equilibrium, the amount of each force is proportional to the sine of the angle formed by the other two.

8. Three forces **a**, **b**, **c** act on a particle, the angle opposite **a** being α, opposite **b** being β, and opposite **c** being γ. Prove analytically that the length S of their sum is given by the formula,

$$S^2 = a^2 + b^2 + c^2 + 2bc \cos \alpha + 2ca \cos \beta + 2ab \cos \gamma.$$

9. A tripod is formed with the legs $OA = a$, $OB = b$, $OC = c$, rigidly attached to each other at O. The three feet A, B, C are set on a horizontal plane and a weight W is placed at O. Show that the amount of the force in each leg is proportional to the product of the length of that leg and the area of the projection on the hori-

zontal of the triangle formed by the other two legs. Also show that the amount of the force in each leg is proportional to the product of the sine of the angle between the other two legs and the cosine of the angle which their plane makes with the horizontal.

10. If the legs of the tripod in the above problem are of equal length, show that the amount of the force in each leg is proportional to the sine of twice the corresponding angle of the triangle ABC.

11. A cylinder of weight W rests on two smooth inclined planes whose intersection is horizontal and parallel to the axis of the cylinder. If the planes make angles α and β respectively with the horizontal, find the pressure on each plane.

$$\textit{Ans.}\quad f = \frac{\sin\beta}{\sin(\alpha+\beta)}\,W, \qquad g = \frac{\sin\alpha}{\sin(\alpha+\beta)}\,W$$

64. Statics of a Set of Particles.

We have seen that a necessary and sufficient condition that a single particle at rest be in equilibrium is that the sum of all the forces acting upon it be zero. If we now consider a set consisting of several particles, the set remains at rest when and only when every particle of it remains at rest, so that it is clear that,

A set of particles at rest is in equilibrium only if the total force acting on each particle is zero.

To determine when this occurs it is necessary to know to which particle each force is applied. In other words the forces are no longer completely characterized by the free vectors previously employed for a single particle, but each force must be regarded as an attached vector, i.e. a vector associated with a definite point. If we then represent each force acting on a particle of the set by a vector attached to that point, we shall have equilibrium when and only when each sum obtained by adding together the vectors attached to the same point is zero.

The forces acting on any one particle of the set may be divided into two classes. Firstly we have the *exterior forces* which are regarded as imposed upon the particle from outside the set. These forces, which we shall represent by the letter $\mathbf{f}$, may be quite arbitrary in every respect. There are also the forces exerted on each particle which we regard as due to the presence of the other particles of the set. As examples of such forces we may think of the mutual attraction or repulsion of the particles due to their gravitation, electric charges upon them, their magnetic properties or to light emitted by them or even to their

being in actual contact. We call these the *interior forces* of the set and represent them by the letter $\mathbf{g}$. An important theorem concerning the forces of this class is,

The interior forces of a set of particles constitute a null system of sliding vectors.

This theorem, which holds whether the set is in equilibrium or not, is an immediate consequence of Newton's third law which states in effect (§ 2) that forces always occur in pairs, the two forces being equal in magnitude and acting in opposite senses along the same straight line. Thus if the force exerted by the particle P_2 on the particle P_1 is $\mathbf{g}_1$ and the force exerted by P_1 on P_2 is $\mathbf{g}_2$, then we have,

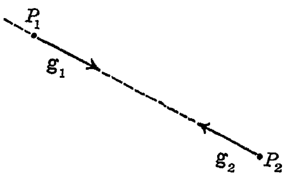

FIG. 58

$$\mathbf{g}_1 + \mathbf{g}_2 = 0.$$

Since this holds for every pair of interior forces of the system the sum $\mathbf{T}$ of all interior forces must be zero. Furthermore since $\mathbf{g}_1$ and $\mathbf{g}_2$ as sliding vectors have the same line of action through P_1 and P_2 we may, using elementary operations, slide one along the line until it has the same point of application as the other and then replace them by their sum zero. Thus by elementary operations the entire system of interior forces can be reduced to a system containing no sliding vectors and is by definition a null system. Consequently the moment $\mathbf{N}_O$ of the interior forces with respect to any point O is zero. We may now prove the important theorem,

A necessary condition for the equilibrium of a set of particles is that the exterior forces form a null system of sliding vectors.

Let us consider the set of particles $P_1, P_2, \cdots, P_n$ and indicate by $\mathbf{f}_i$ and $\mathbf{g}_i$ respectively the sum of all the exterior forces and of the interior forces acting on the particle P_i. If P_i is to remain at rest we must have,

$$\mathbf{f}_i + \mathbf{g}_i = 0, \tag{1}$$

and if we sum this equation for all particles of the set we have,

$$\mathbf{S} + \mathbf{T} = 0 \qquad \text{where} \qquad \mathbf{S} = \Sigma\, \mathbf{f}_i, \quad \mathbf{T} = \Sigma\, \mathbf{g}_i.$$

But we have just observed that $\mathbf{T} = 0$ and consequently we now have $\mathbf{S} = 0$. Now let O be any point and cross-multiply both

members of equation (1) by $\mathbf{p}_i = OP_i$, obtaining $\mathbf{p}_i \times \mathbf{f}_i + \mathbf{p}_i \times \mathbf{g}_i = 0$. If we sum this equation for all particles of the set we have,

$$\mathbf{M}_O + \mathbf{N}_O = 0 \qquad \text{where} \qquad \mathbf{M}_O = \Sigma\, \mathbf{p}_i \times \mathbf{f}_i, \quad \mathbf{N}_O = \Sigma\, \mathbf{p}_i \times \mathbf{g}_i.$$

But we have just observed that $\mathbf{N}_O = 0$ and consequently we now have $\mathbf{M}_O = 0$. Since then both the sum $\mathbf{S}$ and the moment $\mathbf{M}_O$ of the exterior forces are zero the system of exterior forces is a null system.

If we are to have equilibrium in any given set of particles it is clear that we must also have equilibrium in every sub-set consisting of a portion of the particles of the original set. The condition proven necessary for equilibrium in the above theorem consequently also holds for the exterior forces of every sub-set of a set which is in equilibrium and if it does hold for every sub-set it is sufficient for the equilibrium of the given set. For clearly we might take as sub-sets the individual particles of the given set and these would obviously be in equilibrium if the forces acting on each of them formed a null system.

We shall see in the next article that the condition upon the exterior forces here proven *necessary* for the equilibrium of any set of particles will also be *sufficient* in case the particles constitute a rigid body. From this point of view the theorem proven above is often known as,

The Principle of Rigidification. If any set of particles is in equilibrium under the action of a system of exterior forces, it will continue in equilibrium under these exterior forces if the interior forces are so altered as to render the set rigid.

For if the set is in equilibrium under the action of certain exterior forces, these forces must constitute a null system and we shall see that this is sufficient for the equilibrium of a rigid body. Or in other words,

Any set of conditions upon the exterior forces which are necessary for the equilibrium of a rigid body are also necessary for equilibrium if this body is not rigid.

For if this were not so there would exist systems of exterior forces not satisfying these conditions for which the body would be in equilibrium when not rigid and not in equilibrium when rigid, contrary to the statement just proven.

The principle seems intuitively evident. For example, if a flexible cotton cord be dipped in melted wax and hung up by its

ends while still hot, we know that its curve of equilibrium will not be greatly altered as it cools and stiffens.

EXERCISES

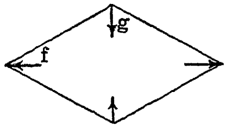

FIG. 59

A toggle joint press is formed by hinging four rods of length 53 inches in the form of a rhombus. If two opposite vertices are 9.2 inches apart and are drawn together with a force of 23 pounds, with what force are the other two vertices pushed apart?

Ans. 264 pounds

2. In the braced bridge beam shown the rods are of equal length and negligible weight and freely pivoted at the joints. If a load of 1000 pounds be placed at the mid-point, determine the stress in each rod and whether it is a tension or compression.

Ans. $f = 577.4$, $g = 577.4$,
$h = 288.7$, $k = 577.4$

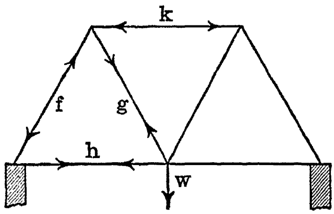

FIG. 60

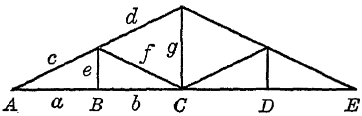

FIG. 61

3. Find the stresses in each member of the braced beam AE carrying a load of 10 tons at each of the points B, C, D. The panels are 12 feet wide and the middle vertical is 10 feet long.

Ans. $= 36$, $b = 36$, $c = 39$, $d = 26$, $f = 13$, $g = 20$

4. A light smooth cord of length 12 has one end fastened at the origin O and the other at the point $D \equiv (0, 2, 4)$. Three beads A, B, C are free to slide on the cord and are arranged in the order O, A, B, C, D. The following forces are applied to the beads.

$$\mathbf{f}_A \equiv (2, 0, 0), \qquad \mathbf{f}_B \equiv (0, 4, 0), \qquad \mathbf{f}_C \equiv (0, 0, 4).$$

Find the position of equilibrium of the beads and the tension g in the cord.

Ans. $A \equiv (2, 4, 4)$, $g = 3$

5. A heavy circular ring is suspended from a point by equal symmetrically placed smooth light cords and a smaller ring of equal weight rests on the cords at their mid-points. Show that the plane of the smaller ring is 2/3 as far from the point of support as the plane of the larger ring.

65. Statics of a Rigid Body.

As a special case of a set of particles we have the rigid body which we have defined as a set of particles such that the distance

between every pair of them remains constant for all motions of the set. The theorem of the preceding article is of course here applicable and we know that a necessary condition for equilibrium is that the exterior forces should form a null system of sliding vectors. We shall show that for a rigid body this condition is also sufficient. The proof of this theorem rests on four principles as follows:

1. *The motion of a rigid body is not affected if two forces are applied to it having the same magnitude and acting in opposite senses along the same line.*

We shall give no proof of this principle, regarding it as an additional postulate of the rigid body. It would seem to be a natural consequence of the fact that the distance between every pair of particles is unalterable and there seems to be no reason why a proof could not be given, although it might well be very complicated.

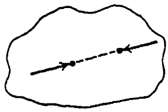

FIG. 62

2. *The motion of a rigid body is not affected if any force applied to it is moved along its line of action to a new point of application.*

This is a consequence of Principle 1. For suppose a force **f** to be applied to a point A of the body and we wish to move it along its line of action to a point B. We may bring this about without affecting the motion of the body by applying a force **f** at B and a force $-$ **f** at A as permitted by Principle 1. The forces **f** and $-$ **f** at A then annul each other and the desired effect has been produced.

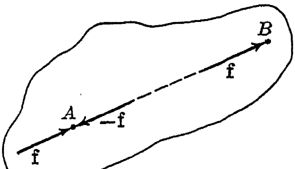

FIG. 63

3. *If two or more forces acting on a rigid body have their lines of action concurrent they may, without affecting the motion of the body, be replaced by a force which is their vector sum and concurrent with them.*

This is a consequence of Principle 2 and of the fact implied in Newton's second law that forces acting on a particle are subject to vector addition. For by Principle 2 the concurrent forces may all be shifted along their lines of action to the point of concurrence and since they then act on the same particle they may be replaced by their sum and this single force may then be shifted to any point along its line of action.

4. *Any force acting on a rigid body may, without affecting the motion of the body, be replaced by others concurrent with it and of which it is the sum.*

This is an immediate consequence of Principle 3, for the operation described in Principle 4 could be annulled by a properly chosen operation of Principle 3 and since the latter does not affect the motion of the body the former could not.

Principle 2 states in effect that forces acting on a rigid body are completely characterized by giving their values as sliding vectors. For although a sliding vector has merely a line of action without a specific point of application, still by Principle 2 it is not necessary in determining the effect of a force on a rigid body to specify the point of application provided the line of action is given. If we then represent by sliding vectors the various forces acting on a rigid body, we may apply to these forces all the discussion carried through in Chapter VI concerning sliding vectors. Principles 3 and 4 are then seen to state specifically that the elementary operations of sliding vectors performed on the forces acting on a rigid body do not affect the motion of that body. Since any two equivalent systems of sliding vectors are interchangeable by means of the elementary operations we now have the important general principle,

Two equivalent systems of forces acting on a rigid body have the same effect on its motion.

We may now prove the *principal theorem* on the statics of a rigid body.

A necessary and sufficient condition that a rigid body be in equilibrium is that the exterior forces constitute a null system of sliding vectors.

We have already seen that the condition is necessary. That it is also sufficient is at once apparent, for if the exterior forces form a null system we may replace them by a system consisting of no forces at all without affecting the motion of the body. And Newton's first law (§ 2) states that if a body at rest is acted upon by no forces, it will remain at rest. The equilibrium is thus established. It follows that if we represent by $\mathbf{S}$ and $\mathbf{M}_O$ the sum and moment with respect to O of the exterior forces acting on the rigid body, the necessary and sufficient condition for equilibrium is,

$$\mathbf{S} = \mathbf{M}_O = 0,$$

where the point O may be chosen arbitrarily.

In the light of the general principle stated above the categories of reduction, central axis of the system and lines of zero moment which were concepts developed for systems of sliding vectors may now be applied to the system of exterior forces acting on a rigid body. Some aspects of this application will appear in the following paragraphs.

In addition to applying forces to a rigid body, we may impose certain conditions upon its motion. Thus we may require that a certain point of the body remain fixed at a point in space. This additional condition modifies the accelerations of the particles of the body and thus in effect introduces an additional exterior force applied at the point which, as in the case of a single particle, we shall call the *constraint* $\mathbf{k}$. The other exterior forces are then called *applied forces* $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$. In the case here considered $\mathbf{k}$ is by hypothesis an exterior force acting on the body at a given point O and always keeping the particle of the body at O in equilibrium. We have now the theorem,

The necessary and sufficient condition that a rigid body with a fixed point O be in equilibrium is that the moment of the applied forces with respect to O be zero.

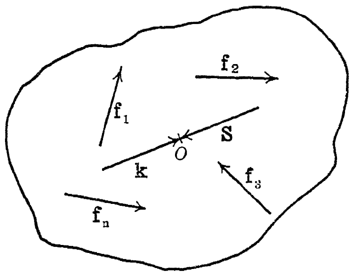

FIG. 64

Let us represent by $\mathbf{S}$ and $\mathbf{M}_O$ the sum and moment of the applied forces $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$. If the body is in equilibrium then by the general theorem the exterior forces $\mathbf{k}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$ must constitute a null system and their sum and moment with respect to O must both vanish. Thus we have

$$\mathbf{S} + \mathbf{k} = 0, \qquad \mathbf{M}_O = 0,$$

the moment of $\mathbf{k}$ with respect to O being zero. The condition $\mathbf{M}_O = 0$ is thus necessary for equilibrium of the body. This condition is also sufficient. For if we have $\mathbf{M}_O = 0$ and also $\mathbf{S} = 0$ the applied forces form a null system and we have equilibrium at once, no constraint being necessary, while if $\mathbf{M}_O = 0$ and $\mathbf{S} \neq 0$ the system of applied forces falls in Category II of § 59 since the invariant $\mathbf{S} \cdot \mathbf{M} = 0$. The applied forces are thus equivalent to a unique sliding vector $\mathbf{S}$ applied at O. If we now think of the

rigid body with no applied forces acting except this force $\mathbf{S}$ applied at O, it is clear that the constraint $\mathbf{k}$ must take the value $-\mathbf{S}$, since by hypothesis O remains at rest. Thus the sum $\mathbf{S}+\mathbf{k}$ of the exterior forces is zero and since their moment with respect to O is also zero they constitute a null system and the body is in equilibrium. The theorem is thus established.

Similarly we may show that,

The necessary and sufficient condition that a rigid body with a fixed axis Δ be in equilibrium is that the moment of the applied forces with respect to Δ be zero.

The requirement that the axis Δ remain fixed in effect introduces constraints exactly sufficient to maintain the axis at rest when the applied forces $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$ act on the body. It is clear that the axis will remain at rest if two of its points O and A remain at rest and we may regard this as being brought about by a constraint $\mathbf{k}$ acting at O and a constraint $\mathbf{l}$ acting at A. We shall as before represent by $\mathbf{S}$ and $\mathbf{M}_O$ the sum and the moment with respect to O of the applied forces $\mathbf{f}_1$, $\mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$. Then if the body is in equilibrium the exterior forces must, by the general theorem, form a null system and their sum and moment with respect to O must both vanish. Thus we have,

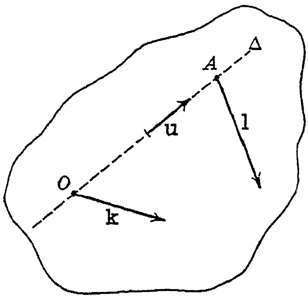

Fig. 65

$$\mathbf{S}+\mathbf{k}+\mathbf{l}=0, \qquad \mathbf{M}_O + OA \times \mathbf{l} = 0,$$

the moment of $\mathbf{k}$ with respect to O being zero. Now let $\mathbf{u}$ be a unit vector with the direction and sense of the fixed axis Δ and form the dot-product of $\mathbf{u}$ with the second of the above equations. Since $\mathbf{u}$ and OA are parallel this yields $\mathbf{u}\cdot\mathbf{M}_O = 0$ or,

$$M_\Delta = 0,$$

where M_Δ is the moment of the applied forces with respect to Δ. The condition stated in the theorem is thus necessary for equilibrium of the body.

This condition $M_\Delta = 0$ is also sufficient, for if $M_\Delta = \mathbf{u}\cdot\mathbf{M}_O = 0$, then since OA is parallel to $\mathbf{u}$ we also have $OA\cdot\mathbf{M}_O = 0$

and A is in the plane through O perpendicular to $\mathbf{M}_O$. This being the case, we may by Lemma II of § 58 reduce the applied forces to a system of two forces, one $\mathbf{S}_1$ applied at the point O and the other $\mathbf{S}_2$ applied at A. If we now think of the rigid body at rest with no applied forces acting except the force $\mathbf{S}_1$ at O and the force $\mathbf{S}_2$ at A, then it is clear that the constraints $\mathbf{k}$ and $\mathbf{l}$ must take such values that $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{k}$, $\mathbf{l}$ form a null system since the axis Δ by hypothesis remains at rest. But since $\mathbf{S}_1$ and $\mathbf{S}_2$ form a system equivalent to the applied forces $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$ it appears that $\mathbf{k}$ and $\mathbf{l}$ likewise form a null system with these applied forces and the rigid body must remain at rest. The theorem is thus established.

Instead of merely permitting the rigid body to rotate about the axis Δ we may also permit it to slide freely along this axis, in which case we say the body has a *sliding axis*. The body is not so fully constrained as when it had a fixed axis and it is therefore necessary to impose additional conditions upon the applied forces in order to insure the equilibrium of the body. We know that the conditions upon the applied forces which were necessary for equilibrium of the body with a fixed axis are still necessary for the body with a sliding axis, but the conditions upon the applied forces which were sufficient for equilibrium of the body with a fixed axis will no longer be sufficient for equilibrium of the body with a sliding axis. In fact we may show that,

The necessary and sufficient condition that a rigid body with a sliding axis Δ be in equilibrium is that the measure of the sum of the applied forces on the axis be zero and that the moment of the applied forces with respect to the axis be zero.

The requirement that Δ be a sliding axis for the rigid body in effect introduces constraints sufficient to maintain two points O and A of the body on this axis Δ. Let us suppose this done by a constraint $\mathbf{k}$ acting at O and a constraint $\mathbf{l}$ acting at A. Since these constraints are applied without friction it follows from our last theorem of § 63 that $\mathbf{k}$ and $\mathbf{l}$ are perpendicular to the axis Δ. If now as before we let $\mathbf{S}$ and $\mathbf{M}_O$ be the sum and the moment with respect to O of the applied forces $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \cdots, \mathbf{f}_n$ which act on the body, then if the body is in equilibrium all the exterior forces must form a null system and their sum and moment with respect to O must vanish. Thus we have,

$$\mathbf{S} + \mathbf{k} + \mathbf{l} = 0, \qquad \mathbf{M}_O + OA \times \mathbf{l} = 0,$$

the moment of $\mathbf{k}$ with respect to O being zero. If as before we represent by $\mathbf{u}$ a unit vector having the direction and sense of the axis Δ and form the dot-product of $\mathbf{u}$ with both of the above equations we find,

$$\mathbf{u} \cdot \mathbf{S} = 0, \qquad \mathbf{u} \cdot \mathbf{M}_O = M_\Delta = 0,$$

it being remembered that $\mathbf{k}$ and $\mathbf{l}$ are perpendicular to $\mathbf{u}$ and that OA is parallel to $\mathbf{u}$. The conditions stated in the theorem are thus necessary for equilibrium.

These conditions are also sufficient. For if $\mathbf{u} \cdot \mathbf{M}_O = 0$, then since $\mathbf{u}$ is parallel to OA we also have $OA \cdot \mathbf{M}_O = 0$ and A is in the plane through O perpendicular to $\mathbf{M}_O$. This being the case, we may by Lemma III of § 58 reduce the applied forces to a force $\mathbf{S}_1$ applied at O and a force $\mathbf{S}_2$ applied at A where the force applied at A is perpendicular to OA. Since in this case $\mathbf{u} \cdot \mathbf{S} = 0$ it follows that $\mathbf{S}_1$ is likewise perpendicular to the axis Δ. If we now think of the rigid body as at rest with no applied forces acting except $\mathbf{S}_1$ at O and $\mathbf{S}_2$ at A, then the axis of the body is acted upon by the forces $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{k}$, $\mathbf{l}$ all of which are perpendicular to Δ and can consequently produce no motion of the axis of the body along Δ, while by hypothesis $\mathbf{k}$ and $\mathbf{l}$ maintain the axis of the body on the fixed axis Δ. Thus the axis of the body can not move at all and the forces $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{k}$, $\mathbf{l}$ must form a null system. But then $\mathbf{S}$, $\mathbf{k}$, $\mathbf{l}$ must likewise form a null system and the whole rigid body is in equilibrium. The theorem is thus established.

In both of the last two theorems it was merely for clarity in the proof that we considered the constraints as being applied at definite points O and A of the axis. The essential feature was merely that in the first theorem the constraints applied to the axis Δ held it fixed while in the second the constraints applied to the axis acted at right angles to it.

If a rigid body moves so that a plane fixed in the body slides freely upon a plane π fixed in space, we say that the rigid body has a *sliding plane*. Since we think of the plane of the body as sliding without friction on the fixed plane π it follows that the constraints which keep the two planes in coincidence constitute a system of forces with points of application all on the sliding

plane and all of them perpendicular to it. We might think of these constraints as reduced to a system of three forces having non-collinear points of application on the plane π, but it will be sufficient in deriving the conditions of equilibrium to consider the system of constraints as a whole and observe that since the constraints are all perpendicular to the plane π, it follows that the sum $\mathbf{T}$ of the system of constraints is also perpendicular to π and their moment $\mathbf{N}_O$ with respect to any point O is parallel to π. We now have the theorem,

The necessary and sufficient condition that a rigid body with a sliding plane π be in equilibrium is that the sum of the applied forces be perpendicular to the plane and the moment of the applied forces with respect to any axis which is perpendicular to the plane be zero.

For if the body be in equilibrium under the action of the constraints mentioned above together with a system of applied forces having a sum $\mathbf{S}$ and a moment $\mathbf{M}_O$, then by the general theorem we must have,

$$\mathbf{S} + \mathbf{T} = 0, \qquad \mathbf{M}_O + \mathbf{N}_O = 0.$$

If now $\mathbf{u}$ be a unit vector perpendicular to the plane and if we cross-multiply the first equation by $\mathbf{u}$ and dot-multiply the second equation by $\mathbf{u}$ we find,

$$\mathbf{u} \times \mathbf{S} = 0, \qquad \mathbf{u} \cdot \mathbf{M}_O = M_\Delta = 0,$$

due to the fact mentioned above that $\mathbf{T}$ is perpendicular to π and $\mathbf{N}_O$ is parallel to π. Here Δ is the axis through O perpendicular to π and may be arbitrarily chosen since O is any point. The conditions stated in the theorem are thus necessary for equilibrium.

These conditions are also sufficient for if they hold we have evidently $\mathbf{S} \cdot \mathbf{M} = 0$ and the system of applied forces is reducible by § 59 to a single vector perpendicular to the fixed plane π, or to an elementary couple with its moment $\mathbf{M}$ parallel to π, or to a null system. In the case of the null system we have equilibrium at once, no constraints being required. In the case of the elementary couple we could take its two sliding vectors as having any direction perpendicular to the moment $\mathbf{M}$ and so we may take them perpendicular to the plane π since $\mathbf{M}$ is parallel to π. Thus in every case the applied forces are equivalent to a system of forces applied to the sliding plane and perpendicular

to it. The same being true by hypothesis of the constraints, it appears that there is no force causing the plane of the body to slide on the fixed plane, and since the constraints keep the plane of the body coincident with the fixed plane it follows that the body plane can not move at all. But then the applied forces and constraints must form a null system and the whole body is consequently in equilibrium. The theorem is thus established.

The above four theorems on the equilibrium of a rigid body subject to constraints illustrate a general principle applicable to all cases of the motion of a rigid body restricted by bilateral constraints, i.e. constraints capable of acting in both senses along their lines of action. Since in all such cases the constraints and the applied forces together form a null system and since the constraints are bilateral it follows that the limitations which are placed upon the sum and moment of the system of constraints by the hypotheses concerning these constraints become necessary and sufficient conditions for equilibrium when applied to the sum and moment of the system of applied forces.

Let us consider the following examples.

The ends A and B of a weightless rod rest on two smooth planes which intersect in a horizontal line O and make angles α and β with the horizontal. The rod is perpendicular to the line O and bears a weight $\mathbf{W}$ at a point C on it at distances a and b from A and B. Determine the angle θ which the rod makes with the horizontal when in equilibrium and determine the amounts f and g of the forces exerted upon it by the two planes.

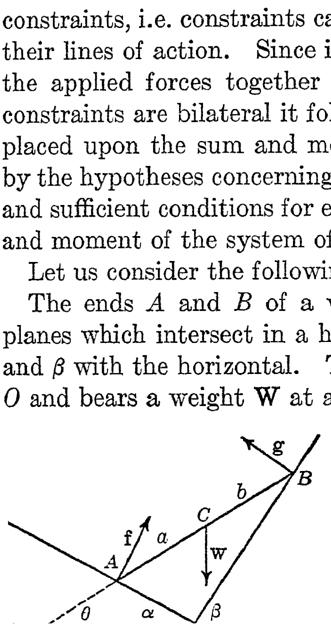

Fig.

Three forces $\mathbf{f}$, $\mathbf{g}$, $\mathbf{W}$ act on the rod, the forces $\mathbf{f}$ and $\mathbf{g}$ being perpendicular to their planes since the planes are smooth and the force $\mathbf{W}$ acting downward. For equilibrium we must have the sum of these forces and their moment with respect to some one point equal to zero. Choosing this point as the point C we have,

$$\mathbf{f} + \mathbf{g} + \mathbf{W} = 0, \qquad CA \times \mathbf{f} + CB \times \mathbf{g} = 0.$$

If we take the vertical plane containing the rod as the XY-plane with the X-axis running horizontally and measure the

above equations on the coördinate axes, we find the three scalar equations,

$$f \sin \alpha - g \sin \beta = 0, \qquad f \cos \alpha + g \cos \beta - W = 0,$$
$$af \cos (\alpha + \theta) - bg \cos (\beta - \theta) = 0.$$

These yield,

$$f = \frac{\sin \beta}{\sin (\alpha + \beta)} W, \qquad g = \frac{\sin \alpha}{\sin (\alpha + \beta)} W,$$
$$\tan \theta = \frac{a \cot \alpha - b \cot \beta}{a + b}.$$

A homogeneous right circular cylinder of length $2l$, radius r and weight W rests on a rough horizontal floor and leans against a smooth vertical wall, the axis of the cylinder being perpendicular to the line of intersection of the floor and wall. If the axis of the cylinder makes an angle θ with the horizontal, determine the forces $\mathbf{f}$ and $\mathbf{g}$ with which the floor and wall act on the cylinder. Also if the wall is rough and the coefficients of friction of the floor and wall are λ and μ respectively, determine the minimum value of θ for which the cylinder can remain in equilibrium. If the coördinate axes be taken as indicated in the figure, we may set,

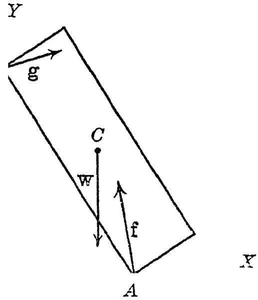

FIG. 67

$$\mathbf{f} = (-f_1, f_2, 0), \qquad \mathbf{g} = (g_1, g_2, 0), \qquad \mathbf{w} = (0, -w, 0),$$

where $\mathbf{w}$ is the force of gravity which will act as if the entire mass of the cylinder were concentrated at its center C. For equilibrium we must have the sum of the forces zero and their moment about some point, as C, must be zero. That is,

$$(1) \qquad \mathbf{f} + \mathbf{g} + \mathbf{w} = 0, \qquad CA \times \mathbf{f} + CB \times \mathbf{g} = 0,$$

where

$$CA = (l \cos \theta - r \sin \theta, -l \sin \theta - r \cos \theta, 0),$$
$$CB = (-l \cos \theta - r \sin \theta, l \sin \theta - r \cos \theta, 0).$$

Measuring equations (1) on the coördinate axes yields the scalar

equations,

$$-f_1 + g_1 = 0, \qquad f_2 + g_2 - w = 0,$$

$$(2) \quad (l \sin\theta + r\cos\theta) f_1 - (l\cos\theta - r\sin\theta) f_2 + (l\sin\theta - r\cos\theta) g_1 + (l\cos\theta + r\sin\theta) g_2 = 0.$$

For the first case where the wall is smooth we have $g_2 = 0$ and equations (2) yield,

$$f_1 = \tfrac{1}{2}(r/l - \cot\theta)\, w, \qquad f_2 = w, \qquad g_1 = \tfrac{1}{2}(r/l - \cot\theta)\, w.$$

For the second case in which the coefficients of friction of the floor and wall are λ and μ we will have at the position of minimum slant,

$$f_1 = \lambda f_2, \qquad g_2 = \mu g_1,$$

and equations (2) then yield,

$$\tan\theta = \frac{(1 - \lambda\mu)\, l}{2\lambda l + (1 + \lambda\mu)\, r}.$$

It is interesting to note that if the floor were smooth the cylinder could only stand with C directly above A.

EXERCISES

1. If a light rod or cord is in equilibrium under the action of forces applied only at its ends, show that the sum of the forces at each end must have the direction of the rod or cord.

2. A light rod AB of length l has one end A fastened by a hinge so that the rod is free to turn in a vertical plane. A weight W is suspended from a point C of the rod, $AC = a$. A cord BD attached to the end B of the rod holds it in equilibrium in a horizontal position, the angle ABD being 150°. Determine the tension $\mathbf{t}$ in the cord and the force $\mathbf{f}$ exerted by the hinge.

   Ans. $t = mW$, $f = \sqrt{1 - m + m^2}\, W$, $\tan\varphi = \dfrac{\sqrt{3}m}{2 - m}$ where $m = 2a/l$ and φ is the angle which $\mathbf{f}$ makes with the vertical

3. A uniform rod of length l and weight W is freely hinged at one end A at a distance a from a smooth vertical wall and the other end B rests against the wall. What is the amount f of the force exerted by the rod on the wall? Discuss the variation of f as a varies.

   Ans. $f = \dfrac{a}{2\sqrt{l^2 - a^2}}\, W$

4. A smooth uniform rod of length $2l$ and weight W rests one end A on a smooth horizontal plane and one of its points D on a support

of height h. A cord of length a attaches A to the bottom E of the support. Determine the tension t in the cord and the amounts f and g of the forces acting at A and D.

Ans. $t = alh\,(a^2 + h^2)^{-3/2}\,W$, $\quad f = \{1 - a^2\,l\,(a^2 + h^2)^{-3/2}\}\,W$,
$g = al\,(a^2 + h^2)^{-1}\,W$

5. A ladder 25 feet long and weighing 100 pounds is set up against a wall with its bottom 7 feet from the base of the wall. If the coefficient of friction of the ground is 1/5 and of the wall is 1/7, what fraction of the entire height of the ladder may a 180 pound man climb before the ladder starts to slip? *Ans.* 65/81

6. A uniform rod of length $2l$ and weight W rests one end A on a smooth floor and the other end B against a smooth vertical wall. The rod is perpendicular to the line O of intersection of the floor and the wall and is attached to the nearest point of that line by a light cord OD of length c, where D is a point of the rod at a distance a from A. Determine in terms of l, a, c, W the tension t in the cord and the amounts f and g of the forces acting at A and B. Evaluate t, f, g for $l = 35$, $a = 20$, $c = 34$, $W = 24$.
Ans. $t = 17$, $f = 32$, $g = 15$

7. Show that when a smooth sphere resting on any supports and acted upon by the force of gravity is in equilibrium, its center and its centroid must be on the same vertical line.

8. Six equal uniform rods each of weight W are freely hinged together to form the edges of a regular tetrahedron which is placed with one face resting on a smooth horizontal floor. Find the tension g in the horizontal rods. *Ans.* $g = 0.20412W$

9. A homogeneous hemisphere of weight W rests its curved face on a smooth horizontal floor and a weight f is attached to a point A of its edge. Determine the angle θ which the plane face of the hemisphere makes with the horizontal. Evaluate θ for $W = 8$, $f = 3$. (Note: The centroid of a homogeneous hemisphere is at a distance 3/8 of the radius from the plane face.) *Ans.* $\theta = 45°$

10. A homogeneous hemisphere of radius r has a cord of length l attached to a point of its edge and the other end of the cord is fastened to a point of a smooth vertical wall. What must be the value of $m = l/r$ in order that the hemisphere may rest its curved face against the wall with its flat face making a given angle θ with the wall. Evaluate m for $\tan\theta = 3/4$. (See note on Problem 9.)
Ans. $m = 1.139$

11. A homogeneous hemisphere is placed with its curved face on a rough plane inclined at an angle θ to the horizontal. What is the maximum value of θ for which the hemisphere will remain on its curved face? Is the inclination of the plane face of the hemisphere always uniquely determined by the value of θ? (See note on Problem 9.) *Ans.* $\theta \leqq 22°\,1'$

12. A smooth hemispherical bowl of radius r has its edge in a horizontal plane and a uniform rod rests on the edge of the bowl with one end touching the inner surface. If the length of the rod within the bowl is a, determine its entire length $2l$. Evaluate $2l$ for $a = 5/3$, $r = 1$. What is the length of the shortest rod that can rest in this position in this bowl? *Ans.* $2l = 28/15$, $2l = 1.633r$

13. Three equal smooth spheres rest on a horizontal plane and are held in contact by a band placed about them at the level of their centers. If a fourth sphere of the same size and weight W is placed upon them, how much is the tension in the band increased? Answer the same question when there are four spheres in the bottom layer with their centers at the vertices of a square.
Ans. $0.13608W$, $0.17678W$

14. A homogeneous right circular cylinder of radius r and weight W rests one base on a horizontal plane with coefficient of friction μ. What will be the amount of the moment of a force just sufficient to rotate the cylinder about its axis; about an element of the cylindrical surface? *Ans.* $\frac{4}{3}\mu r W$, $\frac{32}{9}\mu r W$

15. A uniform heavy rod rests tangentially upon a smooth curve in a vertical plane and one end presses against a smooth vertical plane perpendicular to the first. Determine the curve so that the rod may be in equilibrium in any such position.
Ans. A four-cusped hypocycloid

16. Two uniform heavy rods have their ends connected by two light strings and the whole system is supported by a string attached to the mid-point of one rod. Show that the figure formed by the rods and strings is a trapezoid or parallelogram.

66. Centroid of a Rigid Body.

We have regarded a rigid body as made up of a set of n particles $P_1, P_2, \cdots, P_n$ whose mutual distances remain constant. If the radius vectors of these particles from the origin are $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n$ and their masses $m_1, m_2, \cdots, m_n$, we shall define the *centroid of the body* as a fictitious particle P_0 having a radius vector $\mathbf{p}_0$ and a mass m_0 given by the formulas,

$$m_0 = \Sigma m_i, \qquad m_0 \mathbf{p}_0 = \Sigma m_i \mathbf{p}_i.$$

We shall see that in certain connections the centroid P_0 is a sort of representative particle capable of replacing the whole set $P_1, P_2, \cdots, P_n$. This is true when the set of particles constitutes a part of a larger set whose centroid is to be determined. The theorem here is,

The centroid of a set of particles is not altered if any sub-set chosen from the given set is replaced by its centroid.

We need give no proof of this theorem as it is readily seen to be exactly equivalent to the theorem proven in § 62 concerning the centroid of a set of points $A_1, A_2, \cdots, A_n$ and associated scalars $l_1, l_2, \cdots, l_n$. As another example we have the theorem,

If the particles of a rigid body are acted upon by forces all having the same direction and sense and with the amount of each force proportional to the mass of the particle acted upon, then the effect of the forces is not altered if the particles are replaced by their centroid in calculating the forces.

To see this we recall from § 65 that two equivalent systems of forces have the same effect on the motion of a rigid body. Consequently our theorem will be proven if we show that the forces acting on the particles of the body form a system equivalent to the force which would act on the centroid. Since the amounts of the forces acting on the particles are proportional to their masses we may write,

$$l_i = km_i,$$

where l_i is the amount of the force acting on the particle P_i and k is a constant. We saw in § 62 that a system of parallel sliding vectors such as these forces is equivalent to a single sliding vector of length $l_0 = \Sigma l_i$ applied at the point P_0 where,

$$OP_0 = \mathrm{p}_0 = \Sigma\, l_i\, \mathrm{p}_i/l_0.$$

But this sliding vector constitutes exactly the force which would act on the centroid of the body, since the amount of this force would be $km_0 = k\Sigma m_i = \Sigma l_i = l_0$ while the location of the centroid of the body is given by,

$$OP_0 = \mathrm{p}_0 = \Sigma\, m_i\, \mathrm{p}_i/m_0 = k\Sigma\, m_i\, \mathrm{p}_i/km_0 = \Sigma\, l_i\, \mathrm{p}_i/l_0.$$

The theorem is thus established.

This is a theorem of the greatest practical importance, for the force of gravity of the earth acts on the particles of a rigid body at its surface in directions which are sensibly parallel when the body is of ordinary size and with amounts which are strictly proportional to their masses. We may therefore by the theorem regard the force of gravity acting on a rigid body as a single force

acting downward, proportional to the mass of the body and applied at its centroid. With this fact in mind the following statements become self-evident applications of the theorems of the preceding article.

A necessary and sufficient condition for the equilibrium under the action of gravity of a rigid body with one fixed point is that the fixed point and the centroid of the body be on the same vertical line.

As a special case of this we have the fact that,

If the centroid of a rigid body is a fixed point the body is in equilibrium under the action of gravity in any position.

A necessary and sufficient condition for the equilibrium under the action of gravity of a rigid body with a fixed axis is that the fixed axis and the centroid of the body lie in the same vertical plane.

As a special case of this we have the fact that,

If any centroidal line of a rigid body is a fixed axis, the body is in equilibrium under the action of gravity in any position.

A *centroidal line or plane* is any line or plane through the centroid. The reader may easily formulate the corresponding conditions for equilibrium under the action of gravity of a rigid body having a sliding axis or a sliding plane.

In the case of the approximately rigid bodies actually found in nature the number of particles forming the body is ordinarily so great that it would be quite impossible to perform the summations required for the direct determination of the centroid of the body. In that case we proceed as follows. We regard the space T occupied by the body as cut into a large number n of subdivisions $\Delta\tau_1, \Delta\tau_2, \Delta\tau_3, \cdots, \Delta\tau_n$. We represent by $\rho_i \Delta\tau_i$ the mass of the particles contained in the ith subdivision, ρ_i being known as the *mean density* of that subdivision. We are of course still quite unable to locate the centroid of the particles in the subdivision but if each subdivision be taken as convex and very small in all its dimensions it will suffice to observe that the centroid is at some point P_i within the subdivision. We now apply the first theorem of this section determining the centroid of the whole body by first replacing the particles of each of the subdivisions

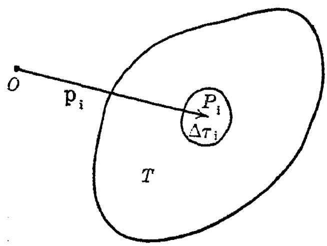

FIG. 68

by its centroid. Thus we have,

$$m_0 = \Sigma \rho_i \, \Delta\tau_i, \qquad m_0 \, \mathbf{p}_0 = \Sigma \rho_i \, \mathbf{p}_i \, \Delta\tau_i.$$

Since the centroids of the subdivisions are not accurately known this method will usually only give an approximate result, the degree of accuracy depending in general on the number of subdivisions employed. But it may happen that when we take the subdivisions sufficiently small the quantities ρ_i approximately take on the values assumed by a continuous function $\rho(\mathbf{p})$ for $\mathbf{p} = \mathbf{p}_i$. This function is known as the *density of the body at the point P.* If we assume that no matter how small the subdivisions are taken the mean density ρ_i of each subdivision is the value assumed by $\rho(\mathbf{p}_i)$ at some point P_i of that subdivision, then it is shown in works on the integral calculus that the above formulas will become in the limit, as the dimensions of the subdivisions approach zero,

$$m_0 = \iiint\limits_T \rho(\mathbf{p}) \, d\tau, \qquad m_0 \, \mathbf{p}_0 = \iiint\limits_T \rho(\mathbf{p}) \, \mathbf{p} \, d\tau.$$

The second members of these equations are *volume integrals* extended over the volume T and may be evaluated by triple integrals as explained in works on the calculus. In fact if we represent the coördinates of any point P of the body and of the centroid P_0 by $P \equiv (x, y, z)$ and $P_0 \equiv (x_0, y_0, z_0)$, the above formulas may be written,

$$m_0 = \iiint\limits_T \rho(x, y, z) \, dx \, dy \, dz,$$

$$m_0 \, x_0 = \iiint\limits_T \rho(x, y, z) \, x \, dx \, dy \, dz,$$

$$m_0 \, y_0 = \iiint\limits_T \rho(x, y, z) \, y \, dx \, dy \, dz,$$

$$m_0 \, z_0 = \iiint\limits_T \rho(x, y, z) \, z \, dx \, dy \, dz.$$

If the rigid body possesses a point, line or plane of symmetry, the centroid of the body will be on that point, line or plane. For in that case, corresponding to every particle of the body there will

be another particle of the body of equal mass symmetrically situated to the first with respect to the point, line or plane and if we replace every such pair of particles by their centroid, the resulting centroids will all lie on the point, line or plane of symmetry and the centroid of these centroids will consequently lie there also. If the rigid body is *homogeneous* the density $\rho(\mathbf{p})$ is a constant and therefore may be written in front of the integral signs in the above formulas. This factor then cancels out in the determination of $\mathbf{p}_0$ and consequently the location of the centroid in this case depends only on the shape and position of the body. We may thus speak of the centroid of a geometric figure without the idea of mass entering explicitly and the centroid may then be regarded as a point rather than a particle. The rigid body may take the form of a thin sheet or wire and in that case the integrations in the above formulas need be extended only over the corresponding surface or curve, the density ρ becoming the mass per unit area or per unit length respectively.

Let us consider the following example:

Determine the centroid of a homogeneous hemisphere of unit density and radius a.

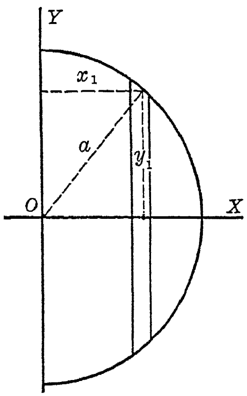

FIG. 69

We place the flat face of the hemisphere in the YZ-plane with the center of the face at the origin. We then divide the volume of the hemisphere into n approximately disk shaped pieces by planes parallel to the YZ-plane, the thickness of the successive disks being $\Delta x_1, \Delta x_2, \Delta x_3, \cdots, \Delta x_n$. The mass of the particles in the ith disk will then be $\pi y_i^2 \Delta x_i$, where y_i is the value assumed by y at some point where the edge of the disk cuts the XY-plane. We then replace the particles in each disk by their centroid, which for the ith disk will be a particle having the above mass and located at a point on the X-axis having its X-coördinate x_i some value assumed by x in that disk. We then determine the centroid of all these centroids by the formulas,

$$m_0 = \Sigma\, \pi y_i^2\, \Delta x_i, \qquad m_0\, x_0 = \Sigma\, \pi y_i^2\, x_i\, \Delta x_i.$$

It is clear that if the disks are very thin we have approximately $x_i^2 + y_i^2 = a^2$ so that we may replace y_i^2 by $a^2 - x_i^2$. If we now proceed to the limit allowing every Δx_i to approach zero, it is shown in the calculus that in the limit the above formulas take the exact form,

$$m_0 = \int^a \pi\,(a^2 - x^2)\,dx, \qquad m_0\,x_0 = \int^a \pi\,(a^2 - x^2)\,x\,dx,$$

which yield,

$$m_0 = 2\pi a^3/3, \qquad m_0\,x_0 = \pi a^4/4.$$

Thus we have finally,

$$\mathbf{p}_0 \equiv (3a/8, 0, 0), \qquad m_0 = 2\pi a^3/3.$$

EXERCISES

In the following problems the bodies are assumed to be of uniform unit density unless otherwise stated.

1. Find the centroid of the following set of particles both directly and by first replacing certain sub-sets by their centroids.

$$\begin{aligned} \mathbf{p}_1 &\equiv (-1, -3, -1), & m_1 &= 1,\\ \mathbf{p}_2 &\equiv (2, -1, 2), & m_2 &= 2,\\ \mathbf{p}_3 &\equiv (-1, 3, 1), & m_3 &= 3,\\ \mathbf{p}_4 &\equiv (0, -1, 1), & m_4 &= 4. \end{aligned}$$

Ans. $\mathbf{p}_0 \equiv (0,$ ($\qquad m_0 = 10$

2. Show that the centroid of two particles is on the line segment joining them and divides it into segments which are inversely proportionally to the masses of the corresponding particles.

3. Show that the centroid of three equal particles is at the point of intersection of the medians of the triangle having its vertices at the particles.

4. Show that the centroid of a set of particles is always within or on any convex polyhedron containing the set. (Hint: Apply the result of Problem 2.)

5. Prove that the centroid of a triangular area is at the point of intersection of the medians.

6. Prove the *Theorem of Pappus* (circa 300 A.D.) The centroid of a triangular area is at the same point as that of an inscribed triangular area whose vertices divide the sides of the original triangle in the same ratio.

7. A sphere of radius $a/2$ and density 3 is imbedded in a sphere of radius a and density 2, the center of the larger sphere being on the surface of the smaller. Determine the centroid of the resulting body. *Ans.* $\mathbf{p}_0 \equiv (a/34, 0, 0)$, $m_0 = 17\pi a^3/6$

8. A circular hole of radius $a/2$ is cut from a circular disk of radius a, the circumference of the hole passing through the center of the disk. Determine the centroid of the resulting body.
Ans. $\mathbf{p}_0 \equiv (-a/6, 0, 0)$, $\mathbf{m}_0 = 3\pi a^2/4$

9. Show that the centroid of the lateral surface of a pyramid or cone is 2/3 of the way from the vertex to the centroid of the boundary of the base, and that the centroid of the volume is 3/4 of the way from the vertex to the centroid of the area of the base.

10. Determine the centroid of the area in the first quadrant bounded by the parabola $x^{1/2} + y^{1/2} = a^{1/2}$.
Ans. $\mathbf{p}_0 \equiv (a/5, a/5, 0)$, $\mathbf{m}_0 = a^2/6$

11. Determine the centroid of the area in the first quadrant bounded by the four-cusped hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.
Ans. $\mathbf{p}_0 \equiv (256a/315, 256a/315, 0)$, $\mathbf{m}_0 = 3\pi a^2/32$

12. Determine the centroid of the area under one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.
Ans. $\mathbf{p}_0 \equiv (\pi a, 5a/6, 0)$, $\mathbf{m}_0 = 3\pi a^2$

13. Determine the centroid of the area in the first quadrant bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.
Ans. $\mathbf{p}_0 \equiv (4a/3\pi, 4b/3\pi, 0)$, $\mathbf{m}_0 = \pi ab/4$

14. Determine the centroid of the area of a circular sector of radius a and angle 2α. *Ans.* $\mathbf{p}_0 \equiv (2a\sin\alpha/3\alpha, 0, 0)$, $\mathbf{m}_0 = a^2\alpha$

15. Prove that the centroid of a zone of a spherical surface is midway between the two bounding planes.

16. Determine the centroid of the paraboloid of revolution obtained by rotating about the X-axis the area in the first quadrant between the parabola $y^2 = 4x$ and the line $x = 4$.
Ans. $\mathbf{p}_0 \equiv (8/3, 0, 0)$, $m_0 = 32\pi$

17. Determine the centroid of the solid of revolution formed by rotating a sector of a circle of radius a and angle α about one of its bounding radii.

$$Ans.\quad \mathbf{p}_0 \equiv \left\{\frac{3}{8}a(1 + \cos\alpha), 0, 0\right\}; \qquad m_0 = 2\pi \quad (1 - \cos\alpha)$$

18. Determine the centroid of a circular arc of radius a and angle 2α.
Ans. $\mathbf{p}_0 \equiv (a\sin\alpha/\alpha, 0, 0)$, $m_0 = 2a\alpha$

19. The points A and D are opposite ends of a uniform slender rod of length 12. Points B and C are marked on the rod so that $AB = 8$,

$BC = 1$, $CD = 3$ and the rod is bent at B. Show that the centroid of the bent rod lies on the line through A and C.

20. Prove the *First Proposition of Pappus* (circa 300 A.D.): The surface of a solid of revolution is equal to the length of the generating plane arc multiplied by the circumference of the circle described by the centroid of the arc.

21. Prove the *Second Proposition of Pappus:* The volume of a solid of revolution is equal to the generating plane area multiplied by the circumference of the circle traced by the centroid of the area.

22. What are the conditions of equilibrium under the action of gravity of a rigid body having a sliding axis; a sliding plane?

CHAPTER VIII

STATICS OF A FLEXIBLE CORD

67. Discrete Particles on a Light Cord.

It is a familiar fact that there exist in nature sets of particles closely approximating the theoretical rigid body in which the interior forces maintain all the mutual distances of the particles constant. But we also find sets of particles which are not quite so completely bound by the interior forces and we call such a set a *deformable body*. An important case of this is the *light flexible cord with attached particles*. We think here of a set of n particles $P_1, P_2, P_3, \cdots, P_n$ each of which may be acted upon by exterior forces, but where the interior forces of the set are only such as might be produced by attaching the particles in order on a light, inextensible, flexible cord. Thus we assume that the distance from each particle P_i to the next P_{i+1} is prevented from exceeding a certain amount c_i by an interior force $\mathbf{g}_i$ acting on P_i and having the direction and sense of the vector $\mathbf{c}_i = P_i P_{i+1}$, and the force $-\mathbf{g}_i$ acting on P_{i+1}. These forces $\mathbf{g}_i$ are called the *tensions* in the cords c_i. The sum of the exterior forces acting on P_i we shall call $\mathbf{f}_i$. In saying that the cord is light we mean simply that no exterior forces act on the cords except $-\mathbf{g}_0$ and $\mathbf{g}_n$ acting on the first and last cords c_0 and c_n which may connect the set to exterior particles. The particles with their connecting cords form the vertices and sides of what is known as the *funicular polygon* (Latin, *funiculus*, a cord).

A necessary and sufficient condition that the set of particles be in equilibrium is of course that,

$$(1) \qquad -\mathbf{g}_{i-1} + \mathbf{g}_i + \mathbf{f}_i = 0, \qquad i = 1, 2, 3, \cdots, n,$$

these being merely the conditions of equilibrium for the n separate particles. Adding these equations member for member we find the familiar necessary condition for equilibrium that the sum of the exterior forces be zero,

$$(2) \qquad -\mathbf{g}_0 + \mathbf{g}_n + \Sigma\, \mathbf{f}_i = 0.$$

If this condition is satisfied by the exterior forces $-\mathbf{g}_0, \mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n, \mathbf{g}_n$, the equations (1) become consistent and may be solved successively for the tensions $\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_{n-1}$. If then it is found that each cord $\mathbf{c}_i = P_i P_{i+1}$ has the direction and sense of the corresponding tension $\mathbf{g}_i$, the cord will be able to produce this tension and the set of particles will be in equilibrium. We may write this condition in the form,

$$\mathbf{c}_i = k_i \mathbf{g}_i, \qquad k_i > 0, \tag{3}$$

to hold for every value of i for which $\mathbf{g}_i$ is not zero.

The above condition for equilibrium may be given a very satisfactory graphical interpretation. Suppose the set to be in equilibrium and from some point O lay off as radius vectors the tensions $\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_n$, calling their termini $G_0, G_1, G_2, \cdots, G_n$.

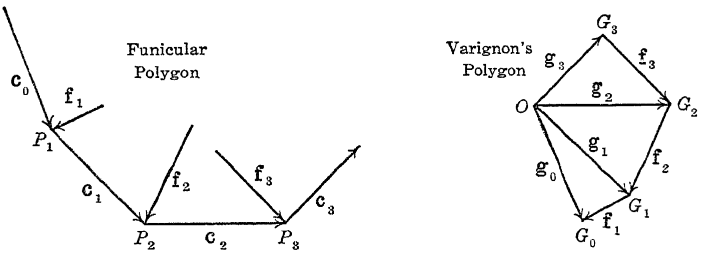

FIG. 70

Since we then have both,

$$-\mathbf{g}_{i-1} + \mathbf{g}_i + G_i G_{i-1} = 0, \qquad \mathbf{g}_{i-1} + \mathbf{g}_i + \mathbf{f}_i = 0,$$

it appears that,

$$G_i G_{i-1} = \mathbf{f}_i,$$

so that the sides of the polygon $O, G_n, G_{n-1}, \cdots, G_0, O$ constitute successively the exterior forces $\mathbf{g}_n, \mathbf{f}_n, \mathbf{f}_{n-1}, \cdots, \mathbf{f}_1, -\mathbf{g}_0$. This polygon is known as *Varignon's polygon* and the point O is called its *pole*. Conversely if the exterior forces $\mathbf{g}_n, \mathbf{f}_n, \mathbf{f}_{n-1}, \cdots, \mathbf{f}_1, \mathbf{g}_0$ be drawn as vectors applied successively end to end, then the set of particles will be in equilibrium only if the resulting polygon closes at some point O and if the diagonals $OG_0, OG_1, \cdots, OG_n$ have the direction and sense of the cords $\mathbf{c}_0, \mathbf{c}_1, \cdots, \mathbf{c}_n$. It is important to note that the sides of Varignon's polygon are vectors proceeding in the opposite sense between the vertices from

the sense of the corresponding sides of the funicular polygon. Thus in Varignon's polygon $\mathbf{f}_2$ runs from G_2 to G_1 while in the funicular polygon $\mathbf{c}_2$ runs from P_2 to P_3.

Certain special cases of the equilibrium of a light cord with attached particles are of particular interest both analytically and graphically.

I. Cord with Free Ends. If the ends of the cord are free the terminal tensions vanish and we have $\mathbf{g}_0 = \mathbf{g}_n = 0$. The first and last of equations (1) then become,

$$\mathbf{g}_1 + \mathbf{f}_1 = 0, \qquad - \mathbf{g}_{n-1} + \mathbf{f}_n = 0.$$

The exterior forces $\mathbf{f}_n, \mathbf{f}_{n-1}, \cdots, \mathbf{f}_1$ form all the sides of Varignon's polygon, which is thus closed with only n sides, the pole O being at the intersection of the sides $\mathbf{f}_1$ and $\mathbf{f}_n$.

II. Cord with Joined Ends. If the ends of the cord are joined together so that the funicular polygon is closed, the terminal tensions $\mathbf{g}_0$ and $\mathbf{g}_n$ must be equal. Equation (2) then reduces to,

$$\Sigma \, \mathbf{f}_i = 0.$$

The vertices G_0 and G_n of Varignon's polygon coincide and we may consequently regard the polygon as closed with the pole O not forming a part of it. The above necessary and sufficient conditions for equilibrium are thus in this case that,

The polygon of the forces $\mathbf{f}_n, \mathbf{f}_{n-1}, \cdots, \mathbf{f}_1$ *be closed.*

There exist a point O such that each vector OG_i have the direction and sense of the side $\mathbf{c}_i$ *of the funicular polygon.*

III. Forces Perpendicular to an Axis. If the set is in equilibrium under the action of the exterior forces $-\mathbf{g}_0, \mathbf{f}_1, \cdots, \mathbf{f}_n, \mathbf{g}_n$ where the $\mathbf{f}_i$ are all perpendicular to a certain axis, then the measures of the tensions $\mathbf{g}_i$ on this axis are all equal. For let $\mathbf{u}$ be a unit vector having the direction and sense of the given axis. Then by hypothesis every $\mathbf{u} \cdot \mathbf{f}_i = 0$ and equations (1) become after dot-multiplying by $\mathbf{u}$,

$$\mathbf{u} \cdot \mathbf{g}_0 = \mathbf{u} \cdot \mathbf{g}_1 = \qquad = \mathbf{u} \cdot \mathbf{g}_n, \tag{4}$$

thus proving the proposition. As an important special case of the above proposition we have the theorem,

If the ends of the cord are attached to two fixed points A and B and if the vector AB as well as the exterior forces $\mathbf{f}_i$ *is perpendicular to a certain axis, then when the set is in equilibrium the funicular*

polygon is a plane polygon in a plane perpendicular to the given axis and the measures of the tensions $\mathbf{g}_i$ *on the axis are zero.*

We shall at first assume that none of the tensions $\mathbf{g}_i$ is zero. Then by equations (3) we may write,

$$(5) \qquad \mathbf{c}_i = k_i\, \mathbf{g}_i, \qquad k_i > 0, \qquad i = 0, 1, 2, \qquad n.$$

If we let K stand for the value of the members of equation (4) we have $\mathbf{u}\cdot\mathbf{g}_i = K$ and on multiplying each of these equations through by the corresponding k_i and adding member for member, we are enabled to write,

$$\mathbf{u}\cdot AB = \mathbf{u}\cdot(\mathbf{c}_0 + \mathbf{c}_1 + \cdots + \mathbf{c}_n) = K\,(k_0 + k_1 + \cdots + k_n).$$

Since $\mathbf{u}\cdot AB$ is by hypothesis zero and since the k_i are all positive, it follows that K is zero and we have,

$$\mathbf{u}\cdot\mathbf{g}_i = 0, \qquad \mathbf{u}\cdot\mathbf{c}_i = 0, \qquad i = 0, 1, 2, \cdots, n.$$

The first of these sets of equations proves the last statement in the theorem. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}_i$ are the radius vectors from the origin of the points A, B, P_i we have from the second set of equations,

$$(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u} = (\mathbf{c}_0 + \mathbf{c}_1 + \cdots + \mathbf{c}_{i-1})\cdot\mathbf{u} = 0.$$

Thus all the vertices of the funicular polygon lie in the plane $(\mathbf{p} - \mathbf{a})\cdot\mathbf{u} = 0$ perpendicular to the given axis. The theorem is thus proven for this case.

If some *one* tension $\mathbf{g}_j$ is zero the theorem still holds even though we can no longer set $\mathbf{c}_j = k_j\, \mathbf{g}_j$. For now one member of equations (4) is zero by hypothesis and the last statement in the theorem follows at once. Furthermore we have,

$$\mathbf{u}\cdot\mathbf{c}_i = \mathbf{u}\cdot k_i\, \mathbf{g}_i = 0, \qquad i \neq j.$$

But by hypothesis $\mathbf{u}\cdot AB = \mathbf{u}\cdot(\mathbf{c}_0 + \mathbf{c}_1 + \cdots + \mathbf{c}_n) = 0$ and on subtracting from this quantity the first member of each of the above set of equations we find $\mathbf{u}\cdot\mathbf{c}_j = 0$. The proof may now be completed as before. If *two or more* of the tensions $\mathbf{g}_i$ are zero the first statement in the theorem holds only in the sense that the funicular polygon may degenerate into two or more plane polygons in planes perpendicular to the given axis, the polygons being joined to each other by cords in which there is no tension.

IV. Forces Parallel to an Axis. If the set of particles on the cord is in equilibrium under the action of the exterior forces

$-\mathbf{g}_0, \mathbf{f}_1, \cdots, \mathbf{f}_n, \mathbf{g}_n$ where the $\mathbf{f}_i$ are all parallel to a certain axis, then the projections of the tensions $\mathbf{g}_i$ on any plane perpendicular to this axis are all equal. For let $\mathbf{u}$ be a unit vector having the direction and sense of this axis. Then by hypothesis every $\mathbf{u} \times \mathbf{f}_i = 0$ and equations (1) become after cross-multiplication by $\mathbf{u}$,

$$(6) \qquad \mathbf{u} \times \mathbf{g}_0 = \mathbf{u} \times \mathbf{g}_1 = \cdots = \mathbf{u} \times$$

Thus $\mathbf{u} \times (\mathbf{g}_j - \mathbf{g}_i) = 0$ and it appears that any two of these tensions differ by a vector which is parallel to the given axis and they must consequently have the same projection on any plane perpendicular to the axis. We may now easily establish the principal theorem for this case.

If the set of particles on the cord is in equilibrium when the forces $\mathbf{f}_i$ *are all parallel to a certain axis, then the funicular polygon is a plane polygon in a plane parallel to this axis and the measures of the tensions* $\mathbf{g}_i$ *on any axis perpendicular to the given axis are all equal.*

In proving the first statement in the theorem we shall at first assume that none of the tensions $\mathbf{g}_i$ is zero so that as before we may write $\mathbf{c}_i = k_i \, \mathbf{g}_i$ for every i. If we let $\mathbf{K}$ stand for the value of the members of equations (6) we have $\mathbf{u} \times \mathbf{c}_i = k_i \mathbf{K}$ and consequently may write,

$$(7) \quad \mathbf{u} \times (\mathbf{p}_i - \mathbf{a}) = \mathbf{u} \times (\mathbf{c}_0 + \mathbf{c}_1 + \cdots + \mathbf{c}_{i-1}) \\ = (k_0 + k_1 + \cdots + k_{i-1}) \, \mathbf{K}.$$

If now $\mathbf{K} = 0$, every vertex of the funicular polygon evidently lies on the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{u} = 0$ parallel to the given axis and the first part of the theorem is proven for this case. If however $\mathbf{K} \neq 0$, we dot-multiply equation (7) through by $\mathbf{p}_i - \mathbf{a}$ and, remembering that the k_i are all positive, we find on dividing out their sum that $(\mathbf{p}_i - \mathbf{a}) \cdot \mathbf{K} = 0$. Thus every vertex of the funicular polygon lies on the plane $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{K} = 0$ which is parallel to the given axis since $\mathbf{K}$ is perpendicular to it. The first part of the theorem is thus proven for this case also. In case one or more of the tensions $\mathbf{g}_i$ is zero the first part of the theorem holds only in the sense that the funicular polygon may degenerate into two or more polygons in planes parallel to the given axis, these polygons being joined by cords in which there is no tension.

To prove the last part of the theorem we let $\mathbf{v}$ be a unit vector having the direction and sense of any axis perpendicular to the given axis. Then since,

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{g}_i) = \mathbf{v} \cdot \mathbf{g}_i,$$

we have on dot-multiplying equations (6) through by $\mathbf{u} \times \mathbf{v}$,

$$\mathbf{v} \cdot \mathbf{g}_0 = \mathbf{v} \cdot \mathbf{g}_1 = \cdots = \mathbf{v} \cdot \mathbf{g}_n, \tag{8}$$

which proves the statement.

The above theorem finds an important application in the theory of the *suspension bridge.* In the usual bridge of this type the weight of the platform of the bridge is so much greater than that of the supporting cables that we may as a first approximation regard it as a weightless cord with heavy equal particles attached to it at points equally spaced horizontally. By the above theorem the funicular polygon lies in a vertical plane and we shall take this plane as the XY-coördinate plane with the X-axis horizontal and the Y-axis vertical. Since the particles are equally spaced horizontally the cords $\mathbf{c}_i$ all have the same X-coördinate and we may write them as $\mathbf{c}_i \equiv (c, h_i, 0)$. If in equations (8) we let $\mathbf{v}$ be the X-coördinate vector $\mathbf{i}$, we see that the tensions $\mathbf{g}_i$ also all have the same X-coördinate. Consequently the scalar multipliers k_i of equations (3) are all equal and we may set $\mathbf{c}_i = k\, \mathbf{g}_i$. The exterior forces $\mathbf{f}_i$ being equal and directed downward we shall write as $\mathbf{f}_i \equiv (0, -f, 0)$. If then we multiply equations (1) through by k and measure the resulting equations on the X-axis, we find,

$$-h_{i-1} + h_i - kf = 0, \qquad i = 1, 2, \cdots, n$$

and adding the first i of these equations gives,

$$h_i = h_0 + ikf. \tag{9}$$

If we take for the left hand terminus of the cord the point $A \equiv (-a, b, 0)$ and call $P_i \equiv (x_i, y_i, 0)$, then since $\mathbf{p}_i = \mathbf{a} + (\mathbf{c}_0 + \mathbf{c}_1 + \cdots + \mathbf{c}_{i-1})$ we have,

$$x_i = -a + ic, \qquad y_i = b + h_0 + h_1 + \cdots + h_{i-1} \tag{10}$$
$$= b + ih_0 + \frac{i\,(i-1)}{2}\, kf.$$

Eliminating i between these equations shows that all the par-

ticles P_i lie on the parabola,

$$y = b + \frac{h_0}{c}(x + a) + \frac{kf}{2c^2}(x + a)(x + a - c),$$

having a vertical axis and opening upward. The amount g_i of the tension $\mathbf{g}_i$ in the cord $\mathbf{c}_i$ will evidently be $1/k$ times the length c_i of that cord. Since $c_i = \sqrt{c^2 + h_i^2}$ and $k = (h_1 - h_0)/f$ we have,

$$g_i = \frac{f}{h_1 - h_0}$$

In particular if we take n an odd number $2m - 1$ and attach the ends of the cord to the two points $A \equiv (-a, b, 0)$, $B \equiv (a, b, 0)$, then B is in effect the point P_{2m} and setting the value of y_{2m} from equation (10) equal to b gives us,

$$2h_0 + (2m - 1)kf = 0.$$

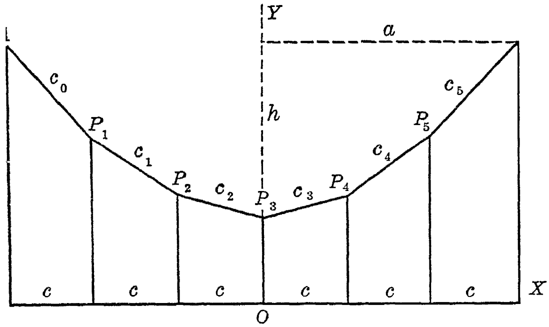

FIG. 71

Evidently P_m is the lowest point of the cord and so the *sag* of the cord is $h = b - y_m$ or,

$$h = - mh_0 - \frac{m(m-1)}{2} kf.$$

These last two equations show that,

$$k = 2h/(m^2 f),$$

and the amount of the tension in the cord $\mathbf{c}_i$ may be written,

$$g_i = \sqrt{a^2 m^2/4h^2 + (i - m + 1/2)^2}\, f.$$

The equation of the circumscribed parabola reduces in this case to,

$$y = b - h + \frac{h}{a^2}x^2.$$

V. Central Forces. If the forces $\mathbf{f}_i$ acting on the particles P_i on the light cord are concurrent at some point we say that they are *central forces.* The principal theorem for this case is,

If the set of particles on the light cord are in equilibrium when the forces $\mathbf{f}_i$ *are concurrent at a point* O*, then the funicular polygon is a plane polygon in a plane through* O *and the moments of the tensions* $\mathbf{g}_i$ *with respect to* O *are all equal.*

If we call $OP_i = \mathbf{p}_i$ then by hypothesis every $\mathbf{p}_i \times \mathbf{f}_i = 0$ and cross-multiplying each of equations (1) by the corresponding $\mathbf{p}_i$ we find,

$$(11) \qquad -\mathbf{p}_i \times \mathbf{g}_{i-1} + \mathbf{p}_i \times \mathbf{g}_i = 0, \qquad i = 1, 2, \cdots, n.$$

If we call the left hand terminus of the cord $A = P_0$ and $OA = \mathbf{a} = \mathbf{p}_0$, then we have every $\mathbf{p}_i - \mathbf{p}_{i-1} = \mathbf{c}_{i-1}$ and since a necessary condition for equilibrium is that every $\mathbf{c}_{i-1} \times \mathbf{g}_{i-1} = 0$ it follows that,

$$(12) \qquad \mathbf{p}_i \times \mathbf{g}_{i-1} - \mathbf{p}_{i-1} \times \mathbf{g}_{i-1} = 0, \qquad i = 1, 2, \cdots, n.$$

From equations (11) and (12) it follows that,

$$\mathbf{p}_i \times \mathbf{g}_i - \mathbf{p}_{i-1} \times \mathbf{g}_{i-1} = (\mathbf{p}_i \times \mathbf{g}_i - \mathbf{p}_i \times \mathbf{g}_{i-1}) + (\mathbf{p}_i \times \mathbf{g}_{i-1} - \mathbf{p}_{i-1} \times \mathbf{g}_{i-1}) = 0$$

or,

$$(13) \qquad \mathbf{p}_0 \times \mathbf{g}_0 = \mathbf{p}_1 \times \mathbf{g}_1 = \cdots = \mathbf{p}_n \times \mathbf{g}_n,$$

thus proving the second part of the theorem.

In proving the first part of the theorem we at first assume that none of the tensions $\mathbf{g}_i$ is zero, which permits the use of equations (3). If we represent by the vector $\mathbf{K}$ the members of equation (13), we have by the use of equations (3),

$$\mathbf{p}_i \times \mathbf{c}_i = k_i \mathbf{K}, \qquad k_i > 0, \qquad i = 0, 1, \cdots, n,$$

or since $\mathbf{c}_i = \mathbf{p}_{i+1} - \mathbf{p}_i$,

$$(14) \qquad \mathbf{p}_i \times \mathbf{p}_{i+1} = k_i \mathbf{K}, \qquad i = 0, 1, \cdots, n.$$

In case $\mathbf{K} = 0$ this equation shows that the $\mathbf{p}_i$ are all parallel

and the funicular polygon is consequently a straight line through O. If $\mathbf{K} \neq 0$ we dot-multiply equation (14) through by $\mathbf{p}_i$ and remembering that the k_i are all positive, we find every $\mathbf{p}_i \cdot \mathbf{K} = 0$, showing that every vertex of the funicular polygon lies in the plane $\mathbf{p} \cdot \mathbf{K} = 0$ passing through O. The theorem is thus proven for this case. If any of the tensions $\mathbf{g}_i$ is zero the first part of the theorem holds only in the sense that the funicular polygon may degenerate into two or more polygons separately satisfying the conclusions of the theorem, these polygons being joined by cords in which there is no tension.

EXERCISES

1. Show that if the funicular polygon is a parallelogram, then the polygon of Varignon has its pole at the intersection of its diagonals.

2. A rhombus of light cord has equal masses m_1 attached at the opposite vertices P_1, P_3 and equal masses m_2 attached at the other vertices P_2, P_4. If these masses are repelled from a point O with forces proportional to the masses and to the distances from O, show that when the set is in equilibrium O will be at the intersection of the diagonals of the rhombus. Show that for $m_1 \neq m_2$ the rhombus reduces to a straight line and that for $m_1 = m_2$ its shape is indeterminate.

3. Show that when the tensions $\mathbf{g}_{i-1}$ and $\mathbf{g}_i$ in two adjacent sections of the light cord are of equal amount g, then the force $\mathbf{f}_i$ acting on the particle P_i between these sections bisects the angle between $\mathbf{g}_{i-1}$ and $-\mathbf{g}_i$ and is of amount $f_i = 2g \cos(\alpha/2)$ where α is the angle between the two sections.

4. A light cord in the vertical XY-plane has its ends fastened at $A \equiv (-5, 0)$, $B \equiv (6, 0)$. Particles of masses m_1, m_2, m_3 attached to the cord at $P_1 \equiv (-4, -3)$, $P_2 \equiv (1, -7)$, $P_3 \equiv (4, -5)$ are acted upon by the force of gravity. If the set is in equilibrium, determine the ratios of the masses. *Ans.* $m_1 : m_2 : m_3 = 6 : 4 : 5$

5. A light cord with heavy attached particles P_i is supported at the ends. Show that if the particles are in equilibrium under the action of gravity, then the vertical line through the centroid of the particles is either concurrent with the lines of the terminal sections of the cord or is parallel to both of them.

6. A light cord A, P_1, P_2, B in the XY-plane has its ends fastened at $A \equiv (-5, 0)$, $B \equiv (5, 0)$. Particles of masses m_1 and m_2 attached to the cord at $P_1 \equiv (-3, 3)$ and $P_2 \equiv (1, 4)$ are repelled from the origin with forces proportional to their masses and their distances from the origin. If the set is in equilibrium, determine the ratio of the masses. *Ans.* $m_1 : m_2 = 2 : 3$

7. A set of particles P_i attached to a light cord with fixed ends are attracted or repelled from a point O by forces proportional to their masses and to their distances from O. Show that when the particles are in equilibrium the line joining O to the centroid P of the particles is either concurrent with the lines of the terminal sections of the cord or is parallel to both of them.

8. A smooth light cord is strung through small equally distant rings $P_1, P_2, \cdots, P_n$ and pulled taut. Show that the amount f_i of the force exerted at the ring P_i is inversely proportional to the radius of the circle through P_{i-1}, P_i, P_{i+1}.

68. Theorems on the Heavy Cord.

If the number of particles on the cord discussed in the preceding article is very large the individual treatment of these particles which we employed becomes awkward and it is then desirable to treat them in groups somewhat as we did in § 66 in determining the centroid of a rigid body. This case arises in nature when we consider the equilibrium of a flexible cord, the particles of the cord itself being acted upon by certain exterior forces. We think of the cord of length l as extending from a point A to a point B and we represent by s the distance along the cord from the end A to any point P of the cord. We assume that the cord is *flexible*, which means that the interior forces exerted by the particles in PB upon the particles in AP may be regarded as a single force $\mathbf{g}$ with the point of application P. The vector $\mathbf{g}$ is regarded as a function of the scalar s and is called the *tension* in the cord. Its length g is the *scalar tension.*

Let us now consider the equilibrium of a segment $P_0 P$ of the cord, the length of the segment being Δs. The exterior forces acting on the particles of the segment will be the tension $\mathbf{g}$ at P and $-\mathbf{g}_0$ at P_0 and all the exterior forces applied to the particles of the segment. We shall assume that these latter forces are of such a nature that as P approaches P_0 they become equivalent to a single force $\mathbf{f}\,\Delta s$ with a point of application P_1 which approaches P_0. The limiting value of the vector $\mathbf{f}$ here introduced is regarded as a function of s evaluated at the point P_0 and known as the *exterior*

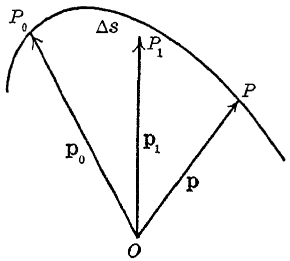

FIG. 72

force per unit length of the curve. Since a necessary condition for equilibrium of any set of particles is that the exterior forces constitute a null system, the forces acting on the segment must have a zero sum and a zero moment about any point O. Thus when the segment is short we may write approximately,

$$\mathbf{g} - \mathbf{g}_0 + \mathbf{f}\,\Delta s = 0, \qquad \mathbf{p}\times\mathbf{g} - \mathbf{p}_0\times\mathbf{g}_0 + \mathbf{p}_1\times\mathbf{f}\,\Delta s = 0,$$

where

$$\mathbf{p}_0 = OP_0, \qquad \mathbf{p} = OP, \qquad \mathbf{p}_1 = OP_1.$$

If we divide through by Δs and indicate by $\Delta\mathbf{g}$ and $\Delta(\mathbf{p}\times\mathbf{g})$ the variations in these quantities from P_0 to P, we have,

$$\mathbf{f} + \frac{\Delta\mathbf{g}}{\Delta s} = 0, \qquad \mathbf{p}_1\times\mathbf{f} + \frac{\Delta(\mathbf{p}\times\mathbf{g})}{\Delta s} = 0,$$

and on proceeding to the limit as P approaches P_0 we find,

$$(1) \qquad \mathbf{f} + \frac{d\mathbf{g}}{ds} = 0, \qquad (2) \qquad \mathbf{p}\times\mathbf{f} + \frac{d(\mathbf{p}\times\mathbf{g})}{ds} = 0$$

as necessary conditions for equilibrium at each point P of the cord. If we expand the derivative in equation (2) and recall that $\frac{d\mathbf{p}}{ds} = \mathbf{t}$, where $\mathbf{t}$ is the unit tangent vector to the cord at P running in the sense of increasing s, we have,

$$\mathbf{p}\times\mathbf{f} + \mathbf{p}\times\frac{d\mathbf{g}}{ds} + \mathbf{t}\times\mathbf{g} = 0.$$

The first two terms reduce to zero by equation (1) and we conclude that $\mathbf{t}\times\mathbf{g} = 0$. The tension in the cord is thus everywhere parallel to the tangent to the cord and consequently $\mathbf{g}$ must always be a scalar multiple of $\mathbf{t}$. We may therefore write,

$$(3) \qquad \mathbf{g} = g\,\mathbf{t},$$

where g is the scalar tension. Since the cord can only pull and never push $\mathbf{g}$ must always have the sense of $\mathbf{t}$ and g is therefore always positive or zero. If we substitute the above expression for $\mathbf{g}$ in equation (1) we have finally,

$$(4) \qquad \mathbf{f} + \frac{d(g\,\mathbf{t})}{ds} = 0.$$

This is the important *intrinsic equation of equilibrium.*

We may throw this intrinsic equation into the form of three scalar equations which make clear the relations of **f** and **g** to the curvature of the cord and to the trihedral of unit vectors **t**, **n**, **b** (§ 32) for the cord at the point considered. Expanding the derivative in equation (4) and remembering that $\frac{d\mathbf{t}}{ds} = \frac{1}{\rho}\mathbf{n}$, where $\frac{1}{\rho}$ is the curvature and **n** is the principal normal, we have,

$$\mathbf{f} + \frac{dg}{ds}\mathbf{t} + \frac{g}{\rho}\mathbf{n} = 0.$$

Consequently we have at once the relations,

$$(5) \qquad \mathbf{f}\cdot\mathbf{t} + \frac{dg}{ds} = 0, \qquad \mathbf{f}\cdot\mathbf{n} + \frac{g}{\rho} = 0, \qquad \mathbf{f}\cdot\mathbf{b} = 0,$$

where **b** is the binormal to the curve at the point. These are the intrinsic equations of equilibrium in scalar form. They have simple and interesting geometric meanings.

If **f** is given in terms of **t** and s and its point of application P, it is theoretically possible to integrate equation (4) and determine the exact form of the curve of equilibrium of the cord, the result depending also on the values of the constants of integration which enter into the solution. Thus if curves of equilibrium exist, equation (4) should serve to determine them. Furthermore it may be shown that equation (4) is not only a necessary condition for equilibrium but is also sufficient. Thus every solution of equation (4) will be a curve of equilibrium. In the general case, however, the solution of this equation turns out to be mathematically equivalent to the solution of a scalar differential equation of the sixth order in one dependent variable and this is usually an extremely difficult problem. However we may in certain cases prove some theorems analogous to those of the preceding article and on the basis of these complete the integration.

I. Forces Perpendicular to an Axis. If the exterior force **f** is everywhere perpendicular to a fixed axis, the measure of the tension **g** on this axis is constant. For let **u** be a unit vector having the direction and sense of the fixed axis. Dot-multiplying both members of equation (1) with **u** gives,

$$\mathbf{u}\cdot\mathbf{f} + \frac{d(\mathbf{u}\cdot\mathbf{g})}{ds} = 0$$

and since by hypothesis $\mathbf{u} \cdot \mathbf{f} = 0$ we have at once,

$$\frac{d(\mathbf{u} \cdot \mathbf{g})}{ds} = 0 \qquad \text{and} \qquad \mathbf{u} \cdot \mathbf{g} = K \text{ (constant)},$$

which proves the statement. As a special case of this we have the theorem,

If the force $\mathbf{f}$ *remains everywhere perpendicular to a fixed axis and if the line joining the end points A and B of the cord is also perpendicular to this axis, then the curve of equilibrium of the cord is a plane curve in a plane perpendicular to this axis and the measure of the tension* $\mathbf{g}$ *on the axis is everywhere zero.*

Evidently we may employ the above result and write,

$$\mathbf{u} \cdot \mathbf{g} = \mathbf{u} \cdot g\,\mathbf{t} = K.$$

We shall at first assume that the tension $\mathbf{g}$ is nowhere zero. If now O is any fixed point and $OP = \mathbf{p}$, then $\mathbf{t} = \frac{d\mathbf{p}}{ds}$ and the above equation becomes,

$$\mathbf{u} \cdot \frac{d\mathbf{p}}{ds} = \frac{K}{g} \qquad \text{or} \qquad \mathbf{u} \cdot d\mathbf{p} = \frac{K}{g}\, ds.$$

Integrating this from one end of the curve to the other we have,

$$\mathbf{u} \cdot (\mathbf{b} - \mathbf{a}) = K \int_0^l \frac{ds}{g} \qquad \text{where} \qquad OA = \mathbf{a}, \qquad OB = \mathbf{b}.$$

The left member is zero by hypothesis and since g is everywhere positive it follows that $K = 0$. Then we have $\mathbf{u} \cdot \mathbf{g} = 0$ which proves the last statement in the theorem. Also,

$$\mathbf{u} \cdot \frac{d\mathbf{p}}{ds} = 0 \qquad \text{or} \qquad \mathbf{u} \cdot d\mathbf{p} = 0$$

and on integrating this along the curve from A to P we obtain,

$$(\mathbf{p} - \mathbf{a}) \cdot \mathbf{u} = 0.$$

This being the equation of a plane perpendicular to $\mathbf{u}$, the theorem is proven for this case.

If the tension $\mathbf{g}$ vanishes at a finite number of points of the cord, the theorem still holds although we can no longer set $\mathbf{u} \cdot \mathbf{t} = K/g$. For in this case $\mathbf{u} \cdot \mathbf{g} = K = 0$ and the last statement in the

theorem follows. And if we now mentally cut out from the cord arbitrarily short segments containing all the points at which g vanishes, then throughout each of the remaining segments we have $\mathbf{u} \cdot \mathbf{t} = 0$ and can prove as before that each segment is a plane curve in a plane perpendicular to **u**. But since these planes are parallel and separated from each other by arbitrarily small amounts they must coincide and the theorem still holds. If g vanishes throughout two or more segments of the curve the first statement of the theorem may hold only in the sense that it holds for the remaining segments separately.

II. Forces Parallel to an Axis. The principal theorem for this case is,

If the force **f** *is everywhere parallel to a fixed axis, the curve of equilibrium of the cord lies in a plane parallel to this axis and the measure of the tension* g *on any axis perpendicular to the given axis is a constant.*

Let **u** be a unit vector having the direction and sense of the fixed axis. Cross-multiplying both members of equation (4) by **u** gives,

$$\mathbf{u} \times \mathbf{f} + \frac{d(g\,\mathbf{u} \times \mathbf{t})}{ds} = 0.$$

The first term is zero by hypothesis and consequently,

$$\frac{d(g\,\mathbf{u} \times \mathbf{t})}{ds} = 0 \qquad \text{or} \qquad g\,\mathbf{u} \times \mathbf{t} = \mathbf{K} \text{ (constant)}.$$

In proving the first statement in the theorem we shall at first assume that the tension g is nowhere zero. Since $\mathbf{t} = \dfrac{d\mathbf{p}}{ds}$ we may write the above equation as,

$$\mathbf{u} \times d\mathbf{p} = \mathbf{K}\frac{ds}{g}$$

and upon integrating along the curve from A to P we have,

$$\mathbf{u} \times (\mathbf{p} - \mathbf{a}) = \mathbf{K}\int_0^s \frac{ds}{g}.$$

If $\mathbf{K} = 0$ the curve of equilibrium is therefore on the line $(\mathbf{p} - \mathbf{a}) \times \mathbf{u} = 0$, which being parallel to **u** shows that the theorem holds

in this case. For $\mathbf{K} \neq 0$ we dot-multiply both members of the above equation by $\mathbf{p} - \mathbf{a}$ and remembering that g is everywhere positive we have $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{K} = 0$. This is the equation of a plane perpendicular to $\mathbf{K}$ and therefore parallel to $\mathbf{u}$ and the first statement in the theorem follows in this case. In case the tension $\mathbf{g}$ vanishes at certain points or along certain segments of the cord the first statement in the theorem may hold only in the sense that it applies to the remaining segments separately.

To prove the last part of the theorem we let $\mathbf{v}$ be a unit vector having the direction and sense of any axis perpendicular to the given axis. Then evidently,

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{g}) = \mathbf{v} \cdot \mathbf{g},$$

and on dot-multiplying both members of the above equation $\mathbf{u} \times \mathbf{g} = \mathbf{K}$ by $\mathbf{u} \times \mathbf{v}$ we find,

$$\mathbf{v} \cdot \mathbf{g} = [\mathbf{u}\ \mathbf{v}\ \mathbf{K}],$$

which proves the statement.

III. Central Forces. The principal theorem for this case is,

If the line of action P, $\mathbf{f}$ of the applied force always passes through a fixed point O, then the curve of equilibrium of the cord is a plane curve in a plane through O and the moment of the tension $\mathbf{g}$ with respect to O is a constant.

By hypothesis $\mathbf{p} \times \mathbf{f} = 0$ where $\mathbf{p} = OP$, and hence on cross-multiplying both members of equation (4) by $\mathbf{p}$ we have,

$$\mathbf{p} \times \frac{d(g\ \mathbf{t})}{ds} = 0.$$

If we now form,

$$\frac{d(\mathbf{p} \times g\ \mathbf{t})}{ds} = \mathbf{p} \times \frac{d(g\ \mathbf{t})}{ds} + \frac{d\mathbf{p}}{ds} \times g\ \mathbf{t},$$

we see that the first term of the second member has just been shown to be zero while the second term is zero due to the fact that $\mathbf{t} = \dfrac{d\mathbf{p}}{ds}$. Hence we have,

$$\mathbf{p} \times g\ \mathbf{t} = \mathbf{K} \ \text{(constant)}. \tag{6}$$

This proves the second statement in the theorem. In proving the first statement in the theorem we shall at first assume that

the tension $\mathbf{g}$ is nowhere zero. For $\mathbf{K} \neq 0$ we dot-multiply both members of the above equation by $\mathbf{p}$ and find $\mathbf{p} \cdot \mathbf{K} = 0$. This being the equation of a plane through O proves the first statement in the theorem for this case. If $\mathbf{K} = 0$, then since g is nowhere zero we have,

$$\mathbf{p} \times \mathbf{t} = \mathbf{p} \times \frac{d\mathbf{p}}{ds} = 0,$$

and consequently $\mathbf{p}$ is a vector of constant direction (§ 27). The curve of equilibrium is thus a straight line through O and the first statement in the theorem holds for this case. If $\mathbf{g}$ is zero at certain points or along certain segments of the cord the first statement in the theorem may hold only in the sense that it applies to the remaining segments separately.

EXERCISES

1. Show that if a cord is in equilibrium with the applied force $\mathbf{f}$ everywhere normal to the cord, then the scalar tension g is constant along the cord.

2. Show that if a cord is in equilibrium, the applied force $\mathbf{f}$ makes everywhere an obtuse or right angle with the principal normal to the cord.

3. Show that if a cord is in equilibrium with the line of action P, $\mathbf{f}$ of the applied force everywhere coplanar with a given axis, then the moment of the tension $\mathbf{g}$ with respect to this axis is constant along the cord.

4. A cord is acted upon at each point P by a force of amount f central at a fixed point O. If φ is the angle which the radius vector OP makes with the tangent to the cord at P and ρ is the radius of curvature at P, show that the condition of equilibrium is,

$$fp \sin^2 \varphi = K/\rho,$$

where $p = OP$ and K is the constant moment of the tension $\mathbf{g}$ with respect to an axis through O perpendicular to the plane of the cord.

5. If the equation of the plane curve of equilibrium of a cord is given, express the measure f of the force applied parallel to the Y-axis and the resulting tension g in terms of the coördinates x, y, the arc s, the inclination α to the X-axis, according to convenience. Apply to $x^2 + y^2 = a^2$, $y^2 = ax$, $x^2 = ay$, $y = a \cosh x/a$, etc.

Ans. $f = K \dfrac{dy'}{ds}$ where $y' = \dfrac{dy}{dx}$, $g = - K \sec \alpha$

For $x^2 + y^2 = a^2$, $f = - Ka/y^2$

6. If the equation of the plane curve of equilibrium of a cord is given, express the measure f of the force applied along the radius vector $\mathbf{p} = OP$ and the resulting tension g in terms of the polar coördinates p, θ, the arc s, according to convenience. Apply to $x^2 - y^2 = a^2$, $p = 2a \cos \theta$, etc.

$$Ans.\quad g = \frac{K}{p^2}\frac{ds}{d\theta}, \qquad f = -\frac{dg}{dp}$$

For $x^2 - y^2 = a^2$, $\qquad f = - K/a^2$

7. A flexible cord with non-uniform linear density is supported at the ends and subject to the force of gravity. Show that the vertical line through the centroid of the cord is concurrent with the tangents at the ends.

8. A flexible cord with non-uniform linear density is fastened at the ends and attracted or repelled from a fixed point O by a force which at each point P of the cord is proportional to the linear density at that point and to the distance OP. Show that the line through O and the centroid of the cord is concurrent with the tangents at the ends.

69. Integration of the Equation of Equilibrium.

On the basis of the general theorems proven in the preceding article it is possible to determine completely the curve of equilibrium in certain cases. For instance if the applied force $\mathbf{f}$ is a known function of the arc s and if we know the tension $\mathbf{g}_0$ in the cord at some given point P_0, then the curve of equilibrium and the tension at any point of it may be determined as follows. We write equation § 68, (1) as $d\mathbf{g} = - \mathbf{f}\, ds$ and integrate along the curve from P_0 to P obtaining,

$$\mathbf{g} = -\int_{s_0}^{s} \mathbf{f}\, ds + \mathbf{g}_0.$$

The scalar tension g being essentially positive is given by $g = |\mathbf{g}|$ and then $\mathbf{t}$ by $\mathbf{t} = \mathbf{g}/|\mathbf{g}|$. But since $\mathbf{t} = \frac{d\mathbf{p}}{ds}$ we have $d\mathbf{p} = \frac{\mathbf{g}}{g} ds$, and integrating between corresponding limits yields,

$$\mathbf{p} = \int_{s_0}^{s} \frac{\mathbf{g}}{g}\, ds + \mathbf{p}_0.$$

This gives us the vector $\mathbf{p} = OP$ as a function of the arc s and

thus determines the form of the curve of equilibrium while the tension **g** has also been found.

One of the most interesting cases is that in which the applied force **f** is a constant. This is the case of a flexible cord or chain of constant linear density hanging from supports and subject to the force of gravity. The above method of integrating the equation of equilibrium may be followed with some simplifications introduced by the fact that from Case II of the preceding article we know that the curve of equilibrium lies in a vertical plane and that the measure of the tension **g** on any horizontal axis is constant. Let us take the plane of the curve as the XY-plane with the Y-axis upward. Then we may write,

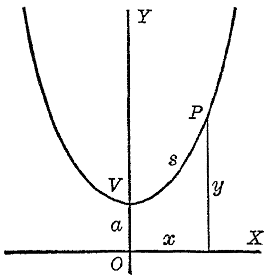

FIG. 73

$$\mathbf{f} \equiv (0, -w, 0), \qquad \mathbf{t} \equiv (\cos\alpha, \sin\alpha, 0),$$

where α is the angle which **t** makes with the X-axis and where w is the weight per unit length of the cord or chain. If we now measure equation (4) on the X and Y-axes we have,

$$\frac{d(g\cos\alpha)}{ds} = 0, \qquad -w + \frac{d(g\sin\alpha)}{ds} = 0.$$

The first of these yields at once,

$$g\cos\alpha = g_0 \text{ (constant)},$$

as was to be expected from the last statement in the theorem of Case II. If the value $g = g_0/\cos\alpha$ be substituted in the second of the above equations it takes the simple form,

$$dm = \frac{1}{a}\,ds \qquad \text{where} \qquad m = \tan\alpha, \qquad a = \frac{g_0}{w}.$$

But for a plane curve we have $ds = \sqrt{1+m^2}\,dx$ so that our last equation may be written,

$$\frac{dx}{a} = \frac{dm}{\sqrt{1+m^2}}.$$

If we integrate both members of this equation between corre-

sponding limits we obtain,

$$\frac{x}{a} = \log\,(m + \sqrt{1 + m^2}) \qquad \text{or} \qquad e^{x/a} = m + \sqrt{1 + m^2},$$

it being assumed that the Y-axis is so located in the plane that $m = 0$ for $x = 0$. Solving for m we find,

$$m = \frac{dy}{dx} = \frac{1}{2}\,(e^{x/a} - e^{-x/a})$$

and multiplying through by dx and integrating between corresponding limits yields,

$$y = \frac{a}{2}\,(e^{x/a} + e^{-x/a}),$$

where it is assumed that the X-axis is so located that $y = a$ for $x = 0$. This then is the equation of the curve of equilibrium of a flexible cord of uniform linear density subject to the force of gravity. The curve is called a *catenary* (Latin, *catena*, a chain) and in this position the X-axis is called the *directrix* and the point $(0, a)$ the *vertex*. The equation may evidently be written,

$$y = a \cosh \frac{x}{a}\cdot$$

To get an idea of the general appearance of this curve in the neighborhood of the vertex we expand the second member as a power series in x obtaining,

$$y = a + \frac{x^2}{2a} + \frac{x^4}{24a^3} + \cdots.$$

For values of x small in comparison with a we may drop all terms of the second member after that in x^2 and conclude that the curve very closely approximates the parabola,

$$x^2 = 2a(y - a).$$

All catenaries are the same shape, differing only in size.

The Y-coördinate of a point P on the catenary is connected with the angle α which the tangent at the point makes with the X-axis by the simple relation,

$$y = a \sec \alpha,$$

for since,

$$m = \tan\alpha = \tfrac{1}{2}(e^{x/a} - e^{-x/a}) = \sinh\frac{x}{a},$$

it follows that,

$$a\sec\alpha = a\sqrt{1 + \sinh^2\frac{x}{a}} = a\cosh\frac{x}{a} = y.$$

If s represent the length of the arc of the curve from the vertex $V \equiv (0, a)$ to the point $P \equiv (x, y)$ the reader may readily prove that,

$$s^2 = y^2 - a^2,$$

and consequently that,

$$s = a\tan\alpha.$$

For the scalar tension g at any point P of the curve we have,

$$g = g_0 \sec\alpha = aw\sec\alpha = wy.$$

Further properties of the catenary are stated in the problems.

As a final example of integration of the equation of equilibrium we may consider the *flexible cylindrical tank*. The flat end faces of a cylindrical tank are set up in vertical planes and thus the elements of the cylindrical face are horizontal. It is desired to determine the form of the cross section perpendicular to the cylindrical elements so that the cylindrical face shall have no tendency to bend when the tank is supported along the top element and is filled with a liquid of density w to a pressure which would cause the liquid to rise to a height h above the top. Since the pressure exerted by a liquid is everywhere perpendicular to the face we may consider the equilibrium of a strip of unit width running around the tank and treat it as a flexible cord, the amount of the force $\mathbf{f}$ acting on it being proportional to the head of the liquid at the point. If we take our coördinate system in the plane of the cross section with a horizontal X-axis at a height h above the top of the tank and an upward Y-axis through the top of the tank, then we have for equilibrium,

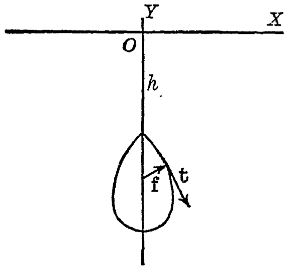

FIG. 74

$$\mathbf{f} + \frac{d(g\,\mathbf{t})}{ds} = 0,$$

where,

$$\mathbf{t} \equiv (\cos\alpha, \sin\alpha, 0), \qquad \mathbf{f} \equiv (wy \sin\alpha, -wy\cos\alpha, 0),$$

α as before being the angle which $\mathbf{t}$ makes with the X-axis. Measuring the equation on the coördinate axes gives,

$$wy \sin\alpha\, ds + d(g\cos\alpha) = 0,$$
$$-wy\cos\alpha\, ds + d(g\sin\alpha) = 0.$$

These equations readily yield the results,

$$dg = 0, \qquad \frac{d\alpha}{ds} = \frac{y}{a} \qquad \text{where} \qquad a = \frac{g}{w},$$

showing that the scalar tension g is a constant and that the curvature $\frac{d\alpha}{ds}$ of the cross section is proportional to the distance from the X-axis. Since $\sin\alpha\, ds = dy$ we also obtain,

$$y\, dy + a\, d(\cos\alpha) = 0,$$

which integrates into,

$$y^2 + 2a\cos\alpha = c,$$

the constant $c = h^2 + 2a\cos\alpha_0$ being determined by h, a and the value α_0 of α along the top ridge of the tank. Then,

$$\frac{dy}{dx} = \tan\alpha = \pm\frac{\sqrt{4a^2 - (c - y^2)^2}}{c - y^2}$$

and since $x = 0$ for $y = -h$, we have on integrating once more,

$$x = \pm\int_{-h}^{y} \frac{c - y^2}{\sqrt{4a^2 - (c - y^2)^2}}\, dy.$$

This is an elliptic integral and can not be evaluated in terms of the elementary functions but it may be reduced to a form involving only the standard elliptic integrals whose values are tabulated, and thus the curve may be plotted.

EXERCISES

1. Determine the curve of equilibrium and the tension in a cord subject to a force directed downward and of amount proportional to the cosine of the angle which the curve makes with the horizontal.
Ans. Parabola with vertical axis

2. Determine the curve of equilibrium and the tension in a cord subject to a force directed downward and of amount $k \sec^2 s$ where k is the tension at the lowest point of the cord and s is the arc measured from that point. *Ans.* Circle of unit radius

3. Determine the form of the catenary of constant stress, i.e. the curve in which a heavy cord of homogeneous material will hang when the cross section is everywhere proportional to the scalar tension. *Ans.* $y = a \log \sec x/a$

4. *The Jumping-Rope Curve.* A uniform heavy cord is fastened at its ends to a fixed axis and rotated about this axis with a constant angular velocity ω. The resulting centrifugal force $\mathbf{f}$ at each point P of the cord is of amount $\omega^2 y$ where y is the distance from P to the axis and $\mathbf{f}$ intersects the axis at right angles. Show that the curve of equilibrium is a plane curve whose equation in a rotating plane is,

$$x = a^2 \int_0^y \frac{dy}{\sqrt{(b^2 - y^2)^2 - a^4}}, \qquad a, b \text{ constant}, \qquad b > a.$$

5. Determine the curve of equilibrium and the scalar tension g of a cord when the applied force $\mathbf{f}$ at each point P of the cord is central at a fixed point O and of an amount f which is a function of the distance $p = OP$,

$$\mathbf{f} = \frac{\varphi(p)}{p}\,\mathbf{p},$$

and in which the scalar tension g_0 at some point P_0 is known.

Ans. $\mathbf{t}\cdot\mathbf{f} + \mathbf{t}\cdot\dfrac{d\mathbf{g}}{ds} = 0$ yields $g = -\displaystyle\int_{p_0}^{p} \varphi(p)dp + g_0$

$g\,[\mathbf{u}\,\mathbf{p}\,\mathbf{t}] = \mathbf{u}\cdot\mathbf{K} = K$ yields $\theta = \displaystyle\int_{p_0}^{p} \frac{K\,dp}{p\sqrt{g^2p^2 - K^2}} + \theta_0$

where $\mathbf{u}$ is a unit vector parallel to $\mathbf{K}$ and where p, θ are plane polar coördinates.

6. Determine the curve of equilibrium and the scalar tension g of a cord when the applied force $\mathbf{f}$ is an attraction of constant amount c to a fixed point O and in which the scalar tension at some point P_0 of the cord is cp_0, ($p_0 = OP_0$).
Ans. An equilateral hyperbola with center at O

7. If s is the length of the arc of the catenary $y = a \cosh \dfrac{x}{a}$ from the

vertex $A \equiv (0, a)$ to the point $P \equiv (x, y)$, show that $s = a \sinh \dfrac{x}{a}$ and hence that $s^2 = y^2 + a^2$.

8. If M is the foot of the perpendicular let fall from the point P of the catenary $y = a \cosh \dfrac{x}{a}$ upon the directrix (X-axis), show that the distance from M to the tangent at P is a.

9. A uniform heavy flexible cord hanging from two supports has weights attached at certain points along it. Show that the segments of the cord are arcs of the same sized catenary and that each weight is as heavy as the corresponding missing arc of the catenary.

10. Show that the radius of curvature ρ at each point P of a uniform flexible cord acted upon by gravity is proportional to the square of the tension g at that point.

11. A loose piece of uniform flexible cord is placed over two smooth pegs. Show that for equilibrium both ends of the cord must just reach to the directrix of the catenary formed by the middle section.

12. A loose piece of uniform flexible cord of length $2c$ is placed over two smooth pegs at the same level $2x$ units apart. If a is the parameter of the catenary formed by the middle section, show that for equilibrium,

$$c = ae^{x/a}.$$

Hence show that the least length of the cord compatible with equilibrium is $2c = 2ex$ and that for cords longer than this there will be exactly two positions of equilibrium.

13. A uniform flexible cord of length $2s$ hangs between two supports on the same level $2x$ units apart and sags an amount h in the middle. Determine x in terms of s and h.

$$Ans. \quad x = \frac{s^2 - h^2}{2h} \log \left(\frac{s + h}{s - h}\right)$$

14. A uniform flexible cord of length $2s$ hangs between two supports on the same level, sags an amount h in the middle and makes an angle α with the horizontal at the ends. Express h in terms of s and α.

$$Ans. \quad h = s \tan \frac{\alpha}{2}$$

15. A uniform flexible cord of length $2s$ hangs between two supports on the same level $2x$ units apart and sags an amount h in the middle. Show that $\dfrac{h}{x}$ is a function of and that calling,

$$\left(\frac{h}{x}\right)^2 = \varphi\left(\frac{s - x}{x}\right)$$

the function φ is given for t sufficiently small by,

$$\varphi(t) = \frac{3}{2}t + \frac{21}{20}t^2 - \frac{27}{1400}t^3 + \frac{27}{2800}t^4 + \cdots$$

or,

$$t = \frac{2}{3}\varphi^2 - \frac{14}{45}\varphi^4 + \frac{278}{945}\varphi^6 + \cdots.$$

16. A cord 32 feet long is supported at the ends by posts of equal height 30 feet apart. Determine the sag h in the middle.
Ans. $h = 4.8527$

17. A flexible wire is supported by two posts of equal height 100 feet apart. If the wire sags 5 feet in the middle how much wire is used?
Ans. 100.6636 feet

18. A uniform flexible cable of length 40 feet and weighing 2 pounds per foot hangs between two supports on the same level. With a load of 1000 pounds attached at the mid-point there is a sag of 10 feet. Find the tension in the cable at the points of support.
Ans. 1050 pounds

70. Cord Constrained to a Surface.

In addition to the applied force **f** acting on a flexible cord we may have at each point a constraint **k** introduced by imposing some further condition upon the cord. The equation of equilibrium of the cord then becomes,

$$\text{(1)} \qquad \mathbf{f} + \mathbf{k} + \frac{d(g\,\mathbf{t})}{ds} = 0.$$

We shall illustrate this by considering the equilibrium of a cord constrained to remain upon a smooth surface. Let the surface be given by its equation,

$$\text{(2)} \qquad F(\mathbf{p}) = 0,$$

where F is a scalar function of the vector $\mathbf{p} = OP \equiv (x, y, z)$; O being the fixed origin and P any point in space. It can be shown (§ 100) that the vector,

$$*\,\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right),$$

when evaluated at any point P of the surface has the direction of the normal to the surface at the point. Since the surface is smooth the constraint **k** introduced by requiring that the cord

* ∇F is read del F and is called the *gradient* of F.

remain on the surface is everywhere normal to the surface and thus we have $\mathbf{k} = \lambda \nabla F$ where λ is a variable scalar as yet undetermined. The equation of equilibrium thus becomes,

$$\mathbf{f} + \lambda \nabla F + \frac{d(g\,\mathbf{t})}{ds} = 0, \tag{3}$$

and equations (2) and (3) determine the curve of equilibrium and the scalars g and λ.

A particularly interesting example is that in which $\mathbf{f}$ is everywhere zero. Expanding the derivative in equation (3) then gives us,

$$\lambda \nabla F + \frac{g}{\rho}\mathbf{n} + \frac{dg}{ds}\mathbf{t} = 0,$$

where $\mathbf{n}$ is the principal normal to the cord and $1/\rho$ is the curvature. Dot-multiplying through by $\mathbf{t}$ and remembering that since the cord is on the surface the tangent to it must be perpendicular to the normal to the surface, we find $\dfrac{dg}{ds} = 0$ and the scalar tension g is therefore a constant. Consequently we have,

$$\lambda \nabla F + \frac{g}{\rho}\mathbf{n} = 0,$$

and so unless the constant g is zero we conclude that the principal normal to the cord always lies along the normal ∇F to the surface. A curve lying on a surface and possessing this property is called a *geodesic* of the surface. We thus have the theorem,

A light cord constrained to a smooth surface lies along a geodesic of that surface unless the tension is everywhere zero.

This gives us intuitive evidence of the familiar theorem of differential geometry that the shortest distance on a surface between any two of its points is along the geodesic through them. The geodesics of a sphere are great circles and of a right circular cylinder are helices.

EXERCISES

1. Obtain the *intrinsic equations of equilibrium of a cord constrained to a smooth surface.* At each point P of the curve of equilibrium let $\mathbf{n}_1$ be the principal normal to the projection of the curve on the tangent plane to the surface at P and let ρ_1 be the radius of curvature of this projection, i.e. the *geodesic radius of curvature* of the

curve of equilibrium. Let $\mathbf{n}_2$ be the principal normal at P to that normal section of the surface which is tangent to the curve of equilibrium at P and let ρ_2 be the radius of curvature of this section. Then $\mathbf{t}$, $\mathbf{n}_1$, $\mathbf{n}_2$ form a trihedral of mutually perpendicular unit vectors and by *Meusnier's theorem,*

$$\rho = \rho_1 \, \mathbf{n} \cdot \mathbf{n}_1, \qquad \rho = \rho_2 \, \mathbf{n} \cdot \mathbf{n}_2.$$

If k is the measure of the constraint on $\mathbf{n}_2$ then the equation of equilibrium,

$$\mathbf{f} + k \, \mathbf{n}_2 + \frac{d\mathbf{g}}{ds} = 0,$$

yields,

$$\mathbf{t} \cdot \mathbf{f} + \frac{dg}{ds} = 0, \qquad \mathbf{n}_1 \cdot \mathbf{f} + \frac{g}{\rho_1} = 0, \qquad \mathbf{n}_2 \cdot \mathbf{f} + \frac{g}{\rho_2} + k = 0.$$

The first two of these equations are the intrinsic equations of equilibrium of the constrained cord, the third serving merely to determine the constraint k.

2. Show as a consequence of Problem 1 that if the smooth surface to which a cord is constrained be deformed in such a way that the lengths of all curves upon it are unaltered and if the applied force $\mathbf{f}$ be so changed that $\mathbf{t} \cdot \mathbf{f}$ and $\mathbf{n}_1 \cdot \mathbf{f}$ retain their former values, then the new curve of equilibrium of the cord is simply the result of this deformation of the original curve of equilibrium. Thus show how the determination of the curve of equilibrium of a cord constrained to a smooth cylinder or cone and acted upon by a force along the elements of the cylinder or cone can be reduced to a simple problem in the plane. Consider some examples.

CHAPTER IX

PRINCIPLE OF VIRTUAL WORK

71. Description and Historical Remark.

In the study of the motion and equilibrium of sets of particles we have not only considered that certain forces were applied to them but that certain conditions were imposed upon their positions or motions. Thus for example we have required that a particle remain on a fixed curve or surface or that a rigid body have a fixed point or axis. Since in the present chapter we are interested ultimately only in questions of equilibrium, we shall consider only imposed conditions which do not vary with the time. Since the imposed conditions generally modify the accelerations of the particles of the set we find it convenient to regard them as brought about by certain additional forces acting on these particles and known as *constraints*. Any motion of the set of particles satisfying the imposed conditions is said to be *compatible with the constraints*.

Since the constraints arise in this fashion they have certain peculiar properties differentiating them from other sets of forces. Thus the constraints are assumed to be just sufficient to maintain the imposed conditions and to act only in maintaining these conditions. In other words,

Motions of the set of particles satisfying the imposed conditions are as a whole neither aided nor opposed by the constraints.

The principle here stated descriptively will be put in a precise form in § 73. For example consider a bead strung on a wire. The force of gravity or other applied force may cause the bead to move while the constraint is the force exerted by the wire compelling the bead to remain upon it. In certain parts of the motion this constraint may require to be a very large force while if the motion of the bead under the applied force alone were along the wire the constraint would reduce to zero. If the wire is very smooth the constraint is practically the only effect the wire produces on the motion of the bead, but if the bead and wire are rough the wire may of course have a strong retarding

effect on the motion, but in this case we regard the effect of the wire as consisting of two forces, namely the constraint which the wire would exert if it were perfectly smooth and the retardation which we regard as one of the applied forces and not a constraint. In other words we assume,

The constraints act without friction.

As another example consider two weights attached to the ends of a flexible cord of fixed length which is supported by a freely turning pulley. Under the action of gravity the weights may swing on the supporting cord and at the same time one weight may descend while the other ascends. The constraint exerted by the cord may decidedly aid the swinging motion of one of the weights but its total effect in this way will be zero. This explains the words, "as a whole" in the above descriptive statement of the principle.

It will be evident from the above examples that the applied forces are usually rather readily measured, but we have often no direct manner of determining the constraints. In fact the constraints were for many years known as the lost forces. Thus in discussing the equilibrium of a set of particles we usually treat the constraints as unknown quantities which are eliminated from the equations in solving for the applied forces. There is however a general principle of mechanics known as the *principle of virtual work* which enables us to set up the equations of equilibrium in terms of the applied forces only.

This principle was employed for the first time by Galileo (1564–1642) but the appreciation of its great value in the solution of problems in statics may be said to date from a letter written by John Bernoulli to Varignon in 1717. The usefulness of the principle was greatly extended in 1743 when d'Alembert showed that problems in kinetics may by a simple device be treated as problems in statics (d'Alembert's Principle). Lagrange in his famous work, Mécanique Analytique, 1788, definitely established the principle of virtual work as a fundamental principle upon which the whole subject of classical mechanics may be made to rest.

72. Definition of Work, Energy.

Let us consider a particle moving along a straight line and acted upon by a constant force **f** having the direction and sense

of the motion. The product of the amount f of the force and the distance s traveled by the particle is called the *work* of the force over the path of the particle,

$$(1) \qquad w = fs.$$

Evidently the work depends only on the amount of the force and the path of the particle and does not involve the speed with which the particle traces out the path. In fact other forces may be acting on the particle during the motion. If the force makes an angle with the path of the particle then in computing the work we include only that component of $\mathbf{f}$ which lies along the path. This component is of length $f \cos \theta$ where θ is the angle which $\mathbf{f}$ makes with the direction and sense of the motion. Thus if $\mathbf{a}$ and $\mathbf{b}$ are the radius vectors from some fixed point O of the particle at the beginning and end of its motion, we have for the work of $\mathbf{f}$,

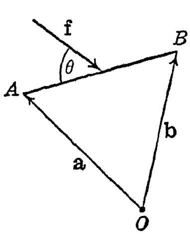

FIG. 75

$$(2) \qquad w = (f \cos \theta)s = \mathbf{f} \cdot (\mathbf{b} - \mathbf{a}).$$

Evidently the first definition of work falls under this second as a special case.

If the force $\mathbf{f}$ varies during the motion or if the path of the particle is curvilinear the above definition can not be directly applied but we may retain essentially the same idea by breaking the path into numerous short sections and treating each section much as if it were straight and the force constant. With this in mind we introduce the concept of *elementary work* defined for the action of a force over a path which we shall ultimately allow to approach zero in length. Suppose the particle to move along a given path from a point P_1 to a point P_2 and to be acted upon by a variable force $\mathbf{f}$. Then the elementary work of this force along this path is defined as the scalar product,

$$(3) \qquad dw = \mathbf{f} \cdot \Delta \mathbf{p},$$

where $\Delta \mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$ and where $\mathbf{f}$ is any value assumed by the force during this motion. It is clear that the elementary work is not uniquely determined by the given conditions since the definition does not specify exactly the value of $\mathbf{f}$ to be employed. However we shall always assume that $\mathbf{f}$ varies continuously along

the path and if the point P_2 approaches P_1 as a limit it is clear that $\mathbf{f}$ must take on the value it has at $\mathbf{P}_1$. Since then the concept of elementary work is applied only to infinitesimal displacements $\Delta\mathbf{p}$, i.e. to paths which are ultimately to approach zero in length, no essential ambiguity is involved in the definition.

We may now define *work* for the general case of a variable force acting on a particle moving over a curvilinear path. Let the path be a continuous curve c running from a point A to a point B. We divide this path into n segments by the points P_1, P_2, $\cdots$, P_{n-1} and form the elementary work for each segment, that for the i^{th} segment being,

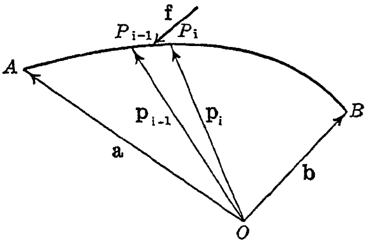

FIG. 76

$$dw_i = \mathbf{f}_i \cdot \Delta\mathbf{p}_i, \qquad (\Delta\mathbf{p}_i = \mathbf{p}_i - \mathbf{p}_{i-1}),$$

where $\mathbf{f}_i$ is any value assumed by $\mathbf{f}$ in the i^{th} segment. It is now possible to show that under quite general conditions the sum of these elementary works for all the segments will approach a uniquely determined limit as the number n of segments increases unlimitedly and the length of each segment approaches zero. When this limit exists it is defined as the work of the force $\mathbf{f}$ over the path c from A to B. Thus we write,

$$(4) \qquad w = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{f}_i \cdot \Delta\mathbf{p}_i = \int \mathbf{f} \cdot d\mathbf{p},$$

the last member being read, "the integral of $\mathbf{f} \cdot d\mathbf{p}$ along c." This is an example of what is known as a *line integral*. If the curve c is given by having $\mathbf{p} = OP$ expressed as a function of some parameter t and if the value of $\mathbf{f}$ at each point is also expressed as a function of t then we may write the work in the form of an ordinary definite integral,

$$(5) \qquad w = \int^{t_2} \mathbf{f} \cdot \mathbf{p}' \, dt, \qquad \left(\mathbf{p}' = \frac{d\mathbf{p}}{dt}\right)$$

where t_1 and t_2 are the values of t for the two end points A and B.

Closely associated with the concept of elementary work $dw = \mathbf{f}\cdot\Delta\mathbf{p}$ is that of the *virtual work* $\delta w = \mathbf{f}\cdot\delta\mathbf{p}$. The only difference is that the *virtual displacement* $\delta\mathbf{p}$ does not need to be the true displacement $\Delta\mathbf{p}$ which the particle actually goes through but may be any displacement whatever. It is understood in both cases that the displacement is to be infinitesimal.

It is clear from equation (1) or (4) that if a constant unit force act over a unit distance in its own direction and sense, the resulting work will also be unity. The work of a pound of force thus acting over a distance of one foot is called a *foot-pound* and the work of a force of one dyne thus acting over a distance of one centimeter is called an *erg*. The meaning of the terms *foot-poundal, kilogram-meter*, etc., will be clear by analogy.

If we consider a set of particles each of which may be acted upon by forces then the elementary work of all the forces is the sum of the elementary works of the separate forces if the displacements take place simultaneously. The same is true of the virtual work and of the work. If w is the work done on a set of particles by a system of forces from a given instant t_0 to any instant t then w will be a scalar function of t and the value of the derivative $\frac{dw}{dt}$ at any instant is called the *power* of the forces at that instant. From equation (5) we have for the power the formula,

$$w' = \mathbf{f}\cdot\mathbf{p}'. \tag{6}$$

The practical engineering unit of power in English-speaking countries is the *horse-power* which is defined as 33,000 foot-pounds per minute.

When a system of forces does work on a set of particles we say that the *energy* of the set has thereby been changed by the amount of the work done. Although we shall not define the total energy of a set of particles, it appears from this statement that energy and work are the same sort of thing and may be measured in the same units. One form in which the energy of a particle may present itself is in its *kinetic energy*, $\frac{1}{2}mv^2$, where m is the mass and v the scalar velocity, but there are many other forms of energy not all of which can be properly discussed in mechanics. It is found in certain of the simpler experiments of physics and chemistry that energy is freely transformed from one form to another but is never created nor destroyed. This

fact is known as the *principle of the conservation of energy.* It is capable of mathematical proof only in very simple theoretical cases but it has a wide range of practical applications even though its complete significance and even its ultimate physical truth are still under investigation.

EXERCISES

1. Show that the work of a central force acting on a particle P over a path C depends only on the end points of the path and not on the form of the path between them, provided that the amount of the force is a function of the distance p from P to the center of force O.
2. Show that the work of a force of constant direction acting on a particle P over a path C depends only on the end points of the path and not on the form of the path between them, provided the amount of the force is a function of the distance r from P to a fixed plane perpendicular to the force.
3. Show that the work of a force acting on a particle P over a path C and constantly directed along the perpendicular from P to a fixed axis depends only on the end points of the path and not on the form of the path between them, provided the amount of the force is a function of the distance r from P to the fixed axis.

73. Proof of the Principle.

Let us consider a set of particles $P_1, P_2, \cdots, P_n$, each particle P_i acted upon by an applied force $\mathbf{f}_i$ and let us assume that certain conditions not varying with the time have been imposed upon the position or motion of the set which cause each particle P_i to be acted upon by a corresponding constraint $\mathbf{k}_i$. The total force acting upon P_i is thus $\mathbf{f}_i + \mathbf{k}_i$. Employing the concept of work introduced in the preceding article we may now put in precise form the characteristic property of the constraints arising from the fact that they act only in bringing about the imposed conditions. To say that the motions of the set of particles satisfying the conditions are as a whole neither aided nor opposed by the constraints we shall interpret as meaning,

In any portion of the motion of the set of particles the total work of the constraints is zero, i.e.,

$$\Sigma \int \mathbf{k}_i \cdot d\mathbf{p}_i = 0, \tag{1}$$

where δt is any time interval during the motion.

This principle may be regarded as a definition of the constraints arising from imposed conditions which do not vary with the time, or at least as a basic postulate concerning such constraints. It will be observed that the applied forces $\mathbf{f}_i$ are not involved.

By introducing the concept of the virtual displacements of the particles we may put the above principle in more convenient form for application. Consider the set of particles at rest at any instant and imagine applied to them an arbitrary system of continuously varying forces $\mathbf{f}_i$ during a time interval δt. The particles moving under these forces and subject to the imposed conditions will have a set of displacements $\delta \mathbf{p}_i$ which will approach zero with δt and which we shall call a set of *virtual displacements compatible with the constraints*. The displacements are *virtual* because they may be quite different from those which the particles would have under the actual system of applied forces. If now the work of each constraint $\mathbf{k}_i$ varies continuously during the time interval δt, the principal part of this infinitesimal work as δt approaches zero will be the virtual work $\mathbf{k}_i \cdot \delta \mathbf{p}_i$ in which $\mathbf{k}_i$ has its value at the beginning of the time interval. Equation (1) then yields,

(2) $$\Sigma\, \mathbf{k}_i \cdot \delta \mathbf{p}_i = 0.$$

These displacements $\delta \mathbf{p}_i$ for which the constraints vary continuousy are said to be *reversible* and the constraints are said to be *bilateral* for these displacements because the above equation would still hold if every $\delta \mathbf{p}_i$ were changed in sign. We may thus state,

The total virtual work of the constraints for any set of reversible displacements is zero.

We have also to consider motions of the set of particles for which some of the constraints vary discontinuously. For example, the constraint exerted by a flexible cord attaching a weight to a fixed point acts only when the cord is taut and will suddenly vanish if the weight is moved in a direction making an acute angle with the constraint, while motions in directions making an obtuse angle with the constraint are incompatible with it and can not take place, and motions at right angles to the constraint are compatible and reversible. Displacements of a set of particles which are compatible with the constraints and become incompatible when reversed in sense are said to be *irre-*

versible and for such displacements the constraints are said to be *unilateral*. As appears from the above example,

The total virtual work of the constraints for any set of irreversible displacements $\delta' \mathbf{p}_i$ *is positive, i.e.,*

$$\Sigma\, \mathbf{k}_i \cdot \delta' \mathbf{p}_i > 0. \tag{3}$$

On the other hand, the actual work of the constraints during any time interval δt continues to be zero as stated in equation (1) even if the motions of some of the particles are irreversible, for in that case the corresponding unilateral constraints vanish as soon as the motion starts and the single non-zero value can not affect the corresponding integral. Thus we have,

$$\Sigma \int \mathbf{k}_i \cdot d' \mathbf{p}_i = 0, \tag{4}$$

where the motions as a whole are irreversible.

We may now easily prove the very important,

Principle of Virtual Work. A necessary and sufficient condition that a set of particles be in equilibrium is that the total virtual work of the applied forces be zero for every set of reversible compatible virtual displacements and negative for every set of irreversible compatible virtual displacements, i.e.,

$$\Sigma\, \mathbf{f}_i \cdot \delta \mathbf{p}_i = 0, \qquad \Sigma\, \mathbf{f}_i \cdot \delta' \mathbf{p}_i < 0. \tag{5}$$

The condition is necessary, for if the particles are in equilibrium the total force $\mathbf{f}_i + \mathbf{k}_i$ acting on each particle must be zero. Thus each $(\mathbf{f}_i + \mathbf{k}_i) \cdot \delta \mathbf{p}_i = 0$ for any virtual displacement whatever and on summing all such equations we have,

$$\Sigma\, \mathbf{f}_i \cdot \delta \mathbf{p}_i + \Sigma\, \mathbf{k}_i \cdot \delta \mathbf{p}_i = 0. \tag{6}$$

If now these displacements are any set of reversible compatible displacements equations (2) and (6) yield the first of equations (5) and if they are irreversible compatible displacements equations (3) and (6) yield the second of equations (5).

To see that the conditions (5) are also sufficient for equilibrium we assume the set of particles to be at rest but not in equilibrium. The particles will then move compatibly with the constraints and during a sufficiently short interval of time δt the direction of motion of each particle will make an acute angle with the total

force acting upon it and we shall have,

$$(7) \quad \Sigma \int (\mathbf{f}_i + \mathbf{k}_i)\cdot d\mathbf{p}_i = \Sigma \int_{\delta t} \mathbf{f}_i\cdot d\mathbf{p}_i + \Sigma \int_{\delta t} \mathbf{k}_i\cdot d\mathbf{p}_i > 0.$$

Whether the motion be reversible or not, the last of the above sums will vanish by equations (1) or (4) and we shall always have,

$$\Sigma \int_{\delta t} \mathbf{f}_i\cdot d\mathbf{p}_i > 0 \qquad \text{or} \qquad \Sigma \int_{\delta t} \mathbf{f}_i\cdot d'\mathbf{p}_i > 0.$$

Now if the time interval δt be allowed to approach zero the principal part of the integral $\int_{\delta t} \mathbf{f}_i\cdot d\mathbf{p}_i$ will be $\mathbf{f}_i\cdot \delta\mathbf{p}_i$ in which $\mathbf{f}_i$ has its value at the beginning of the interval and hence,

$$(8) \qquad \Sigma\, \mathbf{f}_i\cdot \delta\mathbf{p}_i > 0, \qquad \Sigma\, \mathbf{f}_i\cdot \delta'\mathbf{p}_i > 0.$$

These inequalities contradict the conditions (5). Thus the assumption that conditions (5) hold when the set of particles is not in equilibrium leads to a contradiction and conditions (5) must be sufficient for equilibrium. This completes the proof of the principle.

The division of all the forces acting on the set P_i into applied forces $\mathbf{f}_i$ and constraints $\mathbf{k}_i$ is not so definitely determined by the nature of the given problem as might be inferred from the above discussion. We might in many cases call the forces $\mathbf{f}_i$ the forces whose values we wish to determine and the forces $\mathbf{k}_i$ those whose values we do not care to determine. For if we can with any division of all the forces into two classes $\mathbf{f}_i$ and $\mathbf{k}_i$, find a set of virtual displacements $\delta\mathbf{p}_i$ such that the virtual work $\Sigma\, \mathbf{k}_i\cdot\delta\mathbf{p}_i$ of the $\mathbf{k}_i$ is always zero then the hypotheses of the principle for the reversible case will be satisfied and we may use it to determine the forces $\mathbf{f}_i$; the forces $\mathbf{k}_i$ being thus eliminated from the discussion.

EXERCISES

1. Show that if all the virtual displacements of a set of particles which are compatible with the constraints are reversible, then the set of particles is in equilibrium when no forces are applied.

2. *Principle of Rigidification.* If a set of particles is in equilibrium under the action of a certain system of applied forces and under certain imposed conditions, show that the set will still be in equilibrium if additional conditions are imposed.

74. Applications of the Principle.

It is instructive to observe the operation of the principle of virtual work in certain general cases where we are already familiar with the conditions of equilibrium. Let us first consider the case of a free particle, i.e. a particle subject to no constraints. In this case any virtual displacement whatever $\delta\mathbf{p}$ will be compatible with the constraints and the condition that the virtual work $\mathbf{f}\cdot\delta\mathbf{p}$ of the applied force $\mathbf{f}$ be zero yields at once the necessary and sufficient condition, $\mathbf{f} = 0$ previously found in § 63.

Let us next consider the equilibrium of a particle constrained to a curve or surface. For a virtual displacement $\delta\mathbf{p}$ to be compatible with this constraint it must be tangent to the given curve or surface. The virtual work of the applied force $\mathbf{f}\cdot\delta\mathbf{p}$ will thus be zero when and only when this force is normal to the curve or surface, as previously shown in § 63. It will be recalled in this connection that we here consider the constraint $\mathbf{k}$ as having no action along the curve or surface, or in other words we are including in $\mathbf{k}$ only that portion of the force exerted by the curve or surface which it would exert if perfectly smooth, any other force which it exerts being included in the applied force $\mathbf{f}$.

The principle of virtual work yields very readily the conditions for equilibrium of a free rigid body. In this case the applied forces are all the exterior forces applied to the rigid body while the constraints are the interior forces maintaining the particles of the body at fixed distances from each other. Motions compatible with the constraints are then simply all motions for which the body remains rigid and for these we know that the velocity $\mathbf{p}_i'$ of any particle P_i of the body is given at each instant by the formula, § 48,(6),

$$\mathbf{p}_i' = \mathbf{a}' + \boldsymbol{\omega}\times\mathbf{r}_i, \qquad (\mathbf{r}_i = \mathbf{p}_i - \mathbf{a}),$$

where $\mathbf{a}'$ and $\boldsymbol{\omega}$ are independent of the choice of P_i. Let us now give the body any infinitesimal displacement taking place in an infinitesimal time interval δt. Then we have $\delta\mathbf{p}_i = \mathbf{p}_i'\,\delta t$ and the above formula becomes,

$$\delta\mathbf{p}_i = (\mathbf{a}' + \boldsymbol{\omega}\times\mathbf{r}_i)\,\delta t.$$

We form the dot-product of both members of this equation with the force $\mathbf{f}_i$ applied to P_i and sum the resulting equations for all particles of the body,

$$\begin{aligned}\Sigma\, \mathbf{f}_i \cdot \delta \mathbf{p}_i &= \Sigma\, \{\mathbf{a}' \cdot \mathbf{f}_i + \boldsymbol{\omega} \times \mathbf{r}_i \cdot \mathbf{f}_i\}\, \delta t \\ &= \{\mathbf{a}' \cdot \Sigma\, \mathbf{f}_i + \boldsymbol{\omega} \cdot \Sigma\, \mathbf{r}_i \times \mathbf{f}_i\}\, \delta t.\end{aligned}$$

The first member is the virtual work of all the applied forces and a necessary and sufficient condition that it vanish for every choice of $\mathbf{a}'$ and $\boldsymbol{\omega}$ is that,

$$\mathbf{S} = \Sigma\, \mathbf{f}_i = 0, \qquad \mathbf{M}_A = \Sigma\, \mathbf{r}_i \times \mathbf{f}_i = 0.$$

Thus a necessary and sufficient condition for equilibrium of a rigid body is that the applied forces have a zero sum $\mathbf{S}$ and a zero moment $\mathbf{M}_A$ about some point A. This is of course the previously derived condition that the applied forces must form a null system.

The principle of virtual work is especially useful in determining the equilibrium of systems with a concealed mechanism or where the mechanism is extremely complicated. Consider for example the following.

A 5 pound weight descending 2 feet drives a clock for 30 hours. What force must be applied to hold the end of the 3 inch minute hand while the escapement is removed?

To attempt to solve this simple problem by a detailed consideration of all the various pulleys, wheels and pinions of the clock works would be extremely tedious but by use of the principle of virtual work the result is easily obtained. We know that the end of the minute hand will move 180π inches while the weight is moving 24 inches and so if the weight f_1 travels a distance δs_1 the force $\mathbf{f}_2$ applied at the end of the minute hand will act through a distance $\delta s_2 = \dfrac{180\pi}{24}\, \delta s_1$. By the principle of virtual work we have,

$$f_1\, \delta s_1 + f_2\, \delta s_2 = 0$$

and consequently,

$$f_2 = -\frac{24}{180\pi} f_1 = -\frac{2}{3\pi} = -\ .2122 \text{ pounds}.$$

The minus sign indicates that f_1 and $\mathbf{f}_2$ tend to turn the clock in opposite directions.

A uniform heavy rod of length $2l$ has one end A resting against a smooth vertical wall and the other end B is attached by a string of length k to a fixed point O at a distance a from the wall. If A, B, O are in a vertical plane perpendicular to the wall, determine the angle θ which the rod makes with the vertical.

The pressure of the wall on the rod at A must be perpendicular to the wall because the wall is smooth and so if A were to slide about on the wall the pressure would do no work. Likewise the tension in the string at B can do no work because any motion of B would be constantly perpendicular to the string and so perpendicular to the tension. These two forces are thus constraints. The only other force acting on the rod is its weight w acting downward and as if concentrated at the center. By the principle of virtual work we have therefore for any virtual displacement of the rod subject to the given conditions,

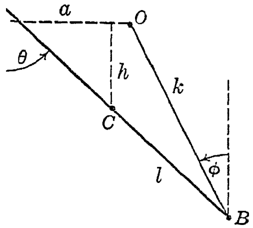

FIG. 77

$$w\,\delta h = 0,$$

h being the vertical distance of the centroid C of the rod below the fixed point O. From the figure we have,

$$2l \sin\theta - k \sin\varphi = a,$$
$$k \cos\varphi - l \cos\theta = h,$$

φ being the angle which the string makes with the vertical. If now the rod be given a virtual displacement subject to the given conditions in which θ and φ change by amounts $\delta\theta$ and $\delta\varphi$ we have by differentiation of the above equations, remembering that a is a constant and that $\delta h = 0$,

$$2l \cos\theta\,\delta\theta - k \cos\varphi\,\delta\varphi = 0,$$
$$l \sin\theta\,\delta\theta - k \sin\varphi\,\delta\varphi = 0.$$

From these we have at once,

$$\tan\varphi = \tfrac{1}{2}\tan\theta,$$

and this together with the equation $2l \sin\theta - k \sin\varphi = a$ will serve to determine θ and φ.

EXERCISES

Apply the principle of virtual work to the solution of the exercises on equilibrium in §§ 63, 64, 65.

1. A man standing on a platform scales balances his weight of 150 pounds by that of a one pound weight on the lever. If the weight is oscillating up and down through one inch, how is the man moving? *Ans.* Down and up through 1/150 inch

2. In lifting a weight by a chain hoist it is found that a pull of 50 pounds on the hand chain is required to start the weight and that the chain must be pulled 20 feet for every foot that the weight rises. If half of the pull is used up in overcoming frictional resistances, how much does the weight weigh? *Ans.* 500 pounds

3. A particle P is constrained to remain on a fixed smooth plane and is acted upon by a given force $a\,\mathbf{u}$ and a variable force $f\,\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are given unit vectors. Show that for equilibrium $\mathbf{u}$ and $\mathbf{v}$ must be coplanar with the normal to the plane and determine the amount f of the variable force and the amount k of the constraint exerted by the plane.
Ans. $f = a \cos \alpha / \cos \beta$, $k = a \sin (\alpha + \beta)/\cos \beta$ where α and β are the angles which $\mathbf{u}$ and $\mathbf{v}$ make with the plane.

4. Two beads P and Q of weights A and B slide on two intersecting smooth straight wires OP and OQ which are in the same vertical plane and make angles α and β with the vertical. The beads are connected by a light cord. What angle θ does the cord make with the vertical when the beads are in equilibrium?
$$\textit{Ans.} \quad \cot \theta = \frac{B \tan \alpha - A \tan \beta}{A + B}$$

5. An elastic band of natural length a, weight W and modulus of elasticity λ is placed around a smooth vertical cone the vertex angle of which is 2α. How far down on the cone will the band slip?
$$\textit{Ans.} \quad h = \frac{a}{2\pi \tan \alpha}\left(1 + \frac{W}{2\pi\lambda \tan \alpha}\right)$$

6. A spherical toy balloon has a radius of 1 inch when there is no pressure within and a radius of 2 inches when the pressure is 1 pound per square inch. What will the radius be when the pressure is $2\frac{1}{2}$ pounds per square inch? Assume that the resistance offered by the balloon to an increase in area is proportional to the amount by which its area exceeds its natural area. *Ans.* $r = 4$

7. A uniform heavy rod rests between two smooth inclined planes which intersect in a horizontal line. Express the angle θ which the rod makes with the horizontal in terms of the angles α and β which the planes make with the horizontal.
$$\textit{Ans.} \quad \tan \theta = \tfrac{1}{2}(\cot \beta - \cot \alpha)$$

8. Four uniform rods AB, BC, CD, DE of equal weight are freely hinged at B, C, D and the ends A and E are freely attached to supports at the same level. If AB and DE are the same length and BC and CD are the same length, show that $\tan \alpha = 3 \tan \beta$ where α and β are the angles which AB and BC make with the horizontal.

9. Two equal heavy rods of length $2a$ are connected by a smooth hinge and placed across a smooth horizontal cylinder of radius r. Show that the hinge must be directly above the axis of the cylinder and set up an equation for the determination of the angle 2φ between the rods. Determine φ for the case $a = 2r$. *Ans.* $\varphi = 45°$

10. Two equal heavy beads slide on a smooth wire in the form of an ellipse with vertical major axis and are connected by a light cord passing through a freely turning pulley at the upper focus of the ellipse. Show that the beads are in equilibrium whenever the cord is taut. What will be the position of equilibrium if the beads are of unequal weight?

11. *De l'Hôpital's Draw Bridge.* A uniform bar AB of length a and weight W is hinged at A so as to be free to rise in a vertical plane and a light cord is attached at B and passes over a freely turning pulley at a point C which is vertically above A a distance a. The other end of the cord is attached to a weight G which is at C when the rod is horizontal. Determine the curve along which the weight may freely slide so that the system will be in equilibrium in any position.

Ans. Pascal's Limaçon $r = a\left(2\sqrt{2} - 4\dfrac{G}{W}\cos\theta\right)$ where $r = CG$ and $\theta = ACG$. For $G = W/\sqrt{2}$ this reduces to the cardioid, $r = 2\sqrt{2}\,a\,(1 - \cos\theta)$.

12. *Leibnitz' Construction for Equilibrium of a Particle* (1693). Let all the forces acting on a particle P be drawn as radius vectors from a point O and let equal particles Q_i be placed at the extremities of these vectors. Show that the condition for equilibrium of the particle P is that the centroid of the equal particles Q_i be at O.

13. Derive from the principle of virtual work the conditions for equilibrium of a rigid body with a fixed point; a fixed axis.

14. Derive from the principle of virtual work the conditions for equilibrium of a light cord with attached particles P_i acted upon by respective exterior forces $\mathbf{f}_i$.

15. N particles P_i move as functions of m independent parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$,

$$\mathbf{p}_i = \mathbf{p}_i(\lambda_1, \cdots, \lambda_m),$$

and in addition are acted upon by the force of gravity. Show that they will be in equilibrium only when the distance r_0 of their

centroid from a fixed horizontal plane has a relative maximum or minimum, i.e.,

$$\frac{\partial r_0}{\partial \lambda_1} = \frac{\partial r_0}{\partial \lambda_2} = \cdots = \frac{\partial r_0}{\partial \lambda_m} = 0.$$

16. *The Theorem of Torricelli.* A rigid body subject to any bilateral constraints upon its position which do not vary with the time and acted upon only by the force of gravity will be in equilibrium only when its centroid is at a relative maximum or minimum distance from a fixed horizontal plane. Establish this theorem by the aid of Problem 15.

75. Lagrange's Multipliers.

In the case where the constraints imposed upon a set of particles $P_1, P_2, \cdots, P_n$ may be expressed by m scalar equations of the form,

$$\varphi_j(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n) = 0, \qquad j = 1, 2, \qquad m,$$

Lagrange has given an interesting method for determining the conditions upon the applied forces which will be necessary and sufficient for equilibrium of the set. For simplicity we shall present his method for a set consisting of three particles P_1, P_2, P_3 subject to the two scalar equations,

$$\text{(1)} \qquad \varphi_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 0, \qquad \varphi_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 0,$$

but the extension to a set of n particles subject to m $(m < 3n)$ equations of constraint requires only an obvious modification of the notation. Let us represent by $\delta\mathbf{p}_1, \delta\mathbf{p}_2, \delta\mathbf{p}_3$ any set of virtual displacements compatible with the conditions (1). Then we must have,

$$\text{(2)} \qquad \begin{aligned} \delta\varphi_1 &= (\nabla_1\varphi_1)\cdot\delta\mathbf{p}_1 + (\nabla_2\varphi_1)\cdot\delta\mathbf{p}_2 + (\nabla_3\varphi_1)\cdot\delta\mathbf{p}_3 = 0, \\ \delta\varphi_2 &= (\nabla_1\varphi_2)\cdot\delta\mathbf{p}_1 + (\nabla_2\varphi_2)\cdot\delta\mathbf{p}_2 + (\nabla_3\varphi_2)\cdot\delta\mathbf{p}_3 = 0, \end{aligned}$$

where,

$$\nabla_i\varphi_j \equiv \left(\frac{\partial\varphi_j}{\partial x_i}, \frac{\partial\varphi_j}{\partial y_i}, \frac{\partial\varphi_j}{\partial z_i}\right), \qquad \mathbf{p}_i \equiv (x_i, y_i, z_i),$$

because it is evident that $\delta\varphi_1$ and $\delta\varphi_2$ are simply the total differentials of φ_1 and φ_2 for the given variations of their arguments.* Now by the principle of virtual work a necessary and sufficient

* The symbol ∇ is due to Sir William Hamilton and is read, "del" and the vector $\nabla_i \varphi_j$ is called the *partial gradient* of φ_j with respect to $\mathbf{p}_i$.

condition that the set of particles be in equilibrium is that the work of the applied forces be zero for every such reversible set of virtual displacements. That is,

$$\mathbf{f}_1 \cdot \delta\mathbf{p}_1 + \mathbf{f}_2 \cdot \delta\mathbf{p}_2 + \mathbf{f}_3 \cdot \delta\mathbf{p}_3 = 0, \tag{3}$$

where $\mathbf{f}_i$ is the force applied to P_i.

The problem of determining the conditions upon the applied forces $\mathbf{f}_i$ which are necessary and sufficient for equilibrium thus resolves itself into stating conditions under which equation (3) will be satisfied by every set of values of $\delta\mathbf{p}_1$, $\delta\mathbf{p}_2$, $\delta\mathbf{p}_3$ which satisfy equations (2). All these equations are linear and homogeneous in the nine scalar variables $\delta x_1, \delta y_1, \cdots, \delta z_3$ and it is proven in algebra that a necessary and sufficient condition that any one such equation be satisfied by all the sets of values of the variables satisfying a given set of such equations is that the coefficients in the one equation be linearly dependent on those of the set of equations. This amounts in our case to saying that for equation (3) to be satisfied by all the values of $\delta\mathbf{p}_1$, $\delta\mathbf{p}_2$, $\delta\mathbf{p}_3$ which satisfy equations (2) it is necessary and sufficient that there exist two scalars λ_1 and λ_2 such that,

$$\begin{aligned} \mathbf{f}_1 + \lambda_1\nabla_1\varphi_1 + \lambda_2\nabla_1\varphi_2 &= 0, \\ \mathbf{f}_2 + \lambda_1\nabla_2\varphi_1 + \lambda_2\nabla_2\varphi_2 &= 0, \\ \mathbf{f}_3 + \lambda_1\nabla_3\varphi_1 + \lambda_2\nabla_3\varphi_2 &= 0. \end{aligned} \tag{4}$$

Between these equations we may eliminate λ_1 and λ_2 and thus determine the conditions upon the applied forces $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ which are necessary and sufficient for equilibrium or we may leave the equations as they are, regarding λ_1 and λ_2 as parameters in terms of which the forces are expressed.

The two scalars λ_1 and λ_2 are known as *Lagrange's multipliers* and equations (4) are *Lagrange's equations for equilibrium.* Lagrange's observation on the problem is essentially that the conditions for equilibrium are given by equations (4) and that these equations may be obtained by multiplying equations (2) respectively by λ_1 and λ_2 and adding the results member for member to equation (3) and then equating the coefficients of $\delta\mathbf{p}_1$, $\delta\mathbf{p}_2$, $\delta\mathbf{p}_3$ in the resulting equation separately to zero.

It will often, but not always, be possible to solve some of equations (4) for λ_1 and λ_2 in terms of the $\nabla_i\varphi_j$ and when this

can be done Lagrange's method gives us at once the constraint imposed upon each particle of the set by each of the conditions (1). For if while the set was in equilibrium the condition $\varphi_j = 0$ were removed and at the same time a constraint $\mathbf{k}_{ij} = \lambda_j \nabla_i \varphi_j$ were applied to each particle P_i the equations of equilibrium (4) would be unaltered and the set of particles would remain in equilibrium. It follows that $\mathbf{k}_{ij} = \lambda_j \nabla_i \varphi_j$ is exactly the constraint imposed upon P_i by the condition $\varphi_j = 0$. In fact since Lagrange's method thus introduces explicitly the constraint upon each particle for each imposed condition, the equations of equilibrium tend to become very lengthy and numerous and the importance of the method is consequently chiefly theoretical.

EXERCISES

1. If x_1, x_2, x_3 are three variables subject to the two conditions,

$$A = a_1x_1 + a_2x_2 + a_3x_3 = 0,$$
$$B = b_1x_1 + b_2x_2 + b_3x_3 = 0,$$

and if y_1, y_2, y_3 are three variables such that,

$$Y = y_1x_1 + y_2x_2 + y_3x_3 = 0,$$

show that the conditions thereby imposed upon y_1, y_2, y_3 may be obtained by equating separately to zero the coefficients of x_1, x_2, x_3 in the expression $Y + \lambda A + \mu B$ and eliminating λ and μ from the resulting equations. Generalize this theorem.

2. Interpret the theorem of Problem 1 in terms of vectors calling,

$$\mathbf{a} \equiv (a_1, a_2, a_3), \qquad \mathbf{b} \equiv (b_1, b_2, b_3),$$
$$\mathbf{x} \equiv (x_1, x_2, x_3), \qquad \mathbf{y} \equiv (y_1, y_2, y_3).$$

CHAPTER X

KINETICS OF PARTICLES

76. Introduction.

Chapters VII, VIII, IX have been concerned primarily with that branch of Dynamics called Statics in which bodies are considered in relation to the forces acting upon them but in which the forces are such that no motion takes place. When motion does take place the study is called *Kinetics* and this forms the other division of Dynamics. We have discussed the kinetics of a single particle in Chapter IV on the Curvilinear Motion of a Particle. We found there that (§ 36) the motion of a particle P is subject to the equation of motion,

$$m\,\mathbf{j} = \lambda\,\mathbf{f},$$

where m is the mass, $\mathbf{j}$ the acceleration, $\mathbf{f}$ the force, and λ a positive scalar constant.

If now we consider a set of n particles m_i, P_i $(i = 1, 2, \cdots, n)$ where m_i are the masses located at the points P_i, then we may have acting on each particle P_i forces arising outside the set whose sum we call the *exterior force* $\mathbf{f}_i$, and forces with which the other particles of the set act on P_i whose sum we call the *interior force* $\mathbf{g}_i$. If we so choose our units of measure that $\lambda = 1$ then by the above result the motions of the n particles of the set are subject to the n equations,

$$(1) \qquad m_i\,\mathbf{p}_i'' = \mathbf{f}_i + \mathbf{g}_i \qquad (i = 1, 2, \cdots, n),$$

where $\mathbf{p}_i = OP_i$ is the radius vector of P_i from the fixed point O.

We have seen (§ 64) that on the assumption of Newton's Third Law of Motion we may show that the interior forces $\mathbf{g}_i$ acting on the set form a null system of sliding vectors. They thus have a zero sum and a zero moment with respect to any point O. That is,

$$(2) \qquad \Sigma\,\mathbf{g}_i = 0, \qquad (3) \quad \Sigma\,\mathbf{p}_i \times \mathbf{g}_i = 0.$$

We shall again find it convenient to refer to the centroid of the set of particles. This, it will be recalled, (§ 66) is a fictitious

particle m_0, P_0 defined by the equations,

$$m_0 = \Sigma\, m_i, \qquad m_0\,\mathbf{p}_0 = \Sigma\, m_i\,\mathbf{p}_i \qquad (\mathbf{p}_0 = OP_0).$$

From these we have at once,

$$(4) \qquad \Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0) = 0,$$

and on differentiating with respect to the time, assuming the masses to be constant, we find,

$$(5) \quad \Sigma\, m_i(\mathbf{p}_i' - \mathbf{p}_0') = 0, \qquad (6) \quad \Sigma\, m_i(\mathbf{p}_i'' - \mathbf{p}_0'') = 0.$$

If in equations (1) all the forces $\mathbf{f}_i$ and $\mathbf{g}_i$ are given in terms of all the positions $\mathbf{p}_i$ and velocities $\mathbf{p}_i'$ of the particles and the time t, we have a set of n vector differential equations of the second order for the determination of the motion of the n particles. Since each vector equation is equivalent to three scalar equations of the same order it follows that equations (1) form a differential system of order $6n$. This means that the solution can be made to depend on the solution of a single differential equation of order $6n$ in one dependent scalar variable. Evidently, for any except very small values of n, the problem bids fair to be almost if not quite insuperably difficult. This is in fact the case and it is even difficult to construct artificially cases in which the solution can be carried through for values of n larger than 2. However, by combining equations (1) in various ways and making use of the properties of the exterior forces and of the centroid mentioned above we can find a set of important theorems concerning the motion of the n particles which throw considerable light on the nature of the motion and which will even in certain very restricted cases enable us to complete the integration.

77. The Theorem of Momentum.

The momentum of a particle has been (§ 36) defined as the product of its mass m and its velocity $\mathbf{p}'$. Since the mass is in nearly every case a positive constant the momentum of a particle is a vector with the same direction and sense as its velocity. The *momentum of a set of particles* is defined as the sum of the momentums of the particles of the set. If we add equations § 76,(1) for all particles of the set we have,

$$\Sigma\, m_i\,\mathbf{p}_i'' = \Sigma\, \mathbf{f}_i + \Sigma\, \mathbf{g}_i.$$

The last term of the second member vanishes by equation § 76, (2) and we have on the assumption that every m_i is constant,

$$(\Sigma\, m_i\, \mathbf{p}_i')' = \Sigma\, \mathbf{f}_i. \tag{1}$$

Since $\Sigma\, m_i\, \mathbf{p}_i'$ is by definition the momentum of the set of particles we may state this equation in words as the important,

Theorem of Momentum. The derivative with respect to the time of the momentum of a set of particles equals the sum of the exterior forces acting on the set.

Equation § 76,(5) may be written,

$$\Sigma\, m_i\, \mathbf{p}_i' = m_0\, \mathbf{p}_0',$$

and by means of this equation (1) takes the form,

$$m_0\, \mathbf{p}_0'' = \Sigma\, \mathbf{f}_i. \tag{2}$$

This is exactly the equation of motion of a particle of mass m_0 acted upon by a force $\Sigma\, \mathbf{f}_i$ and we may therefore also state equation (1) in words by the,

Theorem of the Motion of the Centroid. The centroid of a set of particles moves as if all the exterior forces acting on the set were applied to it.

The significance of these two closely related theorems will be more fully appreciated by the consideration of a few examples. When a shell of shrapnel is shot from a gun we know that the shell travels in an approximate parabola with vertical axis. When the shell bursts and the shrapnel balls fly in all directions great interior forces are set in action but since there is almost no change in the exterior force the centroid of the balls continues to move as before in the original parabola. When a diver leaps from the spring board he can thereafter by the exertion of interior forces twist and turn in the air but he can in no way materially alter the path of his centroid after he has once left the board.

An important example is that in which the exterior force $\Sigma\, \mathbf{f}_i$ remains constantly zero. In this case equation (2) may be integrated (§ 27) and gives us,

$$\mathbf{p}_0' = \mathbf{b}, \qquad \mathbf{p}_0 = \mathbf{a} + \mathbf{b}\, t,$$

where $\mathbf{a}$ and $\mathbf{b}$ are vector constants. If $\mathbf{b}$ is zero we have $\mathbf{p}_0 = \mathbf{a}$ and the centroid remains at rest, while if $\mathbf{b}$ is not zero we have $(\mathbf{p}_0 - \mathbf{a}) \times \mathbf{b} = 0$ which is the equation of a straight line which

will be traced by the centroid with the constant velocity $\mathbf{b}$. This case is illustrated by the solar system. Due to the immense distances of the fixed stars from the solar system and to their roughly symmetric distribution about it we may assume that the forces they exert on the system have in effect a zero sum. If this is the case the centroid of the solar system which is practically at the center of the sun should, by the above argument have a uniform rectilinear motion. This theoretical conclusion is verified by astronomical observations. By using a coördinate system fixed relative to the mean position of a large number of the fixed stars it is found that the solar system is moving in the general direction of the first magnitude star Vega with a velocity of about 12 miles per second. Another illustration of the same case is that of a child skipping rope on the earth. The earth and the child form a set which we may treat as two isolated particles and if we take a coördinate system with its origin at their centroid equation (1) then integrates into,

$$m_1 \mathbf{p}_1 + m_2 \mathbf{p}_2 = 0.$$

Thus if $\mathbf{p}_2$ changes by an amount $\mathbf{k}$, $\mathbf{p}_1$ must change by the amount $(-m_2/m_1)\,\mathbf{k}$, and as the child jumps up and down the earth jumps down and up so that their centroid remains undisturbed. The mass of the earth is 1.32×10^{25} pounds and if the child weighs 80 pounds and jumps 1 foot upward the earth will simultaneously jump downward 6.06×10^{-24} feet.

EXERCISES

1. Prove analytically that if a man stands on perfectly smooth ice he will be able to move his centroid only in a vertical line.

2. If each particle of a set m_i, P_i is attracted toward a fixed point O with a force proportional to its mass and to its distance from O, show that the centroid moves as if it were a particle of the set.

3. A set of beads m_i, P_i slide on smooth parallel wires and attract each other with forces proportional to the product of the two masses and to the distance. Show that they perform simple harmonic motions of the same period relative to the plane through their centroid perpendicular to the wires.

78. The Theorem of Moment of Momentum.

If we cross-multiply each of the equations § 76,(1) by the corresponding $\mathbf{p}_i$ and add the results member for member we

have,

$$\Sigma\, \mathbf{p}_i \times m_i\, \mathbf{p}_i'' = \Sigma\, \mathbf{p}_i \times \mathbf{f}_i + \Sigma\, \mathbf{p}_i \times \mathbf{g}_i.$$

The last term of the second member vanishes by equation § 76, (3) and since $(\mathbf{p}_i \times m_i\, \mathbf{p}_i')' = \mathbf{p}_i \times m_i\, \mathbf{p}_i''$ we may write the above equation as,

$$(1) \qquad (\Sigma\, \mathbf{p}_i \times m_i\, \mathbf{p}_i')' = \Sigma\, \mathbf{p}_i \times \mathbf{f}_i.$$

To interpret this equation we consider the momentum $m_i\, \mathbf{p}_i'$ of each particle of the set as a vector attached to the corresponding particle P_i. Then each of these attached vectors will have a moment $\mathbf{p}_i \times m_i\, \mathbf{p}_i'$ with respect to O and the sum of these moments we call the *moment of momentum* of the set of particles with respect to O. Equation (1) may now be stated as the important,

Theorem of Moment of Momentum. The derivative with respect to the time of the moment of momentum of a set of particles with respect to any fixed point O equals the moment of the exterior forces with respect to O.

It is of course evident that we are here regarding the forces $\mathbf{f}_i$ as vectors attached to their corresponding particles P_i.

As in the previous article we may put our equation into words in several convenient forms. It will be recalled (§ 35) that the rate at which the radius vector $\mathbf{p}$ is sweeping out area about the fixed point O is given by $A' = \frac{1}{2}|\mathbf{p} \times \mathbf{p}'|$. Consequently we naturally call $\frac{1}{2}\, \mathbf{p} \times \mathbf{p}'$ the *vector areal velocity* of P about O, its length being the scalar areal velocity and its direction perpendicular to the plane of $\mathbf{p}$ and $\mathbf{p}'$. Equation (1) then states the,

Theorem of Areal Velocities. The derivative with respect to the time of the sum of the products of the masses of the particles of a set by their respective vector areal velocities about a fixed point O equals one half the moment of the exterior forces with respect to O.

If $\mathbf{u}$ is any constant unit vector we have at once from equation (1),

$$(2) \qquad (\Sigma\, m_i\, \mathbf{u} \cdot \mathbf{p}_i \times \mathbf{p}_i')' = \mathbf{u} \cdot \Sigma\, \mathbf{p}_i \times \mathbf{f}_i.$$

If we now consider an axis through O in the direction and sense of $\mathbf{u}$ we see that the second member of this equation is by definition (§ 57) the moment of the system of exterior forces with respect to this axis. On the other hand the quantity $\frac{1}{2}\, \mathbf{u} \cdot \mathbf{p}_i \times \mathbf{p}_i'$

is the measure of the vector areal velocity of P_i on the given axis and the reader may readily convince himself that this equals the scalar areal velocity of the projection of P_i on a plane through O perpendicular to $\mathbf{u}$; this areal velocity being considered positive when the motion about O is in the positive sense of rotation relative to $\mathbf{u}$. If we represent by A_i the plane area swept out by the projection of P_i in this plane since some given instant, we consequently have,

$$\mathbf{u}\cdot\mathbf{p}_i\times\mathbf{p}_i' = 2A_i',$$

and equation (2) may be written,

$$(3) \qquad 2\Sigma\, m_iA_i'' = \mathbf{u}\cdot\Sigma\,\mathbf{p}_i\times\mathbf{f}_i.$$

Thus we have the theorem,

The sum of the products of the masses of the particles of a set by the areal accelerations about a point O of their projections on a plane through O equals one half the moment of the exterior forces with respect to an axis through O perpendicular to this plane.

In particular if $\mathbf{u}\cdot\Sigma\,\mathbf{p}_i\times\mathbf{f}_i$ is constantly zero equation (3) integrates into,

$$(4) \qquad \Sigma\, m_iA_i' = \tfrac{1}{2}A, \qquad \Sigma\, m_iA_i = \tfrac{1}{2}At + B,$$

where A and B are constants, A being known as the *areal constant* of the set. We therefore have the theorem,

If the moment of the exterior forces acting on a set of particles with respect to a certain axis is constantly zero, the sum of the products of the masses of the particles by the areal velocities about a point O on the axis of their projections on a plane through O perpendicular to the axis is a constant.

As a special case of the Theorem of Areal Velocities we may remark that,

If the moment of the exterior forces of a set of particles with respect to a point O is constantly zero, then the sum of the products of the masses of the particles by their vector areal velocities about O is a constant.

This is an immediate consequence of the fact that when the second member of equation (1) vanishes the equation integrates into,

$$(5) \qquad \Sigma\, m_i\,\mathbf{p}_i\times\mathbf{p}_i' = \mathbf{a},$$

where **a** is a constant vector known as the *vector areal constant.* In this case there will be an areal constant A for every plane through O and on comparing equations (4) and (5) we see that $A = \mathbf{u} \cdot \mathbf{a}$. The constant vector **a** is therefore perpendicular to the plane for which A is a maximum and the length of **a** is the value of this maximum.

As remarked in the previous article, it is natural to assume that the exterior forces exerted by the fixed stars on the solar system have practically a zero sum and it seems likewise probable that their moment with respect to any point in the system is practically zero. Under this hypothesis there exists for the solar system the vector areal constant **a** mentioned above. The plane through the centroid of the solar system perpendicular to this vector is the *invariable plane,* so called by Laplace, on which the product of the masses by the areal velocities of the projections has its maximum value. At present the invariable plane is inclined to the ecliptic at about 2° and the longitude of its ascending node is about 286°. The position of the plane is not known with sufficient accuracy to be of practical use in astronomy but it has been of great value in certain theoretical investigations.

If an object is in the air near the surface of the earth the only exterior forces acting will be the force of gravity and the action of the air. If the latter is negligible the moment of the exterior forces about the centroid is zero (§ 62) and the areal constant A will be zero for every plane through the centroid. Thus a rigid body thrown into the air without any initial motion of rotation will not turn in the air. This being the case it is at first difficult to understand the well known fact that a cat held with its feet upward and dropped from a sufficient height will turn in the air and land on its feet. Slow-motion pictures have solved the puzzle and shown that the cat whirls its tail in a vertical circle in one direction while its body rotates in the opposite direction. It also extends its hind legs and moves them around in a conical motion with the tail, but a cat without a tail has great difficulty in completing the turn.

EXERCISES

1. How may balloonists suspended in an airtight gondola from a spherical balloon cause the gondola to rotate by means of apparatus wholly inside it?

2. Show that the measure on any fixed axis of the vector areal velocity of a particle P about a fixed point O is numerically equal to the scalar areal velocity about O of the projection of P on a plane through O perpendicular to this axis.

3. Two particles move subject to their mutual attraction or repulsion which may vary in any fashion. Two planes are drawn through any fixed point, one passing through each particle and parallel to its velocity. Show that these two planes intersect in a line which remains in a fixed plane.

4. Two particles move subject to their mutual attraction or repulsion which may vary in any fashion. Show that the line joining them remains parallel to a fixed plane.

5. Two equal beads slide on two smooth straight wires intersecting at right angles. If the beads start from rest in any position and attract each other in any fashion, show that they will arrive at the intersection simultaneously.

6. Two particles move subject to their mutual attraction or repulsion which may vary in any fashion. Show that the tangents to their paths will intersect any fixed plane in two points collinear with a fixed point of that plane.

79. The Theorem of Kinetic Energy.

In § 72 we defined the kinetic energy of a particle as one half the product of its mass and the square of its velocity. Thus for a particle m, P the kinetic energy is,

$$T = \tfrac{1}{2}m\,\mathbf{p}'^2.$$

The *kinetic energy of a set of particles* is the sum of the kinetic energies of the individual particles, i.e.,

$$T = \Sigma\,\tfrac{1}{2}m_i\,\mathbf{p}_i'^2.$$

If a force $\mathbf{f}$ acts on a particle P we have seen, § 72,(6), that the time rate at which the force does work, i.e. the power of the force, is given by,

$$w' = \mathbf{f}\cdot\mathbf{p}'.$$

Similarly the *power of a system of forces* $\mathbf{f}_i$ acting respectively on a set of particles P_i is the sum of the powers of the individual forces, i.e.,

$$w' = \Sigma\,\mathbf{f}_i\cdot\mathbf{p}_i'.$$

If we now return to equations § 76,(1) and dot-multiply each one through by the corresponding velocity $\mathbf{p}_i'$ and add the results

member for member we find,

$$\Sigma\, m_i\, \mathbf{p}_i'\cdot\mathbf{p}_i'' = \Sigma\, \mathbf{f}_i\cdot\mathbf{p}_i' + \Sigma\, \mathbf{g}_i\cdot\mathbf{p}_i'.$$

But since $(\mathbf{p}_i'^2)' = 2\mathbf{p}_i'\cdot\mathbf{p}_i''$ it appears that we may write the above equation as,

$$(1) \qquad (\Sigma\, \tfrac{1}{2}m_i\, \mathbf{p}_i'^2)' = \Sigma\, \mathbf{f}_i\cdot\mathbf{p}_i' + \Sigma\, \mathbf{g}_i\cdot\mathbf{p}_i'.$$

With the above definitions in mind we may state this equation as the important,

Theorem of Kinetic Energy. The derivative with respect to the time of the kinetic energy of a set of particles equals the power of all the forces acting on the set.

It should be particularly noted that, unlike the theorems of momentum and moment of momentum, the theorem of kinetic energy involves the interior forces as well as the exterior.

The power being the time rate at which the force does work, it follows that the right member of equation (1) is the derivative w' of the work of all the forces acting on the set of particles and we may write equation (1) as,

$$(2) \qquad T' = w'.$$

If we multiply through by dt this becomes,

$$(3) \qquad dT = dw,$$

which gives us another form in which the theorem of kinetic energy is often stated,

The differential of the kinetic energy of a set of particles equals the elementary work of all the forces acting on it.

If we consider a time interval from t_0 to t and integrate both members of equation (3) with respect to t from t_0 to t, we have,

$$(4) \qquad \int_{t_0}^{t} dT = \int_{t_0}^{t} dw \qquad \text{or} \qquad T - T_0 = w - w_0,$$

showing that,

The change in the kinetic energy of a set of particles during any time interval equals the work of all the forces acting on the set during that interval.

As a simple application of the theorem of kinetic energy we may consider the motion of a body in vacuo near the surface of

the earth. The only force acting is the force of gravity which is proportional to the mass of the body and directed downward. The force is thus $\mathbf{f} = mg\,\mathbf{u}$, where g is a positive scalar constant and $\mathbf{u}$ is a unit vector directed downward. Equation (1) then takes the form,

$$(\tfrac{1}{2}m\,\mathbf{p}'^2)' = mg\,\mathbf{u}\cdot\mathbf{p}'.$$

Integrating between corresponding limits gives,

$$\mathbf{p}'^2 - \mathbf{p}_0'^2 = 2g\,\mathbf{u}\cdot(\mathbf{p} - \mathbf{p}_0),$$

showing that the increase in the square of the velocity is proportional to the distance traveled downward. This is equivalent to equation § 38,(4).

EXERCISES

1. Two beads of equal mass are connected by a light cord. If one of the beads slides on a smooth vertical wire OY and the other on a smooth horizontal wire OX, discuss their motion under the action of gravity.

2. Discuss the motion of two free particles attracting each other with a force proportional to their distance apart.

3. Two beads attract each other with a force proportional to their distance apart. Discuss their motion when one of them slides on a smooth straight wire.

4. Two equal beads attract each other with a force inversely proportional to the square of their distance apart. If they slide on two smooth straight wires intersecting at right angles, discuss their motion. Show that their centroid traces out a conic with focus at the intersection of the wires.

5. Discuss the motion of two particles connected by a light cord and moving in a plane under an attraction to an axis lying in this plane, the amount of the attraction being proportional to the mass of the particle and to its distance from the axis.

6. Two equal beads P and Q are connected by a light cord and Q slides on a smooth straight wire. Initially the cord is taut, Q is at rest and P has a velocity parallel to the wire. Discuss the subsequent motion, showing that the maximum and minimum angular velocities of the cord are proportional to $\sqrt{2}$ and 1, and of the tension in it to 4 and 1.

7. Two particles P and Q are connected by a light cord. The particle P moves on a smooth horizontal plane and the cord passes through a small hole in the plane and supports the particle Q vertically beneath it. Discuss the motion, showing that the distance r from P to the hole varies in general between two limits r_1 and r_2 and that

the path of P is alternately tangent to the two circles $r = r_1$, and $r = r_2$.

8. Three equal particles are attached at equal intervals to a light cord which is initially straight. The middle particle has initially a velocity of amount v at right angles to the cord and the other two particles are at rest. Show that the two end particles will have velocities of amount $\frac{2}{3}v$ when they meet.

80. Theorems Relative to the Centroid.

In the theorems proven in the previous articles of this chapter we have been concerned with the absolute motion of the set of particles; that is we have employed a fixed coördinate system. This may also be expressed by saying that we have used a Newtonian reference frame, i.e. one in which two or more forces acting on a particle produce the same effect as their vector sum. These theorems will in general not hold if we employ a moving reference frame but it is a very striking fact that those of the last two articles do hold if we use a reference frame which is fixed relative to the centroid of the set of particles studied.

As a preliminary to demonstrating this we shall prove two properties of the *kinematics* of the centroid; that is properties of the centroid which do not involve the forces acting on the set.

Principle 1. The moment of momentum of a set of particles with respect to any point is equal to the moment of momentum of the centroid of the set with respect to that point plus the moment with respect to the centroid of the momentum of the set for motion relative to the centroid.

Let us take as the origin O the point mentioned in the statement of the principle and let the radius vector of the centroid be $OP_0 = \mathbf{p}_0$. If we now call $\mathbf{p}_i - \mathbf{p}_0 = \mathbf{r}_i$, equations § 76, (4), (5), (6) become,

$$(1) \quad \Sigma\, m_i\, \mathbf{r}_i = 0, \qquad (2) \quad \Sigma\, m_i\, \mathbf{r}_i' = 0, \qquad (3) \quad \Sigma\, m_i\, \mathbf{r}_i'' = 0,$$

and the principle to be proven is expressed by the equation,

$$(4) \qquad \Sigma\, \mathbf{p}_i \times m_i\, \mathbf{p}_i' = \mathbf{p}_0 \times m_0\, \mathbf{p}_0' + \Sigma\, \mathbf{r}_i \times m_i\, \mathbf{r}_i'.$$

To prove this we observe successively that,

$$\begin{aligned}\Sigma\, m_i\, \mathbf{p}_i \times \mathbf{p}_i' &= \Sigma\, m_i(\mathbf{p}_0 + \mathbf{r}_i) \times (\mathbf{p}_0' + \mathbf{r}_i') \\ &= \Sigma\, m_i\, \mathbf{p}_0 \times \mathbf{p}_0' + \Sigma\, m_i\, \mathbf{p}_0 \times \mathbf{r}_i' + \Sigma\, m_i\, \mathbf{r}_i \times \mathbf{p}_0' + \Sigma\, m_i\, \mathbf{r}_i \times \mathbf{r}_i' \\ &= (\Sigma\, m_i)\, \mathbf{p}_0 \times \mathbf{p}_0' + \mathbf{p}_0 \times (\Sigma\, m_i \mathbf{r}_i') + (\Sigma\, m_i\, \mathbf{r}_i) \times \mathbf{p}_0' + \Sigma\, m_i\, \mathbf{r}_i \times \mathbf{r}_i'.\end{aligned}$$

The second and third terms of the last member vanish by equations (2) and (1) and equation (4) follows at once.

Analogous to the above is,

Principle 2 (Koenig's Theorem). The kinetic energy of a set of particles is equal to the kinetic energy of its centroid plus the kinetic energy of the set for motion relative to the centroid.

Continuing with the above notation the principle to be proven may be written,

$$(5) \qquad \Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2 = \tfrac{1}{2} m_0\, \mathbf{p}_0'^2 + \Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2.$$

To prove this we observe successively that,

$$\begin{aligned} \Sigma\, m_i\, \mathbf{p}_i'^2 &= \Sigma\, m_i (\mathbf{p}_0' + \mathbf{r}_i')^2 = \Sigma\, m_i\, \mathbf{p}_0'^2 + 2\Sigma\, m_i\, \mathbf{p}_0' \cdot \mathbf{r}_i' + \Sigma\, m_i\, \mathbf{r}_i'^2 \\ &= (\Sigma\, m_i)\, \mathbf{p}_0'^2 + 2(\Sigma\, m_i\, \mathbf{r}_i') \cdot \mathbf{p}_0' + \Sigma\, m_i\, \mathbf{r}_i'^2. \end{aligned}$$

The second term of the last member vanishes by equation (2), and equation (5) follows at once.

With the aid of Principle 1 we may now show that the theorem of moment of momentum already proven for the absolute motion of a set of particles still holds for motion relative to the centroid if the moments be taken with respect to the centroid. That is,

Theorem of Relative Moment of Momentum. The derivative with respect to the time of the moment with respect to the centroid of the momentum of a set of particles for motion relative to the centroid equals the moment of the exterior forces with respect to the centroid.

In the above notation this is expressed by the equation,

$$(6) \qquad (\Sigma\, \mathbf{r}_i \times m_i\, \mathbf{r}_i')' = \Sigma\, \mathbf{r}_i \times \mathbf{f}_i$$

being identical in form to equation § 78,(1). Starting with equation § 76, (1) $m_i\, \mathbf{p}_i'' = \mathbf{f}_i + \mathbf{g}_i$ we cross-multiply member for member with the identity $\mathbf{p}_i - \mathbf{p}_0 = \mathbf{r}_i$ and add the results for all particles of the set. Remembering that $\Sigma\, \mathbf{r}_i \times \mathbf{g}_i = 0$ and that $\Sigma\, m_i\, \mathbf{p}_i'' = m_0\, \mathbf{p}_0''$, we find,

$$\Sigma\, m_i\, \mathbf{p}_i \times \mathbf{p}_i'' - m_0\, \mathbf{p}_0 \times \mathbf{p}_0'' = \Sigma\, \mathbf{r}_i \times \mathbf{f}_i.$$

Differentiating equation (4) with respect to the time gives us after transposition,

$$(\Sigma\, \mathbf{r}_i \times m_i\, \mathbf{r}_i')' = \Sigma\, m_i\, \mathbf{p}_i \times \mathbf{p}_i'' - m_0\, \mathbf{p}_0 \times \mathbf{p}_0'',$$

and adding these last two equations member for member yields the desired equation (6). As in the case of equation § 78, (1)

we may interpret equation (6) in terms of the vector areal velocities of the particles. Thus,

Theorem of Relative Areal Velocity. The derivative with respect to the time of the sum of the products of the masses of the particles of a set by their respective vector areal velocities about the centroid equals one half the moment of the exterior forces with respect to the centroid.

By the use of Principle 2 (Koenig's Theorem) we may likewise show that the theorem of kinetic energy expressed by equation § 79,(1) also holds for motion relative to the centroid. That is,

Theorem of Relative Kinetic Energy. The derivative with respect to the time of the kinetic energy of a set of particles for motion relative to the centroid equals the power of all the forces acting on the set for this motion.

This is expressed by the equation,

$$(7) \qquad (\Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2)' = \Sigma\, \mathbf{f}_i \cdot \mathbf{r}_i' + \Sigma\, \mathbf{g}_i \cdot \mathbf{r}_i',$$

which is identical in form with equation § 79, (1). Starting with equation § 76, (1) $m_i\, \mathbf{p}_i'' = \mathbf{f}_i + \mathbf{g}_i$ we dot-multiply member for member with the identity $\mathbf{p}_i' - \mathbf{p}_0' = \mathbf{r}_i'$ and add the results for all particles of the set. Remembering that $\Sigma\, m_i\, \mathbf{p}_i'' = m_0\, \mathbf{p}_0''$, we find,

$$\Sigma\, m_i\, \mathbf{p}_i' \cdot \mathbf{p}_i'' - m_0\, \mathbf{p}_0' \cdot \mathbf{p}_0'' = \Sigma\, \mathbf{f}_i \cdot \mathbf{r}_i' + \Sigma\, \mathbf{g}_i \cdot \mathbf{r}_i'.$$

Differentiating equation (5) with respect to the time gives after transposition,

$$(\Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2)' = \Sigma\, m_i\, \mathbf{p}_i' \cdot \mathbf{p}_i'' - m_0\, \mathbf{p}_0' \cdot \mathbf{p}_0'',$$

and adding these last two equations member for member yields the desired equation (7).

It is noteworthy that the power of the interior forces $\mathbf{g}_i$ remains unaltered by the change from absolute motion of the set to motion relative to the centroid because we have,

$$(8) \qquad \Sigma\, \mathbf{g}_i \cdot \mathbf{r}_i' = \Sigma\, \mathbf{g}_i \cdot \mathbf{p}_i' - (\Sigma\, \mathbf{g}_i) \cdot \mathbf{p}_0' = \Sigma\, \mathbf{g}_i \cdot \mathbf{p}_i',$$

but this will be true of the exterior forces $\mathbf{f}_i$ only at a moment when $(\Sigma\, \mathbf{f}_i) \cdot \mathbf{p}_0' = 0$, i.e. when $\Sigma\, \mathbf{f}_i = 0$ or $\mathbf{p}_0' = 0$ or when $\Sigma\, \mathbf{f}_i$ is perpendicular to $\mathbf{p}_0'$.

The theorems of relative moment of momentum and of relative kinetic energy may be readily proven by an application of

the theory of relative motion discussed in § 51; the moving coördinate system in this case remaining parallel to the fixed system but having the centroid of the set fixed in it. We take the centroid as the origin of the moving system so that $\mathbf{r}_i$ becomes the relative radius vector of P_i. As shown in § 51, we may study the relative motion of a particle exactly as we study the absolute motion provided we consider the particle to be acted upon by an additional force $\mathbf{k} = -m(\mathbf{j}_1 + \mathbf{j}_3)$ known as the local force, where $\mathbf{j}_1$ is the drag acceleration and $\mathbf{j}_3$ is the complementary acceleration of the particle. Since in our case the moving coördinate system remains parallel to the fixed system its angular velocity ω_2 is zero and consequently, as remarked in § 51, we have $\mathbf{j}_3 = 0$ while $\mathbf{j}_1$ is the same for all points in space and in particular equal to the drag acceleration $\mathbf{p}_0''$ of the moving origin. Thus for the particle P_i we have $\mathbf{k} = -m_i\,\mathbf{p}_0''$ and the equation of its relative motion is,

$$(9) \qquad m_i\,\mathbf{r}_i'' = \mathbf{f}_i + \mathbf{g}_i - m_i\,\mathbf{p}_0''.$$

To prove the theorem of relative moment of momentum we cross-multiply both members of this equation by $\mathbf{r}_i$ and add the resulting equations for all particles of the set, remembering that $(\mathbf{r}_i \times \mathbf{r}_i')' = \mathbf{r}_i \times \mathbf{r}_i''$. This gives,

$$(\Sigma\,\mathbf{r}_i \times m_i\,\mathbf{r})_i'' = \Sigma\,\mathbf{r}_i \times \mathbf{f}_i + \Sigma\,\mathbf{r}_i \times \mathbf{g}_i - (\Sigma\,m_i\,\mathbf{r}_i) \times \mathbf{p}_0''.$$

The last two terms of the second member vanish by equations § 76, (3) and (1), and equation (6) then follows, thus proving the theorem. To prove the theorem of relative kinetic energy we dot-multiply both members of equation (9) by $\mathbf{r}_i'$ and add the results for all particles of the set, remembering that $(\frac{1}{2}\,\mathbf{r}_i'^2)' = \mathbf{r}_i' \cdot \mathbf{r}_i''$. This gives,

$$(\Sigma\,\tfrac{1}{2}m_i\,\mathbf{r}_i'^2)' = \Sigma\,\mathbf{f}_i \cdot \mathbf{r}_i' + \Sigma\,\mathbf{g}_i \cdot \mathbf{r}_i' - (\Sigma\,m_i\,\mathbf{r}_i') \cdot \mathbf{p}_0''.$$

The last term vanishes by equation (2) and equation (7) then follows, thus proving the theorem.

The theorems of relative moment of momentum and of relative kinetic energy are here stated in the forms which are found most useful, but that they could be put in somewhat more general forms appears from Problems 4 and 5 at the end of this article.

EXERCISES

1. Prove that the moment with respect to a point A of the momentum of a set of particles equals its moment with respect to another point O, diminished by the moment with respect to O of a vector passing through A and equal to the momentum of the centroid.

2. Prove that if the centroid of a set of particles remains fixed, the momentum of the set has the same moment with respect to every point of space.

3. Prove that the moment with respect to any axis of the momentum of a set of particles equals its moment with respect to a parallel centroidal axis plus the moment with respect to the given axis of the momentum of the centroid. (A *centroidal axis* is any axis through the centroid.)

4. Prove that the theorem of relative moment of momentum holds for motion relative to a point A and moments taken about A if and only if, (a) A has a constant vector velocity, (b) A is at the centroid of the set of particles, (c) the acceleration of A is parallel to the line joining A with the centroid.

5. Prove that the theorem of relative kinetic energy holds for motion relative to a point A if and only if, (a) A has a constant vector velocity, (b) A is fixed relative to the centroid of the set of particles, (c) the acceleration of A is perpendicular to the velocity of the centroid relative to A.

81. The Problem of N Bodies.

As an application of the theorems developed in this chapter we may briefly discuss the famous problem of n bodies. By the *problem of n bodies* is meant the problem of determining what the motions of n particles will be when their initial positions and vector velocities are given if every particle of the set attracts every other of the set according to the law of gravitation and if no exterior forces are acting. The interest in the problem arises largely from the fact that the solar system is approximately such an isolated set of particles. This is due first to the fact that the bodies of the solar system are nearly spherical, and if we assume them to be homogeneous in concentric spherical layers it may be shown (§ 108) that they attract each other as if each were replaced by its centroid. Furthermore the fixed stars are enormously distant from the solar system and their distribution is roughly symmetric about it so that it can probably be safely assumed that their attractions are negligible.

The force with which the particle m_j, P_j attracts the particle m_i, P_i according to the law of gravitation (§ 8) is a vector having the direction and sense of P_iP_j and the length $k^2m_im_j/r_{ij}^2$, where k^2 is a constant and r_{ij} is the distance from P_i to P_j. In our notation this force is therefore the vector,

$$-\frac{k^2m_im_j}{r_{ij}^3}(\mathbf{p}_i - \mathbf{p}_j) \qquad r_{ij} = |\mathbf{p}_i - \mathbf{p}_j|,$$

and the particle P_i being attracted by all the other particles of the set will have the equation of motion,

$$(1) \qquad m_i\,\mathbf{p}_i'' = -k^2m_i\sum\frac{m_j}{r_{ij}^3}(\mathbf{p}_i - \mathbf{p}_j),$$

the summation Σ being extended to all values of j from 1 to n except $j = i$.

We may throw these equations of motion into a convenient form by introducing a positive scalar variable U known as the *potential function* and defined by the equation,

$$U = k^2\sum\sum\frac{m_im_j}{r_{ij}}$$

the summation $\Sigma\Sigma$ being extended to all values of i from 1 to n and to all values of j from 1 to $i - 1$ so that each combination of the values of i and j appears once. Since k^2 and all the m_i are constants we may regard U as a function of the $3n$ coördinates x_i, y_i, z_i of the n particles or as a function of the n radius vectors $\mathbf{p}_i$. Thus we have,

$$U = k^2\,\Sigma\Sigma\, m_im_j\{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\}^{-1/2}$$

or

$$U = k^2\,\Sigma\Sigma\, m_im_j|\mathbf{p}_i - \mathbf{p}_j|^{-1}.$$

Evidently we have, with the same significance for Σ,

$$\frac{\partial U}{\partial x_i} = -k^2m_i\,\Sigma\, m_j\frac{x_i - x_j}{r_{ij}^3}, \qquad \frac{\partial U}{\partial y_i} = -k^2m_i\,\Sigma\, m_j\frac{}{r_{ij}^3}$$

$$\frac{\partial U}{\partial z_i} = -k^2m_i\,\Sigma\, m_j\frac{z_i - z_j}{r_{ij}^3}$$

so that if we define a vector ∇_iU * by its coördinates,

$$\nabla_iU \equiv \left(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}, \frac{\partial U}{\partial z_i}\right),$$

* Read "del sub i of U" and called the gradient of U with respect to $\mathbf{p}_i$.

we have,

$$\nabla_i U = -k^2 m_i \sum \frac{m_j}{r_{ij}^3}(\mathbf{p}_i - \mathbf{p}_j).$$

It now appears that our equations of motion (1) take the simple form,

$$m_i \, \mathbf{p}_i'' = \nabla_i U. \tag{2}$$

As remarked in § 76, we have here a differential system of the order $6n$. Let us see what integrals of it we may obtain by the aid of the theorems of this chapter. Since there are no exterior forces acting equation § 77, (2) becomes in our case,

$$m_0 \, \mathbf{p}_0'' = 0,$$

which integrates at once into,

$$\mathbf{p}_0' = \mathbf{b}, \qquad \mathbf{p}_0 = \mathbf{a} + \mathbf{b}\, t \qquad (\mathbf{a}, \mathbf{b} \text{ const.}), \tag{3}$$

showing that,

Either ($\mathbf{b} = 0$) *the centroid of the set is at rest or* ($\mathbf{b} \neq 0$) *moving with a constant vector velocity along the straight line* $(\mathbf{p} - \mathbf{a}) \times \mathbf{b} = 0$.

We have here performed two vector integrations and introduced two vector constants which is equivalent to performing six scalar integrations and introducing six scalar constants. These are the *six integrals of the motion of the centroid* given by the theorem of momentum and leave the differential system of order $6n - 6$.

Equation § 78, (1) becomes in our case,

$$(\Sigma \, \mathbf{p}_i \times m_i \, \mathbf{p}_i')' = 0,$$

which integrates at once into,

$$\Sigma \, \mathbf{p}_i \times m_i \, \mathbf{p}_i' = \mathbf{c} \qquad (\mathbf{c} \text{ const.}), \tag{4}$$

showing as in § 78 that,

The sum of the products of the masses of the particles by their vector areal velocities about any fixed point is a constant.

In the same way equation § 80, (6) shows that,

The sum of the products of the masses of the particles by their vector areal velocities about the centroid is a constant.

We have here performed one vector integration and introduced one vector constant which is equivalent to three scalar integra-

tions and introducing three scalar constants. These are the three *integrals of areas* given by the theorem of moment of momentum and leave the differential system of order $6n - 9$.

Remembering that for every particle P_i of the set,

$$\mathbf{f}_i = 0, \qquad \mathbf{g}_i = \nabla_i U,$$

equation § 79, (1) may be written,

$$(\Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2)' = \Sigma\, \nabla_i U \cdot \mathbf{p}_i',$$

the summation Σ extending to all values of i from 1 to n. The second member on expansion in coördinates becomes,

$$\frac{\partial U}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial U}{\partial y_1}\frac{dy_1}{dt} + \cdots + \frac{\partial U}{\partial z_n}\frac{dz_n}{dt},$$

which is exactly the derivative U' of U with respect to t. Thus we have,

$$(\Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2)' = U',$$

which integrates into,

$$\text{(5)} \qquad \Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2 = U + h \qquad (h \text{ const.}),$$

showing that,

The kinetic energy of the set of particles differs from the potential function by a constant.

By the use of equation § 80, (5) or § 80, (7) and § 80, (8) we have similarly,

$$(\Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2)' = U',$$

and therefore,

$$\Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2 = U + k \qquad (k \text{ const.}),$$

showing that,

The kinetic energy of the set of particles for motion relative to the centroid differs from the potential function by a constant.

We have here performed one scalar integration and introduced one scalar constant. This is the *integral of kinetic energy*, also frequently called the *vis viva integral.*

The ten integrals of the problem of n bodies which we have now found leave our differential system of order $6n - 10$ and the question arises as to what other integrals exist. No other integrals holding for all values of n have been found, but Jacobi in 1842 showed that if all but two of the $6n$ integrals were known

then by a method known as Jacobi's last multiplier the remaining two could be found. However Bruns has demonstrated that no new integrals algebraic in the coördinates of the particles exist and Poincaré has shown that no new integrals uniform and transcendental in the elements of the astronomical orbits of the particles exist. There still remains the possibility that integrals may exist which would be uniform and transcendental when expressed in terms of other variables.

When the number of particles in the set is reduced to two our problem becomes the *two body problem* and in this case the integration can be completed. The solution was first carried out by Newton in a geometric form. The equations of motion (1) in this case reduce to,

$$
(6)\qquad \begin{aligned} m_1\,\mathbf{p}_1'' &= -\frac{k^2 m_1 m_2}{r^3}(\mathbf{p}_1 - \mathbf{p}_2), \\ m_2\,\mathbf{p}_2'' &= -\frac{k^2 m_2 m_1}{r^3}(\mathbf{p}_2 - \mathbf{p}_1), \end{aligned} \qquad r = |\mathbf{p}_2 - \mathbf{p}_1|.
$$

From equations (3) we see that as soon as the values of the constants $\mathbf{a}$ and $\mathbf{b}$ have been assigned the value of $\mathbf{p}_0$ at each instant is known and to solve equations (6) for $\mathbf{p}_1$ and $\mathbf{p}_2$ it only remains to determine $\mathbf{r}_1 = \mathbf{p}_1 - \mathbf{p}_0$ and $\mathbf{r}_2 = \mathbf{p}_2 - \mathbf{p}_0$. In other words we know the motion of the centroid and it only remains to determine the motion of the two particles relative to the centroid. Since $\mathbf{p}_0'' = 0$ we have $\mathbf{p}_1'' = \mathbf{r}_1''$ and $\mathbf{p}_2'' = \mathbf{r}_2''$, while from the definition of the centroid we find,

$$
m_2(\mathbf{p}_1 - \mathbf{p}_2) = m_0\,\mathbf{r}_1, \qquad m_1(\mathbf{p}_2 - \mathbf{p}_1) = m_0\,\mathbf{r}_2.
$$

Substituting these values in equations (6) gives for the equations of motion relative to the centroid,

$$
(7)\quad \mathbf{r}_1'' = -\frac{K^2}{r_1^3}\,\mathbf{r}_1, \qquad \mathbf{r}_2'' = -\frac{K_2^2}{r_2^3}\,\mathbf{r}_2 \left(K_1^2 = \frac{k^2 m_2^3}{m_0^2},\ K_2^2 = \frac{k^2 m_1^3}{m_0^2} \right).
$$

Here the variables $\mathbf{r}_1$ and $\mathbf{r}_2$ are separated and the two equations may be integrated separately. In fact we have already completely discussed exactly this equation in § 41 on Planetary Motion and the conclusions reached there are directly applicable to the motion of each of the particles P_1 and P_2 about the centroid.

The *problem of three bodies* has a very extensive literature and although the equations of motion have not been completely

integrated a great deal of information has been obtained concerning the nature of the motions of the particles. This is particularly true of the *restricted problem of three bodies* in which two of the bodies are assumed to revolve in circles about their centroid and the third body is assumed to be of such small mass as not to affect this motion. Even a presentation of the principal results obtained is beyond the scope of this book, but we should mention the interesting fact that as long ago as 1772 Lagrange found certain particular solutions of the general problem of three bodies, one of them being that in which the three bodies, which may be of arbitrary masses, revolve at constant speed in coplanar circles about their centroid maintaining themselves at the vertices of an equilateral triangle.

EXERCISES

1. Prove from the definition of the potential function U that the interior forces $\nabla_i U$ acting on the particles P_i of the set form a null system of sliding vectors.

2. In the problem of two bodies derive the equation of motion of one particle relative to the other and show that the integration can be completed.

3. Three particles P_1, P_2, P_3 of masses m_1, m_2, m_3 attract each other according to the law of gravitation. If $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ are their radius vectors refered to a reference frame with fixed origin rotating with a constant vector angular velocity $\boldsymbol{\omega}$, show that their equations of motion relative to the moving reference frame are,

$$\mathbf{q}_1'' + 2\,\boldsymbol{\omega}\times\mathbf{q}_1' = \boldsymbol{\omega}\times(\mathbf{q}_1\times\boldsymbol{\omega}) + k^2\left(\frac{m_3\,\mathbf{r}_2}{r_2^3} - \frac{m_2\,\mathbf{r}_3}{r_3^3}\right),$$

$$\mathbf{q}_2'' + 2\,\boldsymbol{\omega}\times\mathbf{q}_2' = \boldsymbol{\omega}\times(\mathbf{q}_2\times\boldsymbol{\omega}) + k^2\left(\frac{m_1\,\mathbf{r}_3}{r_3^3} - \frac{m_3\,\mathbf{r}_1}{r_1^3}\right),$$

$$\mathbf{q}_3'' + 2\,\boldsymbol{\omega}\times\mathbf{q}_3' = \boldsymbol{\omega}\times(\mathbf{q}_3\times\boldsymbol{\omega}) + k^2\left(\frac{m_2\,\mathbf{r}_1}{r_1^3} - \frac{m_1\,\mathbf{r}_2}{r_2^3}\right),$$

where $\mathbf{r}_1 = \mathbf{q}_2 - \mathbf{q}_3$, $\mathbf{r}_2 = \mathbf{q}_3 - \mathbf{q}_1$, $\mathbf{r}_3 = \mathbf{q}_1 - \mathbf{q}_2$ and k^2 is the constant of gravitation. Show that these equations are satisfied if P_1, P_2, P_3 are fixed relative to the moving reference frame at the vertices of an equilateral triangle of side a in a plane perpendicular to $\boldsymbol{\omega}$ with the centroid of the particles at the origin, where

$$a^3 = \frac{k^2}{\omega^2}(m_1 + m_2 + m_3).$$

These are the *Lagrangian solutions* of the problem of three bodies.

4. In Problem 3 let the mass m_3 of P_3 reduce to zero and let the other two particles P_1 and P_2 move in their Langrangian positions which now lie on a straight line through the centroid. Then $\mathbf{q}_1$ and $\mathbf{q}_2$ are constant and the equation of motion of the infinitesimal particle is,

$$\mathbf{q}'' + 2\,\boldsymbol{\omega}\times\mathbf{q}' = \boldsymbol{\omega}\times(\mathbf{q}\times\boldsymbol{\omega}) + k^2\left(\frac{m_2\,\mathbf{r}_1}{r_1^3} - \frac{m_1\,\mathbf{r}_2}{r_2^3}\right).$$

The determination of this motion is the *restricted problem of three bodies.* Derive *Jacobi's integral* (1836) of this equation,

$$\mathbf{q}'^2 = (\boldsymbol{\omega}\times\mathbf{q})^2 + 2k^2\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - C,$$

in which C is known as *Jacobi's constant.* From this show that if P_3 be initially sufficiently near to P_1 and with a sufficiently small relative velocity then it can never recede far from P_1.

5. Show that the Lagrangian solutions (Problem 3) of the problem of three bodies will hold when the attraction between each pair of bodies is proportional to the product of their masses and any function of their distance, provided the speed of rotation ω is properly related to the length of the side of the triangle and the total mass of the particles.

6. Prove that when three particles are moving in the Lagrangian solution of the problem of three bodies the total force exerted on each particle is directed toward the centroid and of an amount proportional to the distance from the centroid. What follows concerning the areal velocities of the particles about the centroid?

7. In the restricted problem of three bodies (Problem 4) complete the integration of the equation of motion of the infinitesimal body for the case where the two finite bodies are of equal mass and the infinitesimal body is initially on the line through the centroid perpendicular to their plane of motion and has its velocity along this line.

8. Prove that four bodies subject to their mutual attraction can not move in circles about their centroid and maintain their configuration unless they all lie in the same plane.

CHAPTER XI

KINETICS OF A RIGID BODY

82. Moment of Inertia.

As a special case of the kinetics of a set of particles we have the kinetics of a rigid body. Since the rigid body is merely a set of particles in which the interior forces are such as to maintain the mutual distances of the particles, it follows that the general theorems on the kinetics of a set of particles proven in the previous chapter are immediately applicable to the kinetics of a rigid body, and on the basis of these we may seek other theorems peculiar to the rigid body. In so doing we frequently encounter certain positive scalars known as the moments of inertia of the body about certain lines. In this and the succeeding article we shall make a study of these moments of inertia as a necessary preliminary to the study of the kinetics of a rigid body. In this we continue to treat the rigid body as a special set of particles but if one wishes to regard the rigid body as a continuous distribution of matter it is only necessary to replace the sums given here by the corresponding definite integrals as was done in § 66. The moment of inertia of a body about a line is the only kind of moment of inertia having applications in mechanics but it is desirable to consider also moments of inertia about points and planes as these are useful in computing the first.

We define the *moment of inertia* I of a particle m, P about a given point, line or plane as the product mr^2 of the mass m of the particle by the square of its distance r to the point, line or plane. Similarly the moment of inertia I of a set of particles m_i, P_i $(i = 1, 2, \cdots, n)$ is given by the formula,

$$I = \Sigma\, m_i r_i^2,$$

being the sum of the moments of inertia of the individual particles. Thus the moment of inertia of the set about a given point $A(OA = \mathbf{a})$ is given by,

$$(1) \qquad I = \Sigma\, m_i(\mathbf{p}_i - \mathbf{a})^2,$$

about a line $(\mathbf{p} - \mathbf{a}) \times \mathbf{u} = 0$ by, § 22, (10),

(2) $$I = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a}) \times \mathbf{u}\}^2 \qquad (\mathbf{u}^2 = 1)$$

and about a plane $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{u} = 0$ by, § 22, (11),

(3) $$I = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a}) \cdot \mathbf{u}\}^2. \qquad (\mathbf{u}^2 = 1)$$

There are several simple theorems concerning the moments of inertia of a set of particles about related points, lines and planes which it is desirable to have available. First let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be any three mutually perpendicular unit vectors having a positive sense of rotation so that,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \qquad \mathbf{i} = \mathbf{j} \times \mathbf{k}, \qquad \mathbf{j} = \mathbf{k} \times \mathbf{i}, \qquad \mathbf{k} = \mathbf{i} \times \mathbf{j}.$$

As a special case of the familiar identity, § 21, (3),

$$[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a} - [\mathbf{c}\ \mathbf{d}\ \mathbf{a}]\mathbf{b} + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d} = 0$$

we have,

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{i})\mathbf{i} + (\mathbf{a} \cdot \mathbf{j})\mathbf{j} + (\mathbf{a} \cdot \mathbf{k})\mathbf{k}$$

and consequently,

(4) $$\mathbf{a}^2 = (\mathbf{a} \cdot \mathbf{i})^2 + (\mathbf{a} \cdot \mathbf{j})^2 + (\mathbf{a} \cdot \mathbf{k})^2.$$

As a special case of the identity, § 21, Prob. 1, b,

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

we have,

$$(\mathbf{a} \times \mathbf{i})^2 = \mathbf{a}^2 - (\mathbf{a} \cdot \mathbf{i})^2$$

and consequently,

(5) $$\mathbf{a}^2 = (\mathbf{a} \cdot \mathbf{i})^2 + (\mathbf{a} \times \mathbf{i})^2.$$

Equations (4) and (5) give on subtracting member for member,

(6) $$(\mathbf{a} \times \mathbf{i})^2 = (\mathbf{a} \cdot \mathbf{j})^2 + (\mathbf{a} \cdot \mathbf{k})^2.$$

If we permute the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in this last equation and add the results we have,

$$(\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2 = 2\{(\mathbf{a} \cdot \mathbf{i})^2 + (\mathbf{a} \cdot \mathbf{j})^2 + (\mathbf{a} \cdot \mathbf{k})^2\},$$

which by equation (4) becomes,

(7) $$2\,\mathbf{a}^2 = (\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2.$$

If we subtract two times equation (5) from equation (7) we find

after transposition,

$$(8) \qquad (\mathbf{a}\times\mathbf{j})^2 + (\mathbf{a}\times\mathbf{k})^2 = (\mathbf{a}\times\mathbf{i})^2 + 2(\mathbf{a}\cdot\mathbf{i})^2.$$

If now in each of the equations (4), (5), (6), (7), (8) we replace $\mathbf{a}$ by $\mathbf{p}_i - \mathbf{a}$, multiply the equation through by m_i and sum for all particles of the set we find respectively,

$$(4') \quad \Sigma\, m_i(\mathbf{p}_i - \mathbf{a})^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{i}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{j}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{k}\}^2,$$

$$(5') \quad \Sigma\, m_i(\mathbf{p}_i - \mathbf{a})^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{i}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{i}\}^2,$$

$$(6') \quad \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{i}\}^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{j}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{k}\}^2,$$

$$(7') \quad 2\Sigma\, m_i(\mathbf{p}_i - \mathbf{a})^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{i}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{j}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{k}\}^2,$$

$$(8') \quad \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{j}\}^2 + \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{k}\}^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\times\mathbf{i}\}^2 + 2\,\Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{i}\}^2.$$

On referring to equations (1), (2), (3) we see that equations (4′), (5′), (6′), (7′), (8′) yield respectively the following theorems:

1. *The moment of inertia of a set of particles about a point equals the sum of the moments of inertia of the set about any three mutually perpendicular planes passing through that point.*

2. *The moment of inertia of a set of particles about a point equals the sum of the moments of inertia of the set about any perpendicular plane and line passing through that point.*

3. *The moment of inertia of a set of particles about a line equals the sum of the moments of inertia of the set about any pair of perpendicular planes intersecting in that line.*

4. *The moment of inertia of a set of particles about a point equals one half the sum of the moments of inertia of the set about any three perpendicular lines through that point.*

5. *The sum of the moments of inertia of a set of particles about any two perpendicular lines through a point equals twice the moment of inertia of the set about their plane plus the moment of inertia of the set about a line through the point perpendicular to the plane.*

This last theorem brings out the fact that unless all the particles of the set lie in the plane mentioned the moment of inertia about the third line is less than the sum of the moments of inertia about the first two lines. In fact equation (8′) shows that we

have,

$$(9) \qquad I_j + I_k - I_i \geq 0,$$

where I_i, I_j, I_k are the moments of the set about any three mutually perpendicular lines through the same point.

The above five theorems are concerned with moments of inertia about incident points, lines and planes, i.e. having a common point. The next two theorems show how the moment of inertia of a set varies from point to point in space. Let $(\mathbf{p} - \mathbf{p}_0)\cdot\mathbf{u} = 0$ and $(\mathbf{p} - \mathbf{a})\cdot\mathbf{u} = 0$ $(\mathbf{u}^2 = 1)$ be the equations of two parallel planes, the first of which is a centroidal plane, i.e. a plane passing through the centroid m_0, P_0 of the set of particles. Then by the fundamental theorem of the scalar product we have,

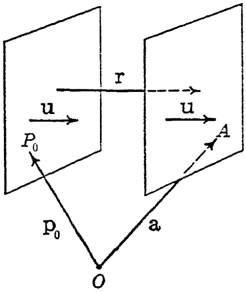

FIG. 78

$$(\mathbf{a} - \mathbf{p}_0)\cdot\mathbf{u} = \mathbf{r}\cdot\mathbf{u},$$

where $\mathbf{r}$ is a vector running perpendicularly from the first plane to the second. Consequently we have,

$$(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u} = (\mathbf{p}_i - \mathbf{p}_0)\cdot\mathbf{u} - \mathbf{r}\cdot\mathbf{u}.$$

Squaring both members gives us,

$$\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u}\}^2 = \{(\mathbf{p}_i - \mathbf{p}_0)\cdot\mathbf{u}\}^2 + \mathbf{r}^2 - 2(\mathbf{r}\cdot\mathbf{u})\{(\mathbf{p}_i - \mathbf{p}_0)\cdot\mathbf{u}\}.$$

We now multiply through by m_i and sum the resulting equations for all particles of the set, obtaining,

$$\Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u}\}^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{p}_0)\cdot\mathbf{u}\}^2 + (\Sigma\, m_i)\,\mathbf{r}^2 - 2(\mathbf{r}\cdot\mathbf{u})\{\Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0)\}\cdot\mathbf{u}.$$

But now the last term vanishes by equation § 76, (4) and we have,

$$(10) \qquad \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u}\}^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{p}_0)\cdot\mathbf{u}\}^2 + m_0 r^2,$$

where r is the distance between the two planes. We have thus established the theorem,

6. *The moment of inertia of a set of particles about a plane equals the moment of inertia of the set about a parallel centroidal plane plus the moment of inertia of the centroid about the first plane.*

The next theorem is highly analogous both in statement and proof to the one just discussed. Let $(\mathbf{p} - \mathbf{p}_0) \times \mathbf{u} = 0$ and $(\mathbf{p} - \mathbf{a}) \times \mathbf{u} = 0$ $(\mathbf{u}^2 = 1)$ be the equations of two parallel lines the first of which is a centroidal line. By the fundamental theorem of the vector product we have,

$$(\mathbf{a} - \mathbf{p}_0) \times \mathbf{u} = \mathbf{r} \times \mathbf{u},$$

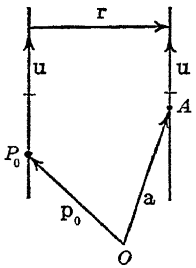

FIG. 79

where $\mathbf{r}$ is a vector running perpendicularly from the first line to the second. Consequently we have,

$$(\mathbf{p}_i - \mathbf{a}) \times \mathbf{u} = (\mathbf{p}_i - \mathbf{p}_0) \times \mathbf{u} - \mathbf{r} \times \mathbf{u}.$$

Squaring both members gives us,

$$\{(\mathbf{p}_i - \mathbf{a}) \times \mathbf{u}\}^2 = \{(\mathbf{p}_i - \mathbf{p}_0) \times \mathbf{u}\}^2 + \mathbf{r}^2 - 2(\mathbf{r} \times \mathbf{u}) \cdot \{(\mathbf{p}_i - \mathbf{p}_0) \times \mathbf{u}\}.$$

We multiply both members of this equation by m_i and sum the resulting equations for all particles of the set, obtaining,

$$\Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a}) \times \mathbf{u}\}^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{p}_0) \times \mathbf{u}\}^2 + (\Sigma\, m_i)\,\mathbf{r}^2 - 2(\mathbf{r} \times \mathbf{u}) \cdot \{\Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0)\} \times \mathbf{u}.$$

But now the last term vanishes by equation § 76, (4) and we have,

$$(11) \quad \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a}) \times \mathbf{u}\}^2 = \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{p}_0) \times \mathbf{u}\}^2 + m_0\, r^2,$$

where r is the distance between the two lines. We have thus established the theorem,

7. *The moment of inertia of a set of particles about a line equals the moment of inertia of the set about a parallel centroidal line plus the moment of inertia of the centroid about the given line.*

When the moment of inertia of a set of particles has been determined it is convenient to factor the expression obtained for it into two factors one of which is the mass m_0 of the set and the other of which we call r_0^2 where r_0 is known as the *radius of gyration* of the set about the point, line or plane about which the moment of inertia was determined. Thus r_0 is the distance which a single particle of mass m_0 must be from the point, line or plane in order that it have the same moment of inertia that the original set has.

We shall illustrate the determination of moments of inertia by integration and by the use of the above theorems in the following simple example.

Find the moment of inertia of a homogeneous sphere of density ρ and radius a about a diametral plane; about a diameter; about a tangent line.

Let the center of the sphere be chosen as the origin of coördinates and let the diametral plane of the problem be chosen as the YZ-plane. We consider first only the right hand half of the sphere and divide this by planes parallel to the YZ-plane into n circular disks of thickness $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$. The principal part of the volume of the i^{th} disk will then be $\pi(a^2 - x_i^2)\Delta x_i$ where x_i is any value of x in the i^{th} disk. The principal part of the mass of this disk will then be $\pi\rho(a^2 - x_i^2)\Delta x_i$ and the principal part of its moment of inertia about the YZ-plane will be $\pi\rho x_i^2(a^2 - x_i^2)\Delta x_i$. We now sum this for all the disks and proceed to the limit allowing the number of disks to become infinite and the thickness of each disk to approach zero. On the application of Duhamel's theorem and the fundamental theorem of the integral calculus we have for the entire sphere,

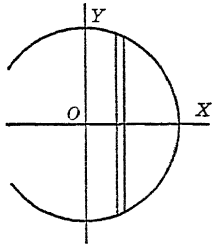

FIG. 80

$$I = 2 \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \pi\rho x_i^2(a^2 - x_i^2)\Delta x_i \right\} = 2 \int_0^a \pi\rho x^2(a^2 - x^2)\, dx$$

$$= \frac{4}{15}\pi\rho a^5 = m_0 a^2/5.$$

Thus the radius of gyration is $r_0 = \frac{1}{\sqrt{5}}\, a$. For the moment of inertia of the sphere about a diameter we apply Theorem 3 and have at once,

$$I = m_0 \frac{2a^2}{5}, \qquad r_0 = \sqrt{\frac{2}{5}}\, a.$$

For the moment of inertia about a tangent line we apply Theorem 7 and obtain,

$$I = m_0 \frac{2a^2}{5} + m_0 a^2 = m_0 \frac{7a^2}{5}, \qquad r_0 = \sqrt{\frac{7}{5}}\, a.$$

EXERCISES

1. Find the moment of inertia of the set of particles,

$$m_1 = 3, \quad P_1 = (0, 1, 2), \qquad m_2 = 2, \quad P_2 = (-1, 1, 0),$$
$$m_3 = 4, \quad P_3 = (2, 3, -3)$$

   (a) about the ZX-plane, the XY-plane, *Ans.* 41, 48
   (b) about the X-axis, *Ans.* 89
   (c) about the point (2, 1, 3), *Ans.* 211
   (d) about the line $\dfrac{x-2}{1} = \dfrac{y-1}{3} = \dfrac{z-3}{-1}$. *Ans.* $158\frac{4}{11}$

2. Find the moment of inertia of a right circular cone of altitude a and radius r about the base, about the axis, about the vertex.

Ans. $m_0 \dfrac{a^2}{10}, \quad m_0 \dfrac{3r^2}{10}, \quad m_0 \left(\dfrac{3a^2}{5} + \dfrac{3r^2}{10}\right)$

3. Find the moment of inertia of a circular arc of radius a and central angle 2α about the center of the circle, about the midpoint of the arc.

Ans. $m_0 a^2, \quad 2m_0 \left(1 - \dfrac{\sin \alpha}{\alpha}\right) a^2$

4. Show that the moment of inertia of three particles, m_1, P_1, m_2, P_2, m_3, P_3, about a centroidal line perpendicular to their plane is,

$$I = \frac{1}{m_0} (m_2 m_3 a_1^2 + m_3 m_1 a_2^2 + m_1 m_2 a_3^2),$$

where a_1, a_2, a_3 are the sides of the triangle P_1, P_2, P_3 opposite the respective particles.

5. Find the moment of inertia of a triangular area about one side of the triangle.

Ans. $m_0 h^2/6$ where h is the altitude perpendicular to the given side

6. Find the moment of inertia of an elliptic area about the major axis, the minor axis. *Ans.* $m_0 b^2/4, \quad m_0 a^2/4$

7. Find the moment of inertia of the area of the lemniscate $p^2 = a^2 \cos 2\theta$ about the origin, about the axis.

Ans. $m_0 \dfrac{a^2}{8}, \quad m_0 \left(\dfrac{3\pi - 8}{48}\right) a^2$

8. Find the moment of inertia of the torus about its axis.

Ans. $m_0(b^2 + \frac{3}{4}a^2)$

9. Knowing that the moment of inertia of a rectangle of sides a and b about the side of length b is $m_0 a^2/3$, determine without integration the moment of inertia of,

   (a) a rectangle about a vertex, about its center, about the midpoint of one side,

(*b*) a rectangular parallelopiped about a face, about an edge, about a vertex, about its center, the center of a face, the midpoint of an edge.

10. Prove *Routh's Rule.* The moment of inertia of a rectangular parallelopiped, an elliptic cylinder, an ellipsoid about an axis of symmetry is given by the formula,

$$I = m_0 \frac{a^2 + b^2}{n},$$

where m_0 is the total mass, a and b are the two semiaxes perpendicular to each other and to the axis of symmetry and n is 3, 4 or 5 according as the body is a rectangular parallelopiped, an elliptic cylinder or an ellipsoid.

11. Prove Theorem 7 of this article by the use of Theorems 3 and 6 and a simple geometric construction.

12. Prove that the moment of inertia of a set of particles about a point equals the moment of inertia of the set about its centroid plus the moment of inertia of the centroid about the point.

13. Without integration prove that the moment of inertia of a spherical shell of radius a about a diameter is $m_0 2a^2/3$.

83. Principal Axes of Inertia. Poinsot's Ellipsoid.

If we have given a set of particles m_i, P_i and a point A, then corresponding to any given vector $\mathbf{u}$ we may find another vector $\varphi_A(\mathbf{u})$ defined by the formula,

$$\varphi_A(\mathbf{u}) = \Sigma\, m_i(\mathbf{p}_i - \mathbf{a}) \times \{\mathbf{u} \times (\mathbf{p}_i - \mathbf{a})\} \tag{1}$$

or the equivalent,

$$\varphi_A(\mathbf{u}) = \Sigma\, m_i((\mathbf{p}_i - \mathbf{a})^2\, \mathbf{u} - \{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u}\}(\mathbf{p}_i - \mathbf{a})).$$

Since all the operations here performed upon the vector $\mathbf{u}$ are commutative with multiplication by a scalar and distributive with respect to addition it follows that if $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ then,

$$\varphi_A(\mathbf{u}) = a\varphi_A(\mathbf{i}) + b\varphi_A(\mathbf{j}) + c\varphi_A(\mathbf{k}).$$

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are any three mutually perpendicular unit vectors and if we call,

$$\varphi_A(\mathbf{i}) = \mathbf{f}, \qquad \varphi_A(\mathbf{j}) = \mathbf{g}, \qquad \varphi_A(\mathbf{k}) = \mathbf{h},$$

then since $\mathbf{u} = (\mathbf{u}\cdot\mathbf{i})\mathbf{i} + (\mathbf{u}\cdot\mathbf{j})\mathbf{j} + (\mathbf{u}\cdot\mathbf{k})\mathbf{k}$, we have,

$$\varphi_A(\mathbf{u}) = (\mathbf{u}\cdot\mathbf{i})\mathbf{f} + (\mathbf{u}\cdot\mathbf{j})\mathbf{g} + (\mathbf{u}\cdot\mathbf{k})\mathbf{h}. \tag{2}$$

In particular if **i**, **j**, **k** are the three coördinate vectors we have in coördinates,

$$\varphi_A(\mathbf{u}) = (f_1u_1 + g_1u_2 + h_1u_3,\ f_2u_1 + g_2u_2 + h_2u_3,\ f_3u_1 + g_3u_2 + h_3u_3).$$

It thus appears that the coördinates of $\varphi_A(\mathbf{u})$ are certain linear homogeneous functions of the coördinates of **u** and for this reason $\varphi_A(\mathbf{u})$ is said to be a certain *linear vector function* of **u**. A linear vector function is also frequently called an *homography*.

The particular linear vector function defined by equation (1) is known as the *homography of inertia* of the set of particles m_i, P_i for the point A because if a line be drawn through A in the direction of a unit vector **u**, then the moment of inertia $I_A(\mathbf{u})$ of the set of particles about this line is given by the simple formula,

$$(3) \qquad I_A(\mathbf{u}) = \mathbf{u}\cdot\varphi_A(\mathbf{u}).$$

For evidently,

$$\begin{aligned}\mathbf{u}\cdot\varphi_A(\mathbf{u}) &= \Sigma\, m_i\mathbf{u}\cdot((\mathbf{p}_i - \mathbf{a}) \times \{\mathbf{u} \times (\mathbf{p}_i - \mathbf{a})\}) \\ &= \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{a}) \times \mathbf{u}\}^2 = I_A(\mathbf{u}).\end{aligned}$$

Thus if we wish to calculate the moments of inertia of the given set of particles m_i, P_i for several lines passing through the same point A we may calculate **f**, **g**, **h** once for all and then $\varphi_A(\mathbf{u})$ and $I_A(\mathbf{u})$ for each individual line follow easily from equations (2) and (3). As a further aid in calculating $\varphi_A(\mathbf{u})$ we have the theorem,

The homography of inertia $\varphi_A(\mathbf{u})$ of a set of particles for a point A equals the homography of inertia $\varphi_0(\mathbf{u})$ of the set for its centroid plus the homography of inertia of the centroid for the point A.

For we have,

$$\begin{aligned}\varphi_A(\mathbf{u}) &= \Sigma\, m_i(\mathbf{p}_i - \mathbf{a}) \times \{\mathbf{u} \times (\mathbf{p}_i - \mathbf{a})\} \\ &= \Sigma\, m_i\{(\mathbf{p}_i - \mathbf{p}_0) + (\mathbf{p}_0 - \mathbf{a})\} \\ &\qquad\qquad \times \{\mathbf{u} \times (\mathbf{p}_i - \mathbf{p}_0) + \mathbf{u} \times (\mathbf{p}_0 - \mathbf{a})\} \\ &= \Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0) \times \{\mathbf{u} \times (\mathbf{p}_i - \mathbf{p}_0)\} \\ &\qquad\qquad + (\Sigma\, m_i)(\mathbf{p}_0 - \mathbf{a}) \times \{\mathbf{u} \times (\mathbf{p}_0 - \mathbf{a})\} \\ &\quad + \{\Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0)\} \times \{\mathbf{u} \times (\mathbf{p}_0 - \mathbf{a})\} \\ &\qquad\qquad + (\mathbf{p}_0 - \mathbf{a}) \times \{\mathbf{u} \times \Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0)\}.\end{aligned}$$

Since by definition of the centroid $\Sigma\, m_i(\mathbf{p}_i - \mathbf{p}_0) = 0$ and

$\Sigma\, m_i = m_0$ this reduces to,

$$(4) \qquad \varphi_A(\mathbf{u}) = \varphi_0(\mathbf{u}) + m_0(\mathbf{p}_0 - \mathbf{a}) \times \{\mathbf{u} \times (\mathbf{p}_0 - \mathbf{a})\}$$

as stated in the theorem.

We shall see later that when the set of particles m_i, P_i and the point A are given there always exist certain vectors $\mathbf{u}$ for which $\mathbf{u}$ and $\varphi_A(\mathbf{u})$ are parallel, that is,

$$(5) \qquad \mathbf{u} \times \varphi_A(\mathbf{u}) = 0.$$

Under these circumstances we say that the direction of $\mathbf{u}$ is a *principal direction* for the point A and the line through A in this direction is called a *principal axis* for the point A. If $\mathbf{u}$ runs in a principal direction for A we have,

$$(6) \qquad \varphi_A(\mathbf{u}) = \{I_A(\mathbf{u})\}\mathbf{u},$$

for in fact by equations (5) and (6),

$$\begin{aligned} 0 = \mathbf{u} \times \{\mathbf{u} \times \varphi_A(\mathbf{u})\} &= \{\mathbf{u} \cdot \varphi_A(\mathbf{u})\}\mathbf{u} - \mathbf{u}^2\varphi_A(\mathbf{u}) \\ &= \{I_A(\mathbf{u})\}\mathbf{u} - \varphi_A(\mathbf{u}). \end{aligned}$$

We now take up the problem of finding the points on a given line for which that line is a principal axis. We first show that there can not be more than one such point on the given line except in one special case.

If any line is a principal axis for two of its points, then the line passes through the centroid of the set of particles and is a principal axis for each of its points.

Let A and B be two points on a line having the direction of the unit vector $\mathbf{u}$ and let this line be a principal axis for both A and B. Then employing equation (4) we have from equation (5),

$$\begin{aligned} 0 = \mathbf{u} \times \varphi_A(\mathbf{u}) &= \mathbf{u} \times \varphi_0(\mathbf{u}) + m_0\{(\mathbf{p}_0 - \mathbf{a}) \cdot \mathbf{u}\}(\mathbf{p}_0 - \mathbf{a}) \times \mathbf{u}, \\ 0 = \mathbf{u} \times \varphi_B(\mathbf{u}) &= \mathbf{u} \times \varphi_0(\mathbf{u}) + m_0\{(\mathbf{p}_0 - \mathbf{b}) \cdot \mathbf{u}\}(\mathbf{p}_0 - \mathbf{b}) \times \mathbf{u}. \end{aligned}$$

Since A and B are both on the line with direction $\mathbf{u}$ we have $(\mathbf{b} - \mathbf{a}) \times \mathbf{u} = 0$ or $\mathbf{b} \times \mathbf{u} = \mathbf{a} \times \mathbf{u}$ and so the second of the above equations may be written,

$$0 = \mathbf{u} \times \varphi_0(\mathbf{u}) + m_0\{(\mathbf{p}_0 - \mathbf{b}) \cdot \mathbf{u}\}(\mathbf{p}_0 - \mathbf{a}) \times \mathbf{u}$$

and on subtracting member for member from the first equation

yields,

$$m_0\{(\mathbf{b} - \mathbf{a})\cdot\mathbf{u}\}(\mathbf{p}_0 - \mathbf{a})\times\mathbf{u} = 0.$$

Since the scalar $m_0(\mathbf{b} - \mathbf{a})\cdot\mathbf{u}$ can not be zero we have,

$$(\mathbf{p}_0 - \mathbf{a})\times\mathbf{u} = 0,$$

showing that the centroid is on the given line. The proof that the line is a principal axis for each of its points now becomes part of the following general theorem on lines passing through the centroid of the set.

If a line passing through the centroid of the set of particles is a principal axis for any of its points it is a principal axis for each of its points.

To see this we cross-multiply both members of equation (4) by $\mathbf{u}$ obtaining,

$$(7)\quad \mathbf{u}\times\varphi_P(\mathbf{u}) = \mathbf{u}\times\varphi_0(\mathbf{u}) + m_0\{(\mathbf{p}_0 - \mathbf{p})\cdot\mathbf{u}\}(\mathbf{p}_0 - \mathbf{p})\times\mathbf{u}.$$

If now a line of direction $\mathbf{u}$ is a principal axis for the point P then we have $\mathbf{u}\times\varphi_P(\mathbf{u}) = 0$ and if this line passes through the centroid we have $(\mathbf{p}_0 - \mathbf{p})\times\mathbf{u} = 0$. Equation (7) then yields $\mathbf{u}\times\varphi_0(\mathbf{u}) = 0$ and the line must be a principal axis for the centroid. Now let P be any point on the line. We still have $(\mathbf{p}_0 - \mathbf{p})\times\mathbf{u} = 0$ and we have just seen that $\mathbf{u}\times\varphi_0(\mathbf{u}) = 0$, so that by equation (7) we have $\mathbf{u}\times\varphi_P(\mathbf{u}) = 0$ and the line is consequently a principal axis for each of its points.

The case of lines not passing through the centroid is covered by the following theorem.

A necessary and sufficient condition that a line of direction $\mathbf{u}$ *not passing through the centroid be a principal axis for one of its points is that* $\varphi_0(\mathbf{u})$ *be parallel to the plane containing the centroid and the line.*

To see that the condition is necessary we assume that P is a point of the given line for which it is a principal axis. Then $\mathbf{u}\times\varphi_P(\mathbf{u}) = 0$ and on multiplying equation (7) through by $(\mathbf{p} - \mathbf{p}_0)\cdot$ the last term vanishes and we have,

$$(8)\quad (\mathbf{p} - \mathbf{p}_0)\cdot\mathbf{u}\times\varphi_0(\mathbf{u}) = 0.$$

Thus the vectors $\mathbf{p} - \mathbf{p}_0$, $\mathbf{u}$, $\varphi_0(\mathbf{u})$ are coplanar and $\varphi_0(\mathbf{u})$ is parallel to the plane containing the centroid and the given line.

To see that condition (8) is also sufficient we consider two cases as follows:

I. $\mathbf{u} \times \varphi_0(\mathbf{u}) = 0$. Under this hypothesis the given line is parallel to a principal axis for the centroid and what we seek to establish is that,

If a line is parallel to a principal axis for the centroid it is a principal axis for the foot of the perpendicular let fall upon it from the centroid.

If P is the foot of the perpendicular then $(\mathbf{p}_0 - \mathbf{p})\cdot\mathbf{u} = 0$, while by hypothesis $\mathbf{u} \times \varphi_0(\mathbf{u}) = 0$ and thus both terms of the second member of equation (7) vanish. Consequently $\mathbf{u} \times \varphi_P(\mathbf{u}) = 0$ and the line is a principal axis for P.

II. $\mathbf{u} \times \varphi_0(\mathbf{u}) \neq 0$. Our hypothesis that $\varphi_0(\mathbf{u})$ is parallel to the plane of the given line and the centroid gives us equation (8) in which P is any point of the given line. Consequently we have,

$$\{(\mathbf{p} - \mathbf{p}_0) \times \mathbf{u}\} \times \{\mathbf{u} \times \varphi_0(\mathbf{u})\} = \{(\mathbf{p} - \mathbf{p}_0)\cdot\mathbf{u} \times \varphi_0(\mathbf{u})\}\mathbf{u} = 0$$

and the vectors $(\mathbf{p} - \mathbf{p}_0) \times \mathbf{u}$ and $\mathbf{u} \times \varphi_0(\mathbf{u})$ are parallel. Since $\mathbf{u} \times \varphi_0(\mathbf{u})$ is not zero we may therefore write,

$$(\mathbf{p} - \mathbf{p}_0) \times \mathbf{u} = \frac{\{(\mathbf{p} - \mathbf{p}_0) \times \mathbf{u}\}\cdot\{\mathbf{u} \times \varphi_0(\mathbf{u})\}}{\{\mathbf{u} \times \varphi_0(\mathbf{u})\}^2}\,\mathbf{u} \times \varphi_0(\mathbf{u}).$$

This reduces the second member of equation (7) to,

$$\mathbf{u} \times \varphi_0(\mathbf{u}) \left\{ 1 + \frac{m_0\{(\mathbf{p} - \mathbf{p}_0)\cdot\mathbf{u}\}\{(\mathbf{p} - \mathbf{p}_0) \times \mathbf{u}\}\cdot\{\mathbf{u} \times \varphi_0(\mathbf{u})\}}{\{\mathbf{u} \times \varphi_0(\mathbf{u})\}^2} \right\}$$

and our theorem will be proven if we can find a point P of the given line for which this vanishes. Let us assume the line to be given by its direction $\mathbf{u}$ and the foot A of the perpendicular let fall from the centroid upon it. Then $(\mathbf{a} - \mathbf{p}_0)\cdot\mathbf{u} = 0$ and if we set $\mathbf{p} = \mathbf{a} + \lambda\,\mathbf{u}$ we find on substitution that the above expression vanishes for,

$$\lambda = \frac{\{\mathbf{u} \times \varphi_0(\mathbf{u})\}^2}{m_0(\mathbf{a} - \mathbf{p}_0)\cdot\varphi_0(\mathbf{u})}\,.$$

Thus,

$$\mathbf{p} = \mathbf{a} + \frac{\{\mathbf{u} \times \varphi_0(\mathbf{u})\}^2}{m_0(\mathbf{a} - \mathbf{p}_0)\cdot\varphi_0(\mathbf{u})}\,\mathbf{u}$$

gives the point on the given line for which it is a principal axis. The theorem is thus proven.

The last theorems have been concerned with the determination of those points on a given line for which it is a principal axis of

inertia for the given set of particles. We shall now consider the determination of the principal axes for a given point. An important property of the homography of inertia in this connection is the fact that if $\mathbf{u}$ and $\mathbf{v}$ are any two vectors then,

$$\mathbf{u}\cdot\varphi_A(\mathbf{v}) = \mathbf{v}\cdot\varphi_A(\mathbf{u}). \tag{9}$$

This becomes apparent on expanding these quantities as they both take the form,

$$\begin{aligned}\mathbf{u}\cdot\varphi_A(\mathbf{v}) &= \mathbf{v}\cdot\varphi_A(\mathbf{u})\\ &= \Sigma\, m_i((\mathbf{p}_i - \mathbf{a})^2\, \mathbf{u}\cdot\mathbf{v} - \{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{u}\}\{(\mathbf{p}_i - \mathbf{a})\cdot\mathbf{v}\}).\end{aligned}$$

An homography possessing this property is known as a *dilatation.* As a consequence of this property the vectors $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ of equation (2) are not independent, for letting $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the coördinate vectors we have,

$$\mathbf{j}\cdot\varphi_A(\mathbf{k}) = \mathbf{k}\cdot\varphi_A(\mathbf{j}), \qquad \mathbf{k}\cdot\varphi_A(\mathbf{i}) = \mathbf{i}\cdot\varphi_A(\mathbf{k}), \qquad \mathbf{i}\cdot\varphi_A(\mathbf{j}) = \mathbf{j}\cdot\varphi_A(\mathbf{i})$$

or,

$$\mathbf{j}\cdot\mathbf{h} = \mathbf{k}\cdot\mathbf{g}, \qquad \mathbf{k}\cdot\mathbf{f} = \mathbf{i}\cdot\mathbf{h}, \qquad \mathbf{i}\cdot\mathbf{g} = \mathbf{j}\cdot\mathbf{f}$$

and in coördinates,

$$h_2 = g_3, \qquad f_3 = h_1, \qquad g_1 = f_2.$$

If now $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are any three mutually perpendicular unit vectors we may write $\mathbf{u} = (\mathbf{u}\cdot\mathbf{i})\mathbf{i} + (\mathbf{u}\cdot\mathbf{j})\mathbf{j} + (\mathbf{u}\cdot\mathbf{k})\mathbf{k}$ and as in equation (2), $\varphi_A(\mathbf{u})$ takes the form,

$$\varphi_A(\mathbf{u}) = (\mathbf{u}\cdot\mathbf{i})\mathbf{f} + (\mathbf{u}\cdot\mathbf{j})\mathbf{g} + (\mathbf{u}\cdot\mathbf{k})\mathbf{h},$$

where,

$$\varphi_A(\mathbf{i}) = \mathbf{f}, \qquad \varphi_A(\mathbf{j}) = \mathbf{g}, \qquad \varphi_A(\mathbf{k}) = \mathbf{h}.$$

By the aid of property (9) we may now see that $\varphi_A(\mathbf{u})$ may also be written in the form,

$$\varphi_A(\mathbf{u}) = (\mathbf{u}\cdot\mathbf{f})\mathbf{i} + (\mathbf{u}\cdot\mathbf{g})\mathbf{j} + (\mathbf{u}\cdot\mathbf{h})\mathbf{k}, \tag{10}$$

for from equation (10) by means of property (9) we have,

$$\mathbf{i}\cdot\varphi_A(\mathbf{u}) = \mathbf{u}\cdot\varphi_A(\mathbf{i}) = (\mathbf{f}\cdot\mathbf{i})(\mathbf{u}\cdot\mathbf{i}) + (\mathbf{g}\cdot\mathbf{i})(\mathbf{u}\cdot\mathbf{j}) + (\mathbf{h}\cdot\mathbf{i})(\mathbf{u}\cdot\mathbf{k}),$$

which is clearly also the value of $\mathbf{i}\cdot\varphi_A(\mathbf{u})$ from equation (2). Similarly equations (2) and (10) yield the same values of $\mathbf{j}\cdot\varphi_A(\mathbf{u})$ and $\mathbf{k}\cdot\varphi_A(\mathbf{u})$ and the two forms of $\varphi_A(\mathbf{u})$ are therefore equivalent.

By definition $\mathbf{u}$ is a principal direction for the point A when $\varphi_A(\mathbf{u})$ is parallel to $\mathbf{u}$ and we have seen that this may be written in the form,

$$\varphi_A(\mathbf{u}) = I\,\mathbf{u}, \tag{6}$$

where I, being the moment of inertia of the set of particles about the line through A in the direction $\mathbf{u}$, is essentially positive. The vector equation (6) is equivalent to the three scalar equations,

$$(\mathbf{f} - I\,\mathbf{i})\cdot\mathbf{u} = (\mathbf{g} - I\,\mathbf{j})\cdot\mathbf{u} = (\mathbf{h} - I\,\mathbf{k})\cdot\mathbf{u} = 0, \tag{11}$$

obtained by multiplying equation (6) through successively by $\mathbf{i}\cdot$, $\mathbf{j}\cdot$, $\mathbf{k}\cdot$ and applying property (9). The necessary and sufficient condition that there exist a non-zero vector $\mathbf{u}$ satisfying these equations is that the following determinant vanish.

$$[\mathbf{f} - I\,\mathbf{i},\ \mathbf{g} - I\,\mathbf{j},\ \mathbf{h} - I\,\mathbf{k}] = 0. \tag{12}$$

The principal directions $\mathbf{u}$ for the point A are now determined by solving the cubic equation (12) for I and then finding for each of these values of I the value or values of $\mathbf{u}$ satisfying equations (11). Equation (12) in I is of the form of the equation in x,

$$\begin{matrix} a_{11} - x & a_{12} & a_{13} \\ a_{21} & a_{22} - x & a_{23} \\ a_{31} & a_{32} & a_{33} - x \end{matrix} \qquad a_{ij} = a_{ji}$$

and is of rather frequent occurrence in many different connections. It is known as the *secular equation*. In treatises on algebra it is shown that the secular equation of degree n, and in which the determinant is consequently of order n, has the following two important properties.*

For real values of the a_{ij} the roots of the secular equation are all

If the a_{ij} are real and x is a root of the equation of multiplicity m then the rank of the determinant in the first member of the secular equation is $n-m$.

On the basis of these facts we may now prove the important theorem,

Through any point A there always pass one or more sets of three mutually perpendicular principal axes of inertia for A.

This theorem rests on the easily demonstrated fact that,

* G. Kowalewski, Einführung in die Determinanten Theorie, § 54, § 115.

Any two principal directions for A corresponding to different values of I are perpendicular.

For let I_1 and I_2 be any two different values of I corresponding to two principal directions $\mathbf{u}_1$ and $\mathbf{u}_2$ for A. Then by equation (6),

$$\varphi_A(\mathbf{u}_1) = I_1 \mathbf{u}_1, \qquad \varphi_A(\mathbf{u}_2) = I_2 \mathbf{u}_2,$$

and consequently,

$$\mathbf{u}_2 \cdot \varphi_A(\mathbf{u}_1) = I_1 \mathbf{u}_1 \cdot \mathbf{u}_2, \qquad \mathbf{u}_1 \cdot \varphi_A(\mathbf{u}_2) = I_2 \mathbf{u}_1 \cdot \mathbf{u}_2.$$

Subtracting these member for member and employing property (9) gives us,

$$0 = (I_2 - I_1)\mathbf{u}_1 \cdot \mathbf{u}_2$$

and since I_1 and I_2 are distinct $\mathbf{u}_1$ and $\mathbf{u}_2$ are perpendicular.

If then the three roots of the cubic equation (12) in I are distinct, for each such root the rank of the determinant in the first member will be 2 and the three vectors $\mathbf{f} - I\,\mathbf{i}$, $\mathbf{g} - I\,\mathbf{j}$, $\mathbf{h} - I\,\mathbf{k}$ will be coplanar but not all parallel. Consequently for each of the three roots there will exist a unit vector $\mathbf{u}$, unique except in sense, satisfying equations (6) and (11). As shown above, these will be mutually perpendicular, and the theorem is proven for this case. If two of the roots of equation (12) are equal and the third is distinct, then for the distinct root equations (11) will be satisfied by some value of $\mathbf{u}$, unique except in sense, as before, but for the double root the rank of the determinant in the first member of equation (12) will be 1 and the three vectors $\mathbf{f} - I\,\mathbf{i}$, $\mathbf{g} - I\,\mathbf{j}$, $\mathbf{h} - I\,\mathbf{k}$ will be parallel but not all zero. Equations (11) will thus be satisfied by any $\mathbf{u}$ perpendicular to these three parallel vectors and evidently we may pick two such vectors $\mathbf{u}$ which will be perpendicular to each other; while as shown above they will both be perpendicular to the value of $\mathbf{u}$ for the distinct root. If all three roots of equation (12) are equal, the rank of the determinant in the first member will be 0 for this value of I and the three vectors $\mathbf{f} - I\,\mathbf{i}$, $\mathbf{g} - I\,\mathbf{j}$, $\mathbf{h} - I\,\mathbf{k}$ will all be zero. Consequently equations (6) and (11) will be satisfied by any vectors $\mathbf{u}$ whatever and we may of course pick three values of $\mathbf{u}$ which will be mutually perpendicular. The theorem is thus established.

Let us now represent by the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ such a set of three mutually perpendicular principal directions for A and call

the corresponding values of I, equal or unequal, I_1, I_2, I_3. Then by equations (6),

$$\mathbf{f} = \varphi_A(\mathbf{i}) = I_1\,\mathbf{i}, \qquad \mathbf{g} = \varphi_A(\mathbf{j}) = I_2\,\mathbf{j}, \qquad \mathbf{h} = \varphi_A(\mathbf{k}) = I_3\,\mathbf{k},$$

and the two formulas (2) and (10) for $\varphi_A(\mathbf{u})$ both yield the simple form,

(13) $$\varphi_A(\mathbf{u}) = I_1(\mathbf{u}\cdot\mathbf{i})\mathbf{i} + I_2(\mathbf{u}\cdot\mathbf{j})\mathbf{j} + I_3(\mathbf{u}\cdot\mathbf{k})\mathbf{k},$$

while by equation (3) the moment of inertia is,

(14) $$I_A(\mathbf{u}) = I_1(\mathbf{u}\cdot\mathbf{i})^2 + I_2(\mathbf{u}\cdot\mathbf{j})^2 + I_3(\mathbf{u}\cdot\mathbf{k})^2.$$

The above discussion of the principal axes of inertia for a given point A receives a simple geometric interpretation by the introduction of *Poinsot's ellipsoid.* To construct this ellipsoid we lay off from A in each direction $\mathbf{u}$ a vector $\mathbf{r} = \mathbf{u}/\sqrt{I_A(\mathbf{u})}$. Then the termini P of these vectors $\mathbf{r}$ form a surface surrounding A. To determine the nature of this surface we take a rectangular coördinate system with origin at A and axes lying along a set of three mutually perpendicular principal axes of inertia for A. With this system we have,

$$x = \mathbf{r}\cdot\mathbf{i}, \qquad y = \mathbf{r}\cdot\mathbf{j}, \qquad z = \mathbf{r}\cdot\mathbf{k}$$

and on dividing equation (14) through by $I_A(\mathbf{u})$ we find,

(15) $$I_1x^2 + I_2y^2 + I_3z^3 = 1.$$

If we call $1/\sqrt{I_1} = a$, $1/\sqrt{I_2} = b$, $1/\sqrt{I_3} = c$ this equation takes the form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which we recognize as an ellipsoid with its center at the origin A and principal axes along the principal axes of inertia for A. This is known as Poinsot's ellipsoid.

The moment of inertia about the line through A in the direction $\mathbf{u}$ being given by the formula,

(16) $$I_A(\mathbf{u}) = \frac{1}{\mathbf{r}^2} = \frac{1}{x^2 + y^2 + z^2},$$

is the reciprocal of the square of the distance from A to either point where the line pierces the ellipsoid. It thus appears that the lines through A for which the moment of inertia is a maximum

or a minimum lie along the principal axes for A. Evidently in certain cases Poinsot's ellipsoid reduces to a spheroid or a sphere and in fact it can not very greatly differ in shape from a prolate spheroid. For equation § 82, (9) shows that,

$$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{1}{c^2}$$

and taking $a \geq b \geq c$ we have, on setting $a = mb$,

$$b^2 \leq \left(\frac{1 + m^2}{m^2}\right) c^2.$$

Thus if the largest semiaxis a is large in comparison with the medium semiaxis b then m is large and consequently b is not greatly larger than the smallest semiaxis c. In fact we may easily see that a/b and b/c can not *both* exceed $1.272 \cdots$ and b/c can never exceed $1.414 \cdots$.

EXERCISES

1. A unit particle is placed at each of the four points,

$$(1, -5, -1), \qquad (3, 3, -3), \qquad (1, 1, 5), \qquad (-5, 1, -1).$$

What is the homography of inertia of the set of particles for its centroid? *Ans.* $\varphi_0(\mathbf{u}) = 72\,\mathbf{u}$

2. Through each vertex of a triangle ABC a line is drawn parallel to the median through the next succeeding vertex and a particle of mass 1 is placed at each vertex of the resulting triangle. Also a particle of mass 12 is placed at the midpoint of each side of the triangle ABC. Show that these two sets of three particles have the same moment of inertia about any line through their common centroid.

3. The vector $\mathbf{a}$ forms a uniform rod of mass m_0. Find its homography of inertia for its centroid. *Ans.* $\varphi_0(\mathbf{u}) = \frac{m_0}{12}\,\mathbf{a}\times(\mathbf{u}\times\mathbf{a})$

4. The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form the sides of a triangle. Find the homography of inertia of the area of the triangle for its centroid.

$$\textit{Ans.} \quad \varphi_0(\mathbf{u}) = \frac{m_0}{36}\{\mathbf{a}\times(\mathbf{u}\times\mathbf{a}) + \mathbf{b}\times(\mathbf{u}\times\mathbf{b}) + \mathbf{c}\times(\mathbf{u}\times\mathbf{c})\}$$

5. Find the moment of inertia of a triangular area of sides a, b, c and mass m_0 about the principal centroidal axes in its plane.

$$\textit{Ans.} \quad I = \frac{m_0}{36}\left(\frac{a^2 + b^2 + c^2}{2} \pm \sqrt{a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2}\right)$$

6. The vectors **a**, **b** form adjacent sides of a parallelogram. Find the homography of inertia of the area of the parallelogram for its centroid.

Ans. $\varphi_0(\mathbf{u}) = \dfrac{m_0}{12}\{\mathbf{a}\times(\mathbf{u}\times\mathbf{a}) + \mathbf{b}\times(\mathbf{u}\times\mathbf{b})\}$

7. Find the moment of inertia of a rectangular area of sides a, b about a diagonal.

Ans. $I = \dfrac{m_0}{6}\dfrac{a^2b^2}{a^2+b^2}$

8. Find the moment of inertia of a rectangular parallelopiped of edges a, b, c about a diagonal.

Ans. $I = \dfrac{m_0}{6}\dfrac{b^2c^2+c^2a^2+a^2b^2}{a^2+b^2+c^2}$

9. A wire of mass m_0 forms a circle of radius r in a plane perpendicular to the unit vector **a**. Find its homography of inertia for the center.

Ans. $\varphi_0(\mathbf{u}) = \dfrac{m_0r^2}{2}\{\mathbf{u} + (\mathbf{a}\cdot\mathbf{u})\mathbf{a}\}$

10. The vectors **a**, **b**, **c**, **d**, **e**, **f** form the edges of a tetrahedron. Find the homography of inertia of the volume of the tetrahedron for its centroid.

$$\textit{Ans.}\quad \varphi_0(\mathbf{u}) = \frac{m_0}{80}\{\mathbf{a}\times(\mathbf{u}\times\mathbf{a}) + \mathbf{b}\times(\mathbf{u}\times\mathbf{b}) + \mathbf{c}\times(\mathbf{u}\times\mathbf{c}) + \mathbf{d}\times(\mathbf{u}\times\mathbf{d}) + \mathbf{e}\times(\mathbf{u}\times\mathbf{e}) + \mathbf{f}\times(\mathbf{u}\times\mathbf{f})\}$$

11. The vectors **a** and **b** form the semiaxes of an elliptical area. Find its homography of inertia for the centroid. Specialize for the circle.

Ans. $\varphi_0(\mathbf{u}) = \dfrac{m_0}{4}\{\mathbf{a}\times(\mathbf{u}\times\mathbf{a}) + \mathbf{b}\times(\mathbf{u}\times\mathbf{b})\}$

12. The vectors **a**, **b**, **c** form the semiaxes of an ellipsoidal volume. Find its homography of inertia for the centroid. Specialize for the sphere.

Ans. $\varphi_0(\mathbf{u}) = \dfrac{m_0}{5}\{\mathbf{a}\times(\mathbf{u}\times\mathbf{a}) + \mathbf{b}\times(\mathbf{u}\times\mathbf{b}) + \mathbf{c}\times(\mathbf{u}\times\mathbf{c})\}$

13. Find the moment of inertia of an elliptic area of semiaxes a and b about a diameter of length $2r$.

Ans. $I = \dfrac{m_0}{4}\dfrac{a^2b^2}{r^2}$

14. Two sets of particles which have the same moment of inertia with respect to every line are said to be *equimomental.* Show that a triangular area is equimomental to a set of three equal particles each of mass one third that of the triangle and placed at the mid-points of its sides.

15. Show that a tetrahedral volume of mass m_0 is equimomental to a set of particles consisting of six equal particles of mass $m_0/10$ placed at the midpoints of the edges of the tetrahedron together with a single particle of mass $2m_0/5$ placed at the centroid of the tetrahedron.

16. Show that a triangular area of mass m_0 is equimomental to a set of particles consisting of three equal particles of mass $m_0/12$ placed at the vertices of the triangle together with a single particle of mass $3m_0/4$ placed at the centroid of the triangle.

17. Show that a tetrahedral volume of mass m_0 is equimomental to a set of particles consisting of four equal particles of mass $m_0/20$ placed at the vertices of the tetrahedron together with a single particle of mass $4m_0/5$ placed at the centroid of the tetrahedron.

18. *Legendre's Ellipsoid.* Show that an ellipsoid can always be found having a homogeneous volume equimomental to any given set of particles.

19. Show that a set of four equal particles can always be found equimomental to any given set of particles and that a set of three equal particles can always be found equimomental to any set of particles in a plane.

20. A set of particles has a total mass unity and its centroid is at the origin. If,

$$\varphi_0(1, 0, 0) = (7, 2, 0), \qquad \varphi_0(0, 1, 0) = (2, 6, -2),$$
$$\varphi_0(0, 0, 1) = (0, -2, 5),$$

determine the principal moments of inertia and the principal directions for the centroid.

Ans. $I_1 = 3,\ \mathbf{u}_1 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}),\quad I_2 = 6,\ \mathbf{u}_2 = (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}),$
$I_3 = 9,\ \mathbf{u}_3 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$

21. For what point, if any, of the line,

$$\frac{x-3}{-1} = \frac{y-3}{2} = \frac{z+6}{2}$$

is this line a principal axis of the set of particles of Problem 20?

Ans. $(2, 5 - 4)$

22. For what point, if any, of the line,

$$\frac{x}{-1} = \frac{y+6}{2} = \frac{z-6}{1}$$

is this line a principal axis of the set of particles of Problem 20?

Ans. $\left(-\frac{17}{18}, -\frac{74}{18}, \frac{125}{18}\right)$

23. A *spherical point* for a set of particles is a point P such that the moment of inertia of the set is the same about all lines through P. If I_1, I_2, I_3 are the principal moments of inertia of a set of particles for the centroid, show that the set of particles will possess spherical points only if $I_1 \geq I_2 = I_3$ and that in this case such points exist only on the principal axis for the centroid corresponding to I_1 and at a distance $\sqrt{(I_1 - I_2)/m_0}$ from the centroid.

24. Find the spherical points for a hemispherical volume of radius a.
Ans. At the center of the plane face and also at a point within the hemisphere on the diameter perpendicular to the plane face and at a distance $3a/4$ from the plane face

25. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are three mutually perpendicular vectors, show that the three vectors $\mathbf{u}\times\varphi_A(\mathbf{u})$, $\mathbf{v}\times\varphi_A(\mathbf{v})$, $\mathbf{w}\times\varphi_A(\mathbf{w})$ are coplanar.

26. Show that for Poinsot's ellipsoid with semiaxes $a \geqq b \geqq c$, either $a/b \leqq 1.272\cdots$ or $b/c \leqq 1.272\cdots$.

84. Free Motion of a Rigid Body; Plane Motion.

We are now in a position to consider the kinetics of a set of particles m_i, P_i forming a rigid body. The first peculiarity of such a set which we shall observe is that the total power $\Sigma\, \mathbf{g}_i\cdot\mathbf{p}_i'$ of the interior forces $\mathbf{g}_i$ is constantly zero. This is easily seen to be a consequence of the definition of the rigid body which requires that the distance $|\mathbf{p}_i - \mathbf{p}_j|$ between every pair of particles P_i, P_j of the body remain constant. For if we represent by $\mathbf{h}_{ij}$ the force with which P_j acts on P_i and by $\mathbf{h}_{ji}$ the force with which P_i acts on P_j, then by Newton's third law (§ 2) these two forces are along the line P_iP_j, equal in amount, and of opposite sense and we may set,

$$\mathbf{h}_{ij} = k(\mathbf{p}_i - \mathbf{p}_j), \qquad \mathbf{h}_{ji} = k(\mathbf{p}_j - \mathbf{p}_i).$$

Then,

$$\mathbf{h}_{ij}\cdot\mathbf{p}_i' + \mathbf{h}_{ji}\cdot\mathbf{p}_j' = k(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}_i' - \mathbf{p}_j') = \frac{k}{2}\{(\mathbf{p}_i - \mathbf{p}_j)^2\}' = 0$$

and consequently,

$$\Sigma\, \mathbf{g}_i\cdot\mathbf{p}_i' = \Sigma\, \mathbf{h}_{ij}\cdot\mathbf{p}_i' = 0.$$

Making use of this fact the three fundamental equations § 77, (1), § 78, (1), § 79, (1) which we have derived for the kinetics of any set of particles, become for the rigid body,

(1) $$(\Sigma\, m_i\, \mathbf{p}_i')' = \Sigma\, \mathbf{f}_i,$$

(2) $$(\Sigma\, \mathbf{p}_i \times m_i\, \mathbf{p}_i')' = \Sigma\, \mathbf{p}_i \times \mathbf{f}_i,$$

(3) $$(\Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2)' = \Sigma\, \mathbf{f}_i\cdot\mathbf{p}_i',$$

while for motion relative to the centroid we have from equations § 80, (2), § 80, (6), § 80, (7), $(\mathbf{r}_i = \mathbf{p}_i - \mathbf{p}_0)$,

(4) $$\Sigma\, m_i\, \mathbf{r}_i' = 0,$$

(5) $$(\Sigma\, \mathbf{r}_i \times m_i\, \mathbf{r}_i')' = \Sigma\, \mathbf{r}_i \times \mathbf{f}_i,$$

(6) $$(\Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2)' = \Sigma\, \mathbf{f}_i\cdot\mathbf{r}_i'.$$

For brevity we indicate the momentum, the moment of momentum and the kinetic energy of the body respectively by $\mathbf{R}$, $\mathbf{L}$, T. Thus,

$$\Sigma\, m_i\, \mathbf{p}_i' = \mathbf{R}, \qquad \Sigma\, \mathbf{p}_i \times m_i\, \mathbf{p}_i' = \mathbf{L}, \qquad \Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2 = T,$$

while the corresponding quantities for motion relative to the centroid are,

$$\Sigma\, m_i\, \mathbf{r}_i' = \mathbf{R}_0, \qquad \Sigma\, \mathbf{r}_i \times m_i\, \mathbf{r}_i' = \mathbf{L}_0, \qquad \Sigma\, \tfrac{1}{2} m_i\, \mathbf{r}_i'^2 = T_0.$$

The sum, moment, and power of the exterior forces $\mathbf{f}_i$ are indicated by,

$$\begin{aligned} \Sigma\, \mathbf{f}_i = \mathbf{S}, \qquad & \Sigma\, \mathbf{p}_i \times \mathbf{f}_i = \mathbf{M}, \qquad & \Sigma\, \mathbf{f}_i \cdot \mathbf{p}_i' = W', \\ & \Sigma\, \mathbf{r}_i \times \mathbf{f}_i = \mathbf{M}_0, \qquad & \Sigma\, \mathbf{f}_i \cdot \mathbf{r}_i' = W_0'. \end{aligned}$$

With this notation the above equations take the form,

$$\begin{aligned} &(1) \quad \mathbf{R}' = \mathbf{S}, \qquad &(2) \quad \mathbf{L}' = \mathbf{M}, \qquad &(3) \quad T' = W', \\ &(4) \quad \mathbf{R}_0 = 0, \qquad &(5) \quad \mathbf{L}_0' = \mathbf{M}_0, \qquad &(6) \quad T_0' = W_0'. \end{aligned}$$

The calculation of several of these quantities is greatly facilitated by the use of the fundamental equation of the kinematics of a rigid body § 48, (6),

$$(7) \qquad (\mathbf{p}_i - \mathbf{a})' = \boldsymbol{\omega} \times (\mathbf{p}_i - \mathbf{a}),$$

in which $\boldsymbol{\omega}$ is the vector angular velocity of the body and A is any point rigidly attached to the body. From this we have at once,

$$(8) \qquad (\mathbf{p}_0 - \mathbf{a})' = \boldsymbol{\omega} \times (\mathbf{p}_0 - \mathbf{a}),$$

and the following important relations are obtained,

$$\begin{aligned} \mathbf{R}_A &= \Sigma\, m_i(\mathbf{p}_i - \mathbf{a})' = \boldsymbol{\omega} \times \Sigma\, m_i(\mathbf{p}_i - \mathbf{a}) = m_0\, \boldsymbol{\omega} \times (\mathbf{p}_0 - \mathbf{a}), \\ \mathbf{L}_A &= \Sigma\, (\mathbf{p}_i - \mathbf{a}) \times m_i(\mathbf{p}_i - \mathbf{a})' \\ &= \Sigma\, m_i(\mathbf{p}_i - \mathbf{a}) \times \{\boldsymbol{\omega} \times (\mathbf{p}_i - \mathbf{a})\} = \varphi_A(\boldsymbol{\omega}), \\ T_A &= \Sigma\, \tfrac{1}{2} m_i(\mathbf{p}_i - \mathbf{a})'^2 = \Sigma\, \tfrac{1}{2} m_i\{\boldsymbol{\omega} \times (\mathbf{p}_i - \mathbf{a})\}^2 \\ &= \tfrac{1}{2}\, \boldsymbol{\omega} \cdot \Sigma\, m_i(\mathbf{p}_i - \mathbf{a}) \times \{\boldsymbol{\omega} \times (\mathbf{p}_i - \mathbf{a})\} \\ &= \tfrac{1}{2}\, \boldsymbol{\omega} \cdot \varphi_A(\boldsymbol{\omega}) = \tfrac{1}{2}\, \omega^2 I_A(\boldsymbol{\omega}), \\ W_A' &= \Sigma\, \mathbf{f}_i \cdot (\mathbf{p}_i - \mathbf{a})' = \Sigma\, \mathbf{f}_i \cdot \boldsymbol{\omega} \times (\mathbf{p}_i - \mathbf{a}) = \boldsymbol{\omega} \cdot \Sigma\, (\mathbf{p}_i - \mathbf{a}) \times \mathbf{f}_i. \end{aligned}$$

If the body has a fixed point we may take it as the above point A and use it as the origin of coördinates and these relations

become,

(9) $\mathbf{R} = m_0\, \boldsymbol{\omega} \times \mathbf{p}_0,$ (10) $\mathbf{L} = \varphi(\boldsymbol{\omega}),$

(11) $T = \frac{1}{2}\, \boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\, \omega^2 I(\boldsymbol{\omega}),$ (12) $W' = \boldsymbol{\omega}\cdot\mathbf{M},$

where $\varphi(\mathbf{u})$ is the homography of inertia for the origin and $I(\boldsymbol{\omega})$ is the moment of inertia of the body about a line through the origin in the direction of $\boldsymbol{\omega}$. On the other hand if we take the centroid of the body as the above point A, the relations become,

(13) $\mathbf{R}_0 = 0,$ (14) $\mathbf{L}_0 = \varphi_0(\boldsymbol{\omega}),$

(15) $T_0 = \frac{1}{2}\, \boldsymbol{\omega}\cdot\mathbf{L}_0 = \frac{1}{2}\, \omega^2 I_0(\boldsymbol{\omega}),$ (16) $W_0' = \boldsymbol{\omega}\cdot\mathbf{M}_0.$

On the basis of these equations we may discuss the motion of a rigid body. Theoretically this discussion may be reduced to the consideration of two simpler problems. The equation $\mathbf{p}_i = \mathbf{p}_0 + \mathbf{r}_i$ makes it clear that if we can determine $\mathbf{r}_i$ and $\mathbf{p}_0$ as functions of the time we shall have $\mathbf{p}_i$ at once. We have therefore the problem of determining $\mathbf{r}_i$ or the *problem of motion relative to the centroid* and the problem of determining $\mathbf{p}_0$ or the *problem of the motion of the centroid.* Equation (1), $m_0\, \mathbf{p}_0'' = \mathbf{S}$ shows that the centroid moves as if it were a particle subject to all the exterior forces acting on the body. When these forces $\mathbf{f}_i$ are known the problem of determining $\mathbf{p}_0$ becomes therefore merely a problem in the motion of a particle as discussed in Chapter IV. There remains the problem of the determination of the $\mathbf{r}_i$. We observed in § 80 that the two theorems of moment of momentum and of kinetic energy hold in this relative motion exactly as if the centroid were a fixed point. We shall see in the next article that by means of the first of these theorems the motion of a rigid body with one fixed point can be completely determined and consequently all the conclusions there reached will be applicable to the relative motion of a body about its centroid.

When the exterior forces $\mathbf{f}_i$ depend only on the relative positions and velocities $\mathbf{r}_i$ and $\mathbf{r}_i'$, the motion of a rigid body may actually be determined by the solution of the problem of motion relative to the centroid and following that, the forces $\mathbf{f}_i$ being now known, the solution of the problem of the motion of the centroid. If however the forces $\mathbf{f}_i$ depend on the absolute positions and velocities $\mathbf{p}_i$ and $\mathbf{p}_i'$, the two problems must be considered simultaneously.

As an example we may discuss the motion of a billiard ball, i.e. the sliding and rolling motion of a sphere of mass m and radius a on a horizontal table. We shall assume that the friction of rolling and pivoting is negligible and that the coefficient of sliding friction is μ. Let $\mathbf{k}$ be a unit upward vector. Then the forces acting on the sphere are the force of gravity, $-mg\,\mathbf{k}$ applied at the center C and the reaction of the table, $mg\,\mathbf{k} - mg\mu\,\mathbf{t}$ applied at the point Q of the sphere which is in contact with the table, $\mathbf{t}$ being a unit vector having the direction and sense of the velocity of Q. Equation (1) then becomes,

$$m\,\mathbf{c}'' = -\,mg\mu\,\mathbf{t}$$

and since $\varphi_0(\mathbf{u}) = \frac{2}{5}ma^2\,\mathbf{u}$, equation (5) becomes,

$$\tfrac{2}{5}ma^2\,\boldsymbol{\omega}' = mg\mu a\,\mathbf{k}\times\mathbf{t}.$$

If we let the scalar velocity of Q be v then by equation (7),

$$v\,\mathbf{t} - \mathbf{c}' = -\,a\,\boldsymbol{\omega}\times\mathbf{k}$$

and on differentiating and reducing by the two previous equations, this yields,

$$(v' + \tfrac{7}{2}g\mu)\mathbf{t} + v\,\mathbf{t}' = 0.$$

As long as the sliding motion continues we therefore have,

$$\mathbf{t}' = 0, \qquad v' + \tfrac{7}{2}g\mu = 0$$

and hence,

$$\mathbf{t} = \mathbf{i}, \qquad v = v_0 - \tfrac{7}{2}g\mu t, \qquad \mathbf{i},\ v_0 \text{ const.}$$

Returning to our first two equations we find,

$$\mathbf{c}'' = -\,g\mu\,\mathbf{i}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_0 + \frac{5g\mu}{2a}\,t\,\mathbf{j}, \qquad \mathbf{j} = \mathbf{k}\times\mathbf{i},$$

showing that during the sliding motion the acceleration of the center is a constant and that the center consequently traces out an arc of a parabola, while the vector angular velocity $\boldsymbol{\omega}$ is a linear function of the time. During the sliding motion the *angular velocity of pivoting* $\boldsymbol{\omega}\cdot\mathbf{k}$ evidently remains constant. At the time $t = 2v_0/7g\mu$ the sliding motion ceases and the motion becomes one of rolling only, in a straight line with constant speed.

As another example we may consider the important case of a rigid body so constrained that all its particles move parallel to

the same fixed plane. In this case of *plane motion* our equations may be reduced to simple coördinate forms very convenient in application. We take the centroidal plane parallel to the motion as the XY-coördinate plane and let the position of the centroid P_0 in this plane be determined by its coördinates (x_0, y_0). If then (X_i, Y_i, Z_i) are the coördinates of the exterior force $\mathbf{f}_i$ we have on measuring equation (1) on the X and Y-axes,

$$(17) \qquad m_0 x_0'' = \Sigma X_i, \qquad m_0 y_0'' = \Sigma Y_i.$$

These equations will serve to determine the motion of the centroid.

Now let θ be the angle which any line in this plane and fixed in the body makes with any line in this plane fixed in the reference frame, θ being counted as positive in the positive sense of rotation relative to the Z-axis. The vector angular velocity ω of the body will then have the coördinates $(0, 0, \theta')$ and the moment of momentum $\mathbf{L}_0$ of the body about its centroid P_0 will by equation (14) have the Z-coördinate,

$$\mathbf{k}\cdot\mathbf{L}_0 = \mathbf{k}\cdot\varphi_0(\omega) = \theta'\Sigma\, m_i(\mathbf{r}_i \times \mathbf{k})^2 = \theta' I_0(\omega),$$

in which $I_0(\omega)$ is the moment of inertia of the body about the centroidal line perpendicular to the plane of the motion. Also if we let (ξ_i, η_i, ζ_i) be the coördinates of the vector $\mathbf{r}_i = \mathbf{p}_i - \mathbf{p}_0$ giving the position of the point of application of the force $\mathbf{f}_i$ relative to the centroid, then the moment $\mathbf{M}_0$ of the forces about the centroid will have the Z-coördinate $\mathbf{k}\cdot\mathbf{M}_0 = \Sigma\, \xi_i Y_i - \eta_i X_i$ and equation (5) when measured on the Z-axis yields,

$$(18) \qquad I_0\theta'' = \Sigma\, (\xi_i Y_i - \eta_i X_i).$$

The three equations (17), (18) completely determine the motion of the body but they may be combined to yield other forms occasionally convenient. Thus equation (3) may by Koenig's Principle, § 80, (5) be written,

$$\{\tfrac{1}{2}m_0\, \mathbf{p}_0'^2 + \Sigma\, \tfrac{1}{2}m_i\, \mathbf{r}_i'^2\}' = \Sigma\, \mathbf{f}_i\cdot\mathbf{p}_i'$$

and by equation (15) this reduces to,

$$\{\tfrac{1}{2}m_0\, \mathbf{p}_0'^2 + \tfrac{1}{2}\,\omega^2 I_0\}' = \Sigma\, \mathbf{f}_i\cdot\mathbf{p}_i',$$

which in our present case becomes,

$$(19) \qquad \{\tfrac{1}{2}m_0(x_0'^2 + y_0'^2) + \tfrac{1}{2}I_0\theta'^2\}' = \Sigma\, (X_i x_i' + Y_i y_i').$$

This is of course the theorem of kinetic energy and is particularly convenient to employ where the points of application of some of the forces remain fixed, as these forces do not affect the second member.

To show how these results work in practice we consider the following example.

A circular disc of unit radius rolls from rest without slipping under the action of gravity down the inclined face of a right-triangular lamina 4 units wide and 3 units high having the same density as the disc. The base of the triangular lamina rests on a smooth horizontal plane and the disc and the lamina remain vertical. Express the distance s rolled by the disc and the distance r slid by the lamina in terms of the time.

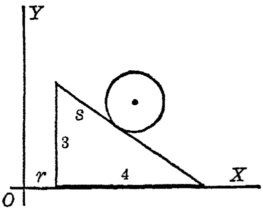

FIG. 81

If θ is the angle through which the disc has turned and x and y are the coördinates of its center, then from the figure,

$$\theta'' = s'', \qquad x'' = r'' + \tfrac{4}{5}s'', \qquad y'' = -\tfrac{3}{5}s''$$

The forces acting on the disc are evidently the force of gravity of amount $\pi\rho g$ (ρ = density) acting downward and the unknown constraint **k** of the lamina. Equations (17) and (18) applied to the disc thus yield,

$$\pi\rho x'' = k_1, \qquad \pi\rho y'' = -\pi\rho g + k_2, \qquad \tfrac{1}{2}\pi\rho\theta'' = -\tfrac{4}{5}k_1 + \tfrac{3}{5}k_2.$$

The only effective force acting on the lamina will evidently be the negative of the horizontal component of the constraint **k** so that the first of equations (17) applied to the lamina yields,

$$6\rho r'' = -k_1.$$

Algebraic reduction of these equations readily gives the relations,

$$(6 + \pi)r'' + \tfrac{4}{5}\pi s'' = 0, \qquad \tfrac{4}{3}r'' + \tfrac{5}{2}s'' = g.$$

Integrating and introducing the initial conditions yields,

$$(6 + \pi)r + \tfrac{4}{5}\pi s = 0, \qquad \tfrac{4}{3}r + \tfrac{5}{2}s = \tfrac{1}{2}gt^2,$$

from which the desired expressions for r and s are obvious.

EXERCISES

1. Derive equation (19) of this section from equations (17) and (18).
2. A uniform rod is placed with one end on a smooth horizontal table and allowed to fall from rest. Show that the velocity with which the midpoint strikes the table is $\sqrt{3gh/2}$ where h is the initial height of the midpoint, and show that as the rod becomes horizontal the pressure on the lower end becomes 1/4 of the weight of the rod.
3. A uniform rod rests one end on a smooth floor and the other on a smooth vertical wall. If the rod starts from rest at an inclination α, show that it will leave the wall when the inclination is θ where $\sin\theta = \frac{2}{3}\sin\alpha$.
4. Four uniform rods are freely jointed at their ends to form a parallelogram which slides on a smooth horizontal plane, one vertex being fixed. The initial shape of the parallelogram is a rectangle and the initial angular velocity of two opposite sides is ω and of the other two is zero. Show that as the figure moves the angular velocity of all the sides will be ω whenever an angle of the parallelogram is a maximum or a minimum.
5. A uniform bar AB of length $2a$ has its ends A and B sliding on two smooth parallel planes whose distance apart is $2b < 2a$. Every particle of the bar is attracted to a point O midway between the two planes by a force proportional to its mass and to its distance from O. Discuss the motion and the form of the surface generated by the bar. Show that in a certain case this surface is a hyperboloid.
6. A bead of mass m slides on a smooth straight wire which has its midpoint O fixed. The system starts from rest in a horizontal position and moves under the action of gravity. If ω is the maximum angular velocity of the wire and θ is the angle it makes with the vertical when this velocity is attained and r is the distance which the bead is then from O, show that,

$$4(mr^2 + I)\omega^4 - 8mgr\omega^2\cos\theta + mg^2\sin^2\theta = 0,$$

where I is the moment of inertia of the wire about O.
7. A smooth wire in the form of a circle of radius a and having a mass M lies flat on a smooth horizontal table and is freely pivoted at one of its points O to the table. A bead P of mass m slides on the wire and is repelled from O with a force proportional to the distance. Discuss the motion assuming that the system starts from rest with the bead very near the fixed point O of the wire.
8. A smooth wire in the form of a circle of radius a and having a mass M lies flat on a smooth horizontal table. A bead P of mass m slides on the wire. If the bead is given an initial velocity v_0 along the wire, discuss the motion and determine the pressure of the bead on the wire.

9. A homogeneous circular disc of radius a and mass m rolls and slides in a vertical plane on a fixed horizontal line. The coefficient of sliding friction is μ and the rolling friction is negligible. Discuss the sliding and rolling phases of the motion and show that there is a position on the disc such that the particle of the disc occupying this position has the same velocity during the entire motion.

10. A homogeneous sphere rolls directly down a rough inclined plane under the action of gravity. Show that the acceleration of the center is constant and 5/7 of what it would be if the plane were smooth.

11. A homogeneous sphere rolls directly down a rough fixed sphere under the action of gravity. Show that the center moves like a bead sliding on a smooth vertical circle under the action of a force of gravity 5/7 of the actual value. If the moving sphere starts from rest very near the top of the fixed sphere, show that the spheres will separate when the line joining their centers makes an angle arccos 10/17 with the vertical.

12. Two equal rough spheres of radius a are balanced on top of each other and the lower sphere rests on a smooth horizontal plane. If slightly disturbed, show that the spheres will remain in contact at the same points and that the line joining their centers will make an angle θ with the vertical satisfying the equation,

$$a(7 + 5\sin^2\theta)\theta'^2 = 2g(1 - \cos\theta).$$

13. A sphere of radius r rolls from rest under the action of gravity down the rough inverted cycloid,

$$x = a(\theta + \sin\theta), \qquad y = a(1 - \cos\theta)$$

from a cusp to an adjacent vertex. Show that the velocity v acquired by its center is given by, $v^2 = \frac{10}{7} g(2a - r)$.

14. A rough cylinder of radius a rotates about its axis which is horizontal with a constant angular acceleration ω'. Show that a sphere can be balanced on it under the action of gravity provided $\omega' \leqq 5g/2a$.

15. A rigid body B is free to rotate about its centroid which is fixed in another rigid body A. If the applied forces acting on B have a zero moment with respect to its centroid, show that the motion of A is the same as if B were replaced by its centroid.

16. Prove *Routh's Theorem.* A homogeneous sphere rolls on a rough plane under the action of any forces whose moment with respect to its center is constantly zero. The motion of the center will be the same as if the plane were smooth and all the forces reduced to 5/7 of their actual value.

85. Motion of a Body with One Fixed Point.

An important case of the motion of a rigid body is that in which one point of the body remains fixed. Not only is this case important in itself but, as explained in the preceding article, it is important because all the conclusions reached hold without modification for the motion of a free rigid body relative to its centroid. We may conveniently take the fixed point of the body as the origin of coördinates O. We shall then have acting on each particle P_i of the body an applied force $\mathbf{f}_i$ and in addition we shall have the constraint $\mathbf{k}$ applied at O and bringing about the fixing of that point. Equations (1), (2), (3) of the preceding article now take the form,

$$(1) \qquad (\Sigma\, m_i\, \mathbf{p}_i')' = \Sigma\, \mathbf{f}_i + \mathbf{k} \qquad \text{or} \qquad \mathbf{R}' = \mathbf{S} + \mathbf{k},$$

$$(2) \qquad (\Sigma\, \mathbf{p}_i \times m_i\, \mathbf{p}_i')' = \Sigma\, \mathbf{p}_i \times \mathbf{f}_i \qquad \text{or} \qquad \mathbf{L}' = \mathbf{M},$$

$$(3) \qquad (\Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2)' = \Sigma\, \mathbf{f}_i \cdot \mathbf{p}_i' \qquad \text{or} \qquad T' = W',$$

where $\mathbf{S}$ and $\mathbf{M}$ and W' are now the sum and the moment about O and the power of the applied forces $\mathbf{f}_i$. The moment and power of the constraint $\mathbf{k}$ do not appear in the right members of equations (2) and (3) because they are evidently zero.

We shall find that equation (2) suffices to determine completely the motion of the body. This is to be expected because we have already observed that the displacement of a rigid body with one fixed point can be completely characterized by the vector $\mathbf{w}$ of Rodrigues' formula, § 44, (2) and a vector equation such as (2) should enable us to determine $\mathbf{w}$. After the motion of the body has been determined by equation (2) we may then employ equation (1) to find the constraint $\mathbf{k}$. By means of the relations (9), (10), (11), (12) of the preceding article our equations (1), (2), (3) take the form,

$$(1') \qquad m_0(\omega \times \mathbf{p}_0)' = \mathbf{S} + \mathbf{k},$$

$$(2') \qquad \{\varphi(\omega)\}' = \mathbf{M},$$

$$(3') \qquad (\tfrac{1}{2}\, \omega \cdot \mathbf{L})' = \{\tfrac{1}{2}\, \omega^2 I(\omega)\}' = \omega \cdot \mathbf{M}.$$

In this connection the moment of momentum $\mathbf{L}$ or $\varphi(\omega)$ is often called the *impulse vector*. We may regard equation (2′) as a vector differential equation for the determination of ω and if we can express ω as a function of the time in such a way as to satisfy

it, we may then seek to determine $\mathbf{w}$ by the equation, § 47, (6),

$$(4) \qquad \mathbf{w}' = \tfrac{1}{2}\{\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\mathbf{w})\mathbf{w} + \boldsymbol{\omega}\times\mathbf{w}\}.$$

And finally if we can integrate this differential equation in $\mathbf{w}$ we may then find the position P of any particle of the body from its initial position P_0 by Rodrigues' formula,

$$(5) \qquad \mathbf{p} + \mathbf{p}_0 = \frac{2}{1+\mathbf{w}^2}\{\mathbf{p}_0 + (\mathbf{w}\cdot\mathbf{p}_0)\mathbf{w} + \mathbf{w}\times\mathbf{p}_0\}.$$

The difficulty in carrying out the procedure outlined above is of course in integrating the differential equations (2′) and (4) for the determination of $\boldsymbol{\omega}$ and $\mathbf{w}$ as functions of the time. Not only are these equations often complicated when expanded in coördinates but the coefficients of the coördinates of $\boldsymbol{\omega}$ in equation (2′) are themselves variables. The difficulties in integrating equation (2′) are greatly reduced by the following device due to Euler. In addition to our fixed coördinate system we consider a moving coördinate system fixed in the moving body and we then write equation (2) in the form,

$$(6) \qquad \mathbf{L}'_2 + \boldsymbol{\omega}\times\mathbf{L} = \mathbf{M},$$

where, as in § 50, (2′), $\mathbf{L}'_2$ is the relative derivative of $\mathbf{L}$ and $\boldsymbol{\omega}\times\mathbf{L}$ is the drag derivative. If in particular we choose as the moving coördinate axes the three principal axes of inertia of the body for the fixed point O and call the coördinates of $\boldsymbol{\omega}$ in this system $(\omega_i, \omega_j, \omega_k)$, then since $\mathbf{L} = \varphi(\boldsymbol{\omega})$ we have by equation § 83, (13),

$$\mathbf{L} \equiv (I_1\omega_i, I_2\omega_j, I_3\omega_k).$$

Consequently,

$$\mathbf{L}'_2 \equiv (I_1\omega'_i, I_2\omega'_j, I_3\omega'_k)$$

and equation (6) becomes when expanded in these coördinates,

Euler's Equations.

$$(7) \qquad \begin{aligned} I_1\omega'_i + (I_3 - I_2)\omega_j\omega_k &= M_i, \\ I_2\omega'_j + (I_1 - I_3)\omega_k\omega_i &= M_j, \\ I_3\omega'_k + (I_2 - I_1)\omega_i\omega_j &= M_k, \end{aligned}$$

where M_i, M_j, M_k are the coördinates of $\mathbf{M}$ in the moving coördinate system. Since I_1, I_2, I_3 are constants these equations involve $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ in a comparatively simple way, but it will of course often happen that $\mathbf{M}$ also depends on $\boldsymbol{\omega}$.

To employ these relative coördinates ω_i, ω_j, ω_k for the determination of the rotation $\mathbf{w}$ which will carry the body from its initial position to its position at the instant t we write $\mathbf{w}'$ in equation (4) in the form $\mathbf{w}_2' + \boldsymbol{\omega} \times \mathbf{w}$ where $\mathbf{w}_2'$ is the relative derivative of $\mathbf{w}$ and $\boldsymbol{\omega} \times \mathbf{w}$ is the drag derivative. This yields,

$$\mathbf{w}_2' = \tfrac{1}{2}\{\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot \mathbf{w})\mathbf{w} - \boldsymbol{\omega} \times \mathbf{w}\}, \tag{8}$$

which may now be readily expanded in relative coördinates and from which the relative coördinates of $\mathbf{w}$ may be determined by integration. As the reader may easily convince himself by a simple geometric argument or by means of formula § 46, (3), the relative coördinates of $\mathbf{w}$ are exactly the same as the absolute coördinates referred to the initial position of the moving system so that the absolute value of $\mathbf{w}$ is known as soon as equation (8) is integrated.

Instead of determining the motion of the body by the above rotation $\mathbf{w}$ we may employ *Euler's angles* which give the position of the moving coördinate system S_2 relative to the fixed system S_1. Referring to the figure of § 46 we see that at each instant the system S_2 has an angular velocity ψ' about OZ_1, φ' about OZ_2 and θ' about ON so that the vector angular velocity $\boldsymbol{\omega}$ is the sum of three vectors of lengths ψ', φ', θ' laid off on OZ_1, OZ_2, ON respectively. The coördinates of these three vectors relative to the moving system S_2 will evidently be respectively,

$$\begin{array}{lccr}
(\sin\varphi \sin\theta\, \psi', & \cos\varphi \sin\theta\, \psi', & \cos\theta\, \psi') \\
(\quad 0 \quad , & 0 , & \varphi' \quad) \\
(\quad \cos\varphi\, \theta' \quad , & -\sin\varphi\, \theta' , & 0 \quad)
\end{array}$$

so that the relative coördinates of their sum $\boldsymbol{\omega}$ are,

$$\begin{aligned}
\omega_i &= \sin\varphi \sin\theta\, \psi' + \cos\varphi\, \theta', \\
\omega_j &= \cos\varphi \sin\theta\, \psi' - \sin\varphi\, \theta', \\
\omega_k &= \cos\theta\, \psi' + \varphi'.
\end{aligned} \tag{9}$$

These are equivalent to the equations,

$$\begin{aligned}
\psi' &= \sin\varphi \csc\theta\, \omega_i + \cos\varphi \csc\theta\, \omega_j, \\
\varphi' &= -\sin\varphi \cot\theta\, \omega_i - \cos\varphi \cot\theta\, \omega_j + \omega_k, \\
\theta' &= \cos\varphi\, \omega_i - \sin\varphi\, \omega_j,
\end{aligned} \tag{10}$$

the integration of which will give ψ, φ, θ from ω_i, ω_j, ω_k and thus determine the motion of the body.

The motion of the body having been determined we may return to equation (1′) and find the constraint $\mathbf{k}$ which maintains the point O fixed. The first member of equation (1′) becomes, on differentiation and reduction by equation (8) of the preceding article, $m_0\,\boldsymbol{\omega}' \times \mathbf{p}_0 + m_0\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}_0)$ and so we have $\mathbf{k}$ given by the formula,

$$\mathbf{k} = m_0\,\boldsymbol{\omega}' \times \mathbf{p}_0 + m_0\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}_0) - \mathbf{S}. \tag{11}$$

In the following sections of this chapter we shall discuss some special cases of the motion of a rigid body with one fixed point in which the integration of equations (7) and (8) or (10) may be carried through and the motion completely determined.

EXERCISES

1. A square lamina of edge $2a$ and mass m has the midpoint of one edge fixed at a point O of a smooth horizontal plane. This edge is free to rotate in the plane and the lamina is free to rotate about the edge. Discuss the motion of the lamina under the force of gravity.
2. A rigid body with one fixed point is acted upon by forces whose moment about the instantaneous axis is constantly zero. In addition the body may be constrained to slide along a smooth surface or have another fixed point. Show that the square of the angular velocity varies inversely as the moment of inertia of the body about the instantaneous axis.

86. Motion Under Forces of Zero Moment.

If a rigid body with one fixed point O is acted upon by applied forces $\mathbf{f}_i$ whose moment $\mathbf{M}$ with respect to O is constantly zero, we have a special case of the motion discussed in the previous article and a case in which the integration of the differential equations may be completely carried out. As in the previous article the conclusions reached are applicable without alteration to the motion of a free rigid body relative to its centroid. Since $\mathbf{M}$ is identically zero we have at once from equations, § 85, (2′), § 85, (3′),

$$(1) \qquad \mathbf{L}' = 0, \qquad (2) \quad T' = 0.$$

Consequently $\mathbf{L}$ is a constant vector and T is a constant scalar.

As in the preceding article we find it convenient to employ the moving coördinate system having for axes the principal axes of inertia of the body for the origin. Equation (1) then yields

a special form of Euler's equations,

$$
(3) \qquad
\begin{aligned}
I_1\omega_i' &= (I_2 - I_3)\omega_j\omega_k,\\
I_2\omega_j' &= (I_3 - I_1)\omega_k\omega_i,\\
I_3\omega_k' &= (I_1 - I_2)\omega_i\omega_j.
\end{aligned}
$$

Since $\mathbf{L}$ is a constant vector with relative coördinates, $\mathbf{L} \equiv (I_1\omega_i, I_2\omega_j, I_3\omega_k)$, we have,

$$
(4) \qquad I_1^2\omega_i^2 + I_2^2\omega_j^2 + I_3^2\omega_k^2 = \mathbf{L}^2,
$$

where $\mathbf{L}^2$ is a constant. By equation § 84, **(11)** we have $2T = \boldsymbol{\omega}\cdot\mathbf{L}$ and hence,

$$
(5) \qquad I_1\omega_i^2 + I_2\omega_j^2 + I_3\omega_k^2 = 2T,
$$

where $2T$ is constant. If we now call $\boldsymbol{\omega}^2 = u$ we have,

$$
(6) \qquad \omega_i^2 + \omega_j^2 + \omega_k^2 = u.
$$

Assuming that the principal moments of inertia I_1, I_2, I_3 are all different, we may solve equations (4), (5), (6) for ω_i^2, ω_j^2, ω_k^2 in terms of the variable u and the constants I_1, I_2, I_3, $\mathbf{L}^2$, $2T$. For simplicity in writing the results we call,

$$
(7) \qquad
\begin{gathered}
\frac{2T(I_2 + I_3) - \mathbf{L}^2}{I_2 I_3} = a, \qquad \frac{2T(I_3 + I_1) - \mathbf{L}^2}{I_3 I_1} = b,\\
\frac{2T(I_1 + I_2) - \mathbf{L}^2}{I_1 I_2} = c,
\end{gathered}
$$

where a little investigation will show that a, b, c are positive constants and that for $I_1 > I_2 > I_3$ we have $b > a$ and $b > c$. In terms of these we find,

$$
(8) \qquad
\begin{aligned}
\omega_i^2 &= \frac{I_2 I_3}{(I_1 - I_2)(I_1 - I_3)}(u - a),\\
\omega_j^2 &= \frac{I_3 I_1}{(I_2 - I_3)(I_2 - I_1)}(u - b),\\
\omega_k^2 &= \frac{I_1 I_2}{(I_3 - I_1)(I_3 - I_2)}(u - c).
\end{aligned}
$$

From equations (3) we see that,

$$
\begin{aligned}
u' = \boldsymbol{\omega}^{2\prime} &= 2(\omega_i\omega_i' + \omega_j\omega_j' + \omega_k\omega_k')\\
&= -2\frac{(I_2 - I_3)(I_3 - I_1)(I_1 - I_2)}{I_1 I_2 I_3}\,\omega_i\omega_j\omega_k
\end{aligned}
$$

and on substituting the values of ω_i, ω_j, ω_k from equations (8) we find,

$$u' = \pm\, 2\sqrt{(u-a)(b-u)(u-c)}.$$

In this differential equation the variables t and u are at once separable and on integrating between corresponding limits we have,

$$t = \pm \frac{1}{2}\int_{u_0}^{u} \frac{du}{\sqrt{(u-a)(b-u)(u-c)}}, \tag{9}$$

where u_0 is the value of u at the instant $t = 0$. The integral in the second member is of the type known as an *elliptic integral* and its value can not in general be expressed in terms of the elementary functions. However if we call,

$$\varphi = \arcsin\sqrt{\frac{b-u}{b-a}}, \qquad k = \sqrt{\frac{b-a}{b-c}}, \qquad (a > c),$$

equation (9) takes the form,

$$t = \pm \frac{1}{\sqrt{b-c}}\{F(k, \varphi) - F(k, \varphi_0)\}, \tag{10}$$

where,

$$F(k, \varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

is known as *Legendre's Elliptic Integral of the first kind* and is a well known tabulated function. Having thus expressed t as an *elliptic integral* in u we have in effect expressed u as an *elliptic function* of t and equations (8) then give us ω_i^2, ω_j^2, ω_k^2 also as functions of t. The initial state of the motion will determine the signs which ω_i, ω_j, ω_k have initially and Euler's equations will determine how these signs vary during the motion. For the special case $\mathbf{L}^2 = 2I_2T$ the constants a and c are equal and in this case the integral in equation (9) takes a form readily evaluated in terms of the elementary functions. The discussion of this case is left as an exercise for the reader.

The coördinates $(\omega_i, \omega_j, \omega_k)$ of the angular velocity $\boldsymbol{\omega}$ relative to the moving coördinate system having been determined, it only remains to determine the position of the body relative to the

fixed coördinate system. This we may conveniently do by the use of Euler's angles, ψ, φ, θ. Since the vector **L** is constant we may take its direction and sense as that of the fixed Z-axis. Calling the length of **L**, L we may then express its coördinates $(I_1\omega_i, I_2\omega_j, I_3\omega_k)$ relative to the moving system as the product of L by the cosines of the angles which it makes with these axes. Thus by table § 46, (8) we have,

$$(11) \qquad I_1\omega_i = L \sin\varphi \sin\theta, \qquad I_2\omega_j = L\cos\varphi\sin\theta,$$
$$I_3\omega_k = L\cos\theta,$$

from which φ and θ are determined at once without integration, it being recalled that θ is an angle of the first or second quadrant and $\sin\theta$ consequently positive. To determine ψ we recall the first two of equations § 85, (9),

$$\omega_i = \sin\varphi\sin\theta\,\psi' + \cos\varphi\,\theta', \qquad \omega_j = \cos\varphi\sin\theta\,\psi' - \sin\varphi\,\theta',$$

from which we have,

$$\psi' = \frac{\omega_i\sin\varphi + \omega_j\cos\varphi}{\sin\theta},$$

while from the first two of equations (11) we find,

$$\omega_i\sin\varphi + \omega_j\cos\varphi = \frac{I_1\omega_i^2 + I_2\omega_j^2}{L\sin\theta}, \qquad L^2\sin^2\theta = I_1\omega_i^2 + I_2\omega_j^2,$$

by which the formula for ψ' becomes,

$$(12) \qquad \psi' = L\,\frac{I_1\omega_i^2 + I_2\omega_j^2}{I_1^2\omega_i^2 + I_2^2\omega_j^2}.$$

From this we have ψ at once by an integration,

$$(13) \qquad \psi = L\int_0^t \frac{I_1\omega_i^2 + I_2\omega_j^2}{I_1\omega_i^2 + I_2^2\omega_j^2}\,dt + \psi_0.$$

The problem of determining the motion of a rigid body with one fixed point under the action of forces of zero moment with respect to this point is thus completely solved except for the special case in which two or more of the principal moments of inertia I_1, I_2, I_3 are equal. We shall discuss this case at the end of this article. However the above analytic solution can not give us a complete picture of the nature of the motion unless

we are familiar with the properties of the elliptic function expressing u in terms of t, and an investigation of this function would be beyond the scope of this book. This difficulty is however largely obviated by a very valuable and interesting geometric treatment of the motion due to Poinsot. We recall that, as explained in § 83, if we lay off from the fixed point O of the body in the direction and sense of every unit vector $\mathbf{u}$ a vector $\mathbf{p} = \mathbf{u}/\sqrt{I_0(\mathbf{u})}$ then the termini P of these vectors form Poinsot's ellipsoid of the body for the point O. The particular point P_1 of the ellipsoid obtained when the unit vector $\mathbf{u}$ has the direction and sense of the angular velocity $\boldsymbol{\omega}$ of the body is called by Poinsot the *pole*. It is one of the points in which Poinsot's ellipsoid for O is pierced by the instantaneous axis. By equation § 84, (11), $2T = \boldsymbol{\omega}\cdot\mathbf{L} = \omega^2 I_0(\boldsymbol{\omega})$ we have,

$$(14) \qquad OP_1 = \mathbf{p}_1 \qquad \boldsymbol{\omega} \qquad = \frac{\boldsymbol{\omega}}{\sqrt{2T}}.$$

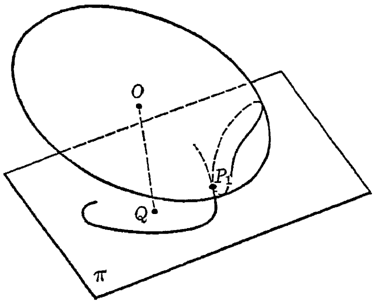

Fig. 82

If we employ the moving coördinate system with axes the principal axes of inertia of the body for O, the equation of the ellipsoid is, § 83, (15),

$$(15) \qquad F(\mathbf{p}) \equiv I_1x^2 + I_2y^2 + I_3z^2 - 1 = 0$$

and the vector ∇F known as the *gradient* of F having at each point of the ellipsoid the direction and sense of the outward normal has the coördinates,

$$\nabla F \equiv \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \equiv (2I_1x,\ 2I_2y,\ 2I_3z).$$

In particular at the pole P_1 the gradient has the value,

$$\nabla F_1 \equiv \frac{1}{\sqrt{2T}}(2I_1\omega_i,\ 2I_2\omega_j,\ 2I_3\omega_k) \equiv \frac{2\mathbf{L}}{2T}$$

The normal to the ellipsoid at the pole is thus parallel to the moment of momentum $\mathbf{L}$ of the body and we have for the equation of the tangent plane π to the ellipsoid at the pole, $(\mathbf{p} - \mathbf{p}_1)\cdot\mathbf{L} = 0$. In the normal form this equation becomes, $(\mathbf{p} - \mathbf{p}_1)\cdot\mathbf{L}/L = 0$

and consequently the distance r from O to this plane is,

$$(16) \qquad r = \frac{\mathbf{p}_1 \cdot \mathbf{L}}{L} = \frac{\sqrt{2T}}{L\sqrt{2T}}$$

In the case under consideration the vector $\mathbf{L}$ and the scalar T are constants and consequently the tangent plane π being perpendicular to a constant vector and at a constant distance from the fixed point O, is a fixed plane. Since the pole P_1 is on the instantaneous axis its drag velocity is always zero and we have established the important,

Poinsot's Theorem. When the forces applied to a rigid body with one fixed point have constantly a zero moment with respect to that point, Poinsot's ellipsoid of the body for that point rolls without slipping on a fixed tangent plane.

The path traced by the pole P_1 on the ellipsoid is known as the *polhode* while its path on the fixed tangent plane is called the *herpolhode*. The polhode is one branch of the locus of those points of the ellipsoid whose tangent planes are at the constant distance r from its center O, the other branch being traced on the ellipsoid by the point diametrically opposite the pole. The equations of this locus in the moving coördinate system are from equations, (14), (4), (5),

$$I_1^2x^2 + I_2^2y^2 + I_3^2z^2 = \frac{L^2}{2T} = \frac{1}{r^2}, \qquad I_1x^2 + I_2y^2 + I_3z^2 = 1,$$

showing that the locus is the intersection of two coaxial ellipsoids. This is in general a curve consisting of two branches each of which is shaped somewhat like the edge of a saddle, but which in special cases may reduce to two ellipses or two points. The herpolhode is a plane curve in the fixed tangent plane π repeatedly encircling the projection Q of O on π. The radial distance QP_1 varies between two extreme values which it takes on alternately, but the curve remains constantly concave towards Q and has no points of inflection.

In the case when two of the principal moments of inertia for the fixed point O are equal, Poinsot's ellipsoid becomes an ellipsoid of revolution. In this case it is evident that the polhode, being one branch of the locus of those points of an ellipsoid of revolution whose tangent planes are at a given distance from its center, is a circle on this ellipsoid in a plane perpendicular to

its axis of figure and hence everywhere at a fixed distance from the center O of the ellipsoid. Since then the distances OP_1 and OQ remain constant, the same must be true of QP_1 and consequently the herpolhode is a circle with center at Q. Furthermore since the vector $OP_1 = \mathbf{p}_1$ is of constant length it follows that the angular velocity $\boldsymbol{\omega} = \sqrt{2T}\,\mathbf{p}_1$ is also of constant length and the scalar angular velocity of the body is constant. Since then the polhode circle is turning at a constant rate while rolling on the herpolhode circle and is maintaining a constant angle between their planes, it appears that the pole P_1 traces out both the polhode and the herpolhode with a constant scalar velocity. Poinsot's geometric treatment thus furnishes a satisfactory description of the motion both in the general and in this special case. The analytic treatment of the special case in which two of the principal moments of inertia for O are equal is comparatively simple and is left as an exercise for the reader.

The motion of the body for the above special case in which two of the principal moments of inertia for the fixed point are equal is often described by saying that the body is rotating with a constant angular velocity about the axis of figure of Poinsot's ellipsoid at the same time that this axis rotates with a constant angular velocity about a fixed axis. This breaking up of the motion into two aspects seems rather artificial but it has special significance when these two rotations are at decidedly different rates. Thus in the case of the motion of the earth relative to its centroid we have the rotation of the earth about its axis taking place in the sidereal day of 23 hours 56 minutes while this axis rotates about a fixed axis in about 25800 years. The angle between these two axes is about $23\frac{1}{2}°$ and the herpolhode circle is over nine million times as large in diameter as the polhode circle.

EXERCISES

1. A uniform thin rod has its midpoint O fixed and is acted upon by forces of zero moment with respect to O. Discuss the motion analytically and by Poinsot's construction showing that the rod describes a plane with constant angular velocity.

2. A uniform right circular cone of radius a and altitude $2a$ has its centroid O fixed and is acted upon by forces whose moment with respect to O is zero. Discuss the motion showing that the vertex traces out a circle.

3. A circular disc rotates about its center as a fixed point and is given an initial angular velocity ω_0 about an axis making an angle α with its plane. If acted upon by no forces, how long will it take the normal to the disc at its center to complete a rotation in space and how much will the disc have rotated in this time?

$$Ans. \quad t = \frac{2\pi}{\omega_0\sqrt{1 + 3\sin^2\alpha}}, \qquad \varphi = \frac{2\pi}{\omega_0}\sqrt{\frac{1 - \omega_0^2}{1 + 3\sin^2\alpha}}$$

4. Complete the analytic discussion in the text for the case $\mathbf{L}^2 = 2I_2T$, showing that for this case $a = c = 2T/I_2$ and that equation (9) integrates into,

$$\sqrt{b - u} = \pm\, \mu \tanh \mu t,$$

where,

$$\mu = \sqrt{b - a} = \sqrt{2T\frac{(I_1 - I_2)(I_2 - I_3)}{I_1I_2I_3}}$$

and where t is counted from the instant at which $\omega_j = 0$. Hence show that,

$$\begin{aligned} \omega_i &= \pm \sqrt{a}\sqrt{\frac{I_2(I_2 - I_3)}{I_1(I_1 - I_3)}}\operatorname{sech} \mu t, \\ \omega_j &= \pm \sqrt{a}\tanh \mu t, \\ \omega_k &= \pm \sqrt{a}\sqrt{\frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}}\operatorname{sech} \mu t. \end{aligned}$$

From this prove that the mean principal axis of inertia for O tends to become parallel to the constant vector $\mathbf{L}$ and that the body tends to rotate about it with a constant angular velocity $\sqrt{a}$.

5. Discuss analytically the motion of a body with one fixed point O and acted upon by forces of zero moment about O for the case $I_1 = I_2$. Show that Euler's equations reduce to,

$$\omega_i' = k\omega_j\omega_k, \qquad \omega_j' = -\,k\omega_k\omega_i, \qquad \omega_k' = 0$$

where,

$$k = \frac{I_2 - I_3}{I_1} = \frac{I_1 - I_3}{I_2}$$

and that these equations yield the integrals,

$$\omega_i = a\sin \mu t, \qquad \omega_j = a\cos \mu t, \qquad \omega_k = \mu/k, \quad (a, \mu \text{ const.})$$

if time be counted from the instant at which $\omega_i = 0$. Hence show that if the fixed axes be chosen as in the text,

$$\psi' = L/I_1, \qquad \varphi' = \mu, \qquad \theta' = 0$$

and the motion is as described at the end of this article.

6. If the forces acting on a body have a zero moment with respect to its centroid, show that,

$$(\omega_i\omega_j\omega_k)(\omega_i\omega_i' + \omega_j\omega_j' + \omega_k\omega_k') + \omega_i'\omega_j'\omega_k' = 0,$$

where ω_i, ω_j, ω_k are the coördinates of the angular velocity relative to the principal axes of inertia of the body for its centroid.

7. If the forces acting on a body have a zero moment with respect to its centroid and if the body has a constant scalar angular velocity, show that either the body is rotating about one of its principal axes of inertia for the centroid as a permanent axis of rotation, or two of the principal moments of inertia for the centroid are equal and the instantaneous axis makes a constant angle with the remaining principal centroidal axis.

8. If a body with one fixed point O is initially at rest and is acted upon by applied forces whose moment $\mathbf{M}$ with respect to O has a constant direction, show that the moment of momentum $\mathbf{L}$ also has this constant direction and that the ratio $T/\mathbf{L}^2$ is constant. Hence show that Poinsot's ellipsoid rolls without slipping on a fixed tangent plane as in the case when $\mathbf{M} = 0$.

9. Show that under the conditions of this article the portion of the normal to Poinsot's ellipsoid at the pole lying between the pole and any one of the principal planes of the ellipsoid is a constant.

10. Show that under the conditions of this article the vector $\boldsymbol{\omega}'$ is parallel to the line of intersection of the fixed tangent plane to Poinsot's ellipsoid and the diametral plane of the constant moment of momentum.

11. Show that under the conditions of this article the scalar angular velocity of the body about the constant moment of momentum is a constant $2T/L$.

12. A body with a fixed point O is acted upon by forces whose moment with respect to O is zero and a plane is drawn through O perpendicular to the constant moment of momentum $\mathbf{L}$. Show that Poinsot's ellipsoid for O intersects this plane in an ellipse of constant area.

13. A body with a fixed point O is acted upon by forces whose moment with respect to O is zero and a line is drawn through O parallel to the constant moment of momentum $\mathbf{L}$. Show that the sum of the squares of the distances of the extremities of the principal axes of Poinsot's ellipsoid for O from this line is a constant. (Poinsot.)

14. If a body with a fixed point O is acted upon by forces whose moment with respect to O is zero, show that the angular velocity Ω of the instantaneous axis satisfies the equation,

$$\boldsymbol{\omega}^2\,\Omega^2 = \frac{4T}{I_1I_2I_3}\{(I_1 + I_2 + I_3)T - \mathbf{L}^2\} - \frac{abc}{\boldsymbol{\omega}^2}\cdot$$

87. Motion of the Top.

A rigid body with one fixed point O which has two of its principal moments of inertia for O equal and the centroid of the body on the remaining principal axis of inertia for O, is called a *symmetric top*. If the only applied force acting on such a body is the force of gravity the body is called a *heavy symmetric top*, or more briefly a top. In 1852 Foucault showed that the rotation of the earth relative to a Newtonian reference frame can be experimentally demonstrated by the motion of a heavy symmetric top in which the centroid is at a point O fixed relative to the earth and he therefore called such a top a *gyroscope*; but at present the term gyroscope is generally used for any heavy symmetric top. The principal axis of inertia for O which passes through the centroid is called the *axis* of the top. The most important case of the motion of a top is that in which it has initially a large angular velocity about its axis.

To determine the motion of the heavy symmetric top we may conveniently employ equations § 85, (2) and § 85, (3) in which the unknown constraint $\mathbf{k}$ does not enter. The only applied force being the force of gravity is equivalent to a single force of amount m_0g acting downward at the centroid P_0. If we represent by $\mathbf{u}$ a unit upward vector we thus have for the sum $\mathbf{S}$ and the moment $\mathbf{M}$ about O of the applied forces,

$$\mathbf{S} = -\, m_0g\,\mathbf{u}, \qquad \mathbf{M} = m_0g\,\mathbf{u}\times\mathbf{p}_0 \qquad (\mathbf{p}_0 = OP_0),$$

where m_0 is the mass of the body and g is the constant of gravity. Equation § 85, (2), $\mathbf{L}' = \mathbf{M}$ then becomes,

$$\text{(1)} \qquad \mathbf{L}' = m_0g\,\mathbf{u}\times\mathbf{p}_0,$$

while equation § 85, (3′), $T' = W'$ becomes,

$$\text{(2)} \qquad (\boldsymbol{\omega}\cdot\mathbf{L})' = 2\boldsymbol{\omega}\cdot\mathbf{M} = 2m_0g\,\boldsymbol{\omega}\cdot\mathbf{u}\times\mathbf{p}_0.$$

From equation (1) we have at once $\mathbf{u}\cdot\mathbf{L}' = 0$ and consequently a first integral of equation (1) is,

$$\text{(3)} \qquad \mathbf{u}\cdot\mathbf{L} = a \qquad (a \text{ const.}).$$

If we now, as in § 85, adopt a fixed coördinate system with origin at O and Z-axis upward and a moving coördinate system with axes along the principal axes of inertia for O, then taking $I_2 = I_1$ and taking the centroid P_0 on the moving Z-axis we

have the following coördinates in the moving system,

$$\begin{aligned} \mathbf{L} &\equiv (I_1\omega_i,\ I_1\omega_j,\ I_3\omega_k), \\ \mathbf{u} &\equiv (\sin\varphi \sin\theta,\ \cos\varphi \sin\theta,\ \cos\theta), \\ \mathbf{p}_0 &\equiv (0,\ 0,\ p_0), \end{aligned}$$

where ψ, φ, θ are Euler's angles giving the position of the moving system relative to the fixed. Expanded in these coördinates equation (2) becomes with the aid of equations § 85, (9)

$$(4)\quad \begin{aligned} (I_1\omega_i^2 + I_1\omega_j^2 + I_3\omega_k^2)' &= 2m_0gp_0(\cos\varphi \sin\theta\, \omega_i - \sin\varphi \sin\theta\, \omega_j) \\ &= A \sin\theta\, \theta', \end{aligned}$$

in which A is the positive constant $2m_0gp_0$. Similarly equation (3) becomes,

$$(5)\quad I_1\omega_i \sin\varphi \sin\theta + I_1\omega_j \cos\varphi \sin\theta + I_3\omega_k \cos\theta = a.$$

Since $\sin\theta\, \theta' = (-\cos\theta)'$ we have from equation (4) a first integral of equation (2),

$$(6)\quad I_1\omega_i^2 + I_1\omega_j^2 + I_3\omega_k^2 + A\cos\theta = b \qquad (b \text{ const.}).$$

Another integral follows from the observation that measuring the members of equation (1) on the moving Z-axis gives $I_3\omega_k' = 0$ and hence,

$$(7)\quad I_3\omega_k = c \qquad (c \text{ const.}).$$

If in our three integrals (5), (6), (7) we substitute for ω_i, ω_j, ω_k their values taken from equations § 85, (9), we find respectively,

$$(8)\quad \begin{aligned} &I_1 \sin^2\theta\, \psi' + c\cos\theta = a, \\ &I_1(\sin^2\theta\, \psi'^2 + \theta'^2) + A\cos\theta = b - c^2/I_3, \\ &I_3(\cos\theta\, \psi' + \varphi') = c. \end{aligned}$$

If we introduce $u = \cos\theta$ as a new variable we then have from these ψ, φ, θ in terms of u by the equations,

$$(9)\quad \psi' = \frac{a - cu}{I_1(1 - u^2)}, \qquad \varphi' = \frac{c}{I_3} - \frac{au - cu^2}{I_1(1 - u^2)}, \qquad \cos\theta = u.$$

Thus it only remains to express u in terms of the time to complete the analytic determination of the motion. Eliminating ψ' and φ' from equations (8), we find,

$$(10)\quad u'^2 = \left(\frac{bI_3 - c^2}{I_3I_1} - \frac{A}{I_1}u\right)(1 - u^2) - \left(\frac{a - cu}{I_1}\right)^2 = f(u),$$

from which u may be obtained in terms of t by an integration.

An examination of the above cubic polynomial $f(u)$ shows that,

$$f(-1) < 0, \qquad f(u_0) \geqq 0, \qquad f(1) < 0, \qquad f(\infty) > 0,$$

where u_0 is the value of $u = \cos\theta$ for $t = 0$ and for which u' must consequently be real. It thus appears that $f(u)$ has in general three real distinct zeros u_1, u_2, u_3 in the intervals,

$$-1 < u_1 < u_0, \qquad u_0 < u_2 < +1, \qquad +1 < u_3 < +\infty$$

and equation (10) yields,

$$t = \pm\sqrt{\frac{I_1}{A}}\int_{u_0}^{u}\frac{du}{\sqrt{(u-u_1)(u_2-u)(u_3-u)}}, \tag{11}$$

This gives t as an elliptic integral in u which by a proper substitution may be expressed in terms of Legendre's elliptic integral of the first kind.

The analytic determination of the motion of the heavy symmetric top is thus complete, and with a knowledge of the properties of the elliptic integral in equation (11) a satisfactory picture of the motion may be obtained. Even without this knowledge several facts are readily apparent. Thus since $f(u)$ can not be negative it follows that u being initially between u_1 and u_2 must remain there, oscillating between these values. Consequently θ oscillates between the corresponding values θ_1 and θ_2. This motion of the axis of the top is called *nutation.* (Latin, *nuto,* to nod.) Referring to equations (9) we see that if $a^2 > c^2$ then ψ' can not vanish and ψ must continually increase or continually decrease. The motion of the top corresponding to a variation in ψ is known as *precession* and so we say in the above case that the *precession* is constantly in one sense. Even if $c^2 > a^2$, still ψ' can never change sign unless the value $u = a/c$ renders $f(u)$ positive and since,

$$f(a/c) = \left(\frac{bI_3 - c^2}{I_3 I_1} - \frac{aA}{cI_1}\right)(1 - a^2/c^2),$$

it follows that for this to occur we must have,

$$\frac{bI_3 - c^2}{I_3} > \frac{aA}{c}, \qquad c^2 > a^2.$$

If both these conditions are satisfied then ψ' may vanish and the precession is first in one sense and then in the other throughout the motion.

A case of particular interest is that in which the top is initially set to rotating about its axis which is inclined from the vertical. In this case we have initially $\omega_i = 0$, $\omega_j = 0$ and by equations (5) and (6) we have,

$$(12) \qquad a = cu_0, \qquad bI_3 - c^2 = AI_3u_0,$$

so that $f(u)$ now takes the form,

$$(13) \qquad f(u) = (u_0 - u)\left\{\frac{A}{I_1}(1 - u^2) - \frac{c^2}{I_1^2}(u_0 - u)\right\}.$$

Thus u' vanishes for $u = u_0$ and will be real only for $u \leqq u_0$, i.e. for $\theta \geqq \theta_0$. Thus in this case the top when released immediately tips away from the upward vertical, the initial value of θ being its minimum value. The first of equations (9) now takes the form,

$$(14) \qquad \psi' = \frac{c(u_0 - u)}{I_1(1 - u^2)},$$

showing that ψ' vanishes for $u = u_0$ but otherwise has the sign of c and ω_k. Thus as often as θ returns to its initial value θ_0 the motion of procession momentarily ceases but otherwise continues always in the same sense about the upward vertical as that of the rotation of the body about that half of its axis on which the centroid lies.

If the body is initially given a *very rapid* rotation about its axis which is inclined from the vertical the results of the last paragraph hold and we may reach some further conclusions. Since u_0 is the larger of those two zeros of $f(u)$ which are actually attained by u it is in fact the zero we previously called u_2. The other zero u_1 must by equation (13) satisfy the relation,

$$(15) \qquad u_0 - u_1 = \frac{AI_1(1 - u_1^2)}{c^2} < \frac{AI_1}{c^2}$$

Since c is now very large we see that $u_0 - u_1$, which is the total variation of u, is small of the order $1/c^2$ and consequently θ remains nearly constant and the nutation of the top is very slight. Furthermore we have $u_0 - u \leqq u_0 - u_1$ and hence from

equations (14) and (15),

$$(16) \qquad |\psi'| \quad \frac{A}{|c|}\frac{1-u_1^2}{1-u^2}.$$

Now initially the axis of the top is inclined to the vertical and so we have $1 - u_0^2 = k$ where k is a positive constant. For sufficiently large values of c^2 we will by inequality (15) have u_1 so close to u_0 that we may write, $(1 - \mu)k < 1 - u_1^2 < (1 + \mu)k$ where μ is an arbitrarily small positive constant. That is, we have,

$$(17) \qquad 1 - \mu < \frac{1-u_1^2}{1-u_0^2} < 1 + \mu.$$

At the same time either u_0 is zero or a sufficiently large value of c^2 will insure that u_1 have the same sign as u_0. In either case since $u_1 \leqq u \leqq u_0$ it follows that u^2 will lie between u_0^2 and u_1^2 and therefore $(1 - u_1^2)/(1 - u^2)$ lies between $(1 - u_1^2)/(1 - u_0^2)$ and 1. By inequality (17) it follows that,

$$(18) \qquad 1 - \mu < \frac{1-u_1^2}{1-u^2} < 1 + \mu.$$

Hence the last factor of the second member of inequality (16) is nearly unity for large values of $|c|$ and consequently $|\psi'|$ is small of the order $|1/c|$. The motion of precession is thus very slow for this case and the axis of the top rotates about the vertical with an angular velocity which, although not in general constant, is always very small.

EXERCISES

1. Can the axis of the heavy symmetric top pass through the vertical position? If so, what must the initial conditions of the motion be?
2. If we lay off from the fixed point O of the heavy symmetric top along the axis of the top a distance $l = r_0^2/p_0$ in the sense OP_0, we locate a point Q known as the *center of oscillation* of the top. Here $p_0 = OP_0$, and r_0 is the radius of gyration of the top about any line through O perpendicular to the axis. Show that the scalar velocity of the center of oscillation is that which it would have acquired in falling from rest from the fixed horizontal plane at an elevation above O of,
$$h = \frac{l}{A}\left(b - \frac{c^2}{I_3}\right),$$
these constants having the same meaning as in the text.

3. If the heavy symmetric top be started so that initially,

$$\theta = 60°, \qquad \theta' = 0, \qquad \psi' = \sqrt{\frac{2A}{3I_1}}, \qquad \varphi' = \frac{3I_1 - I_3}{2I_3}\sqrt{\frac{2A}{3I_1}};$$

show that,

$$a = c = \sqrt{3AI_1} \qquad b = A\left(1 - \frac{3I_1}{2I_3}\right)$$

and that,

$$u'^2 = \frac{\mathcal{A}}{I_1}(1 - u)^2(u - \tfrac{1}{2}).$$

Hence show that,

$$\sec\theta = 1 + \operatorname{sech}\sqrt{\frac{\mathcal{A}}{2I_1}}\, t.$$

4. Show that the motion of a heavy symmetric top may always be duplicated by the motion of a heavy spherical top, ($I_1 = I_2 = I_3$), except for the rate of rotation φ' of the top about its axis.

88. Motion of a Body with a Fixed Axis.

From the practical point of view the most important case of the motion of a rigid body is that in which the body has a fixed axis. The crank shaft of the ordinary gasoline engine is an example of the practical use of such a motion. We may let the position of the fixed axis be determined by a point O upon it and a unit vector **u** along it. The fixing of the axis may be thought of as brought about by the action of a force **k** at the point O and a force **l** at another point A of the axis. Since these points remain at rest the power of these constraints is constantly zero.

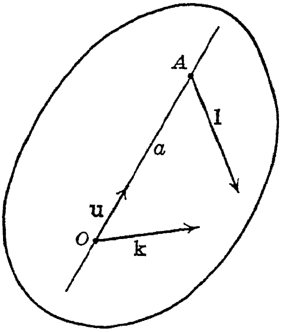

Fig. 83

Equations (1), (2), (3) of § 84 become in this case,

(1) $\mathbf{R}' = \mathbf{S} + \mathbf{k} + \mathbf{l}$,

(2) $\mathbf{L}' = \mathbf{M} + \mathbf{a} \times \mathbf{l} \qquad (\mathbf{a} = OA)$,

(3) $T' = W'$,

where **S**, **M**, W' are the sum and the moment about O and the power of the applied forces. The moment of **k** which should otherwise appear in equation (2) is of course zero as well as the powers of both **k**

and $\mathbf{l}$ which should appear in equation (3). By means of the relations (9), (10), (11), (12) of § 84 these equations take the form,

$$(1') \qquad m_0(\boldsymbol{\omega} \times \mathbf{p}_0)' = \mathbf{S} + \mathbf{k} + \mathbf{l},$$

$$(2') \qquad \{\varphi(\boldsymbol{\omega})\}' = \mathbf{M} + \mathbf{a} \times \mathbf{l},$$

$$(3') \qquad (\tfrac{1}{2}\, \boldsymbol{\omega} \cdot \mathbf{L})' = \{\tfrac{1}{2}\, \omega^2 I(\boldsymbol{\omega})\}' = \boldsymbol{\omega} \cdot \mathbf{M},$$

where $\boldsymbol{\omega}$ is the vector angular velocity and $I(\boldsymbol{\omega})$ is the moment of inertia of the body about the fixed axis.

The motion of the body will be completely determined if we express in terms of the time the angle θ which a plane containing the axis and fixed in the body makes with a plane containing the axis and fixed in space. Since in this case the instantaneous axis lies constantly along the fixed axis we have,

$$\boldsymbol{\omega} = \omega\, \mathbf{u} = \theta'\, \mathbf{u}, \qquad \boldsymbol{\omega}' = \theta''\, \mathbf{u},$$

where $\omega = \theta'$ is the scalar angular velocity counted as positive or negative according as $\boldsymbol{\omega}$ and $\mathbf{u}$ have like or unlike senses. Also in this case $I(\boldsymbol{\omega})$, being the moment of inertia of the body about the fixed axis, is a constant I. Thus equation (3′) reduces to $I\, \boldsymbol{\omega} \cdot \boldsymbol{\omega}' = \boldsymbol{\omega} \cdot \mathbf{M}$ or,

$$(3'') \qquad I\theta'' = \mathbf{u} \cdot \mathbf{M},$$

which may be expressed by the theorem,

For a rigid body with a fixed axis the product of the angular acceleration and the moment of inertia about the axis equals the moment of the applied forces with respect to the axis.

Since equation (3″) is free from the unknown constraints $\mathbf{k}$ and $\mathbf{l}$ it may conveniently be used to determine the angle θ and thus completely determine the motion of the body. Equation (3″) is highly analogous to the familiar equation $ms'' = f$ for the rectilinear motion of a particle and enables us to say in a general way that the moment of inertia plays the same role in rotation about a fixed axis that the mass does in the rectilinear motion of a particle.

The motion having been determined by equation (3″), we may employ equations (1′) and (2′) to evaluate the constraints $\mathbf{k}$ and $\mathbf{l}$. Since the centroid P_0 is fixed in the body we may apply Poisson's formula $\mathbf{p}' = \boldsymbol{\omega} \times \mathbf{p}$ (§ 45) to $\mathbf{p}_0$ as well as to the vectors

$\mathbf{p}_i$ and thus obtain,

$$(4) \qquad m_0(\boldsymbol{\omega}\times\mathbf{p}_0)' = m_0\,\boldsymbol{\omega}'\times\mathbf{p}_0 + m_0\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{p}_0)$$
$$= m_0\theta''\,\mathbf{u}\times\mathbf{p}_0 + m_0\theta'^2\,\mathbf{u}\times(\mathbf{u}\times\mathbf{p}_0).$$

Since $\varphi(\boldsymbol{\omega}) = \Sigma\, m_i\,\mathbf{p}_i\times(\boldsymbol{\omega}\times\mathbf{p}_i)$ we likewise have,

$$(5) \quad \{\varphi(\boldsymbol{\omega})\}' = \Sigma\, m_i\,\mathbf{p}_i'\times(\boldsymbol{\omega}\times\mathbf{p}_i) + \Sigma\, m_i\,\mathbf{p}_i\times(\boldsymbol{\omega}'\times\mathbf{p}_i)$$
$$+ \Sigma\, m_i\,\mathbf{p}_i\times(\boldsymbol{\omega}\times\mathbf{p}_i')$$
$$= \omega'\Sigma\, m_i\,\mathbf{p}_i\times(\mathbf{u}\times\mathbf{p}_i) + \omega^2\Sigma\, m_i(\mathbf{u}\cdot\mathbf{p}_i)(\mathbf{p}_i\times\mathbf{u})$$
$$= \theta''\varphi(\mathbf{u}) + \theta'^2\,\mathbf{u}\times\varphi(\mathbf{u}).$$

Equations (1′) and (2′) thus take the form,

$$(1'') \qquad m_0\theta''\,\mathbf{u}\times\mathbf{p}_0 + m_0\theta'^2\,\mathbf{u}\times(\mathbf{u}\times\mathbf{p}_0) = \mathbf{S} + \mathbf{k} + \mathbf{l},$$

$$(2'') \qquad \theta''\varphi(\mathbf{u}) + \theta'^2\,\mathbf{u}\times\varphi(\mathbf{u}) = \mathbf{M} + a\,\mathbf{u}\times\mathbf{l},$$

where a is the measure of $\mathbf{a}$ on $\mathbf{u}$. From equations (2″) we may determine $\mathbf{l}$ by a vector division (§ 20) except for an arbitrary scalar multiple of $\mathbf{u}$ which may be added to it, and then $\mathbf{k}$ is given by equation (1″). It is impossible to determine $\mathbf{k}$ and $\mathbf{l}$ exactly for if we have any values of $\mathbf{k}$ and $\mathbf{l}$ satisfying equations (1″) and (2″) it is always possible to add to $\mathbf{k}$ and simultaneously subtract from $\mathbf{l}$ any scalar multiple of $\mathbf{u}$ and still satisfy the equations.

A case of particular interest is that in which the applied forces have a zero moment with respect to O, that is $\mathbf{M} = 0$. In this case by equation (3″) we have $\theta'' = 0$ and thus the angular velocity θ' of the body remains constant. Equation (2″) takes the form,

$$(6) \qquad \theta'^2\,\mathbf{u}\times\varphi(\mathbf{u}) = a\,\mathbf{u}\times\mathbf{l}.$$

Unless the body is at rest $\theta'^2 \neq 0$ and so a necessary and sufficient condition that the constraint $\mathbf{l}$ be zero is that $\mathbf{u}\times\varphi(\mathbf{u})$ be zero, which is by definition the condition that the fixed axis be a principal axis of inertia for O. It thus appears that under the condition $\mathbf{M} = 0$ the constraint $\mathbf{l}$ at A could be entirely removed without affecting the motion if and only if the body were at rest, $\theta' = 0$, or rotating about a principal axis of inertia for O, $\mathbf{u}\times\varphi(\mathbf{u}) = 0$. We thus have the theorem,

If a rigid body with one fixed point O is acted upon by applied forces whose moment with respect to O is constantly zero, then a

necessary and sufficient condition that the body when set to rotating about an axis through O should continue to rotate about the same axis is that this axis should be a principal axis of inertia of the body for O.

For this reason the principal axes of inertia of the body for each point O are sometimes called the *permanent axes of rotation* for O.

If we assume that the applied forces acting on the body form a null system we may raise the question as to the conditions under which the constraints $\mathbf{k}$ and $\mathbf{l}$ may *both* be removed without affecting the motion. Since $\mathbf{S}$ and $\mathbf{M}$ are now both zero, equation (1″) takes the form,

$$m_0\theta'^2\,\mathbf{u}\times(\mathbf{u}\times\mathbf{p}_0) = \mathbf{k}+\mathbf{l}. \tag{7}$$

This together with equation (6) shows that a necessary and sufficient condition that $\mathbf{k}$ and $\mathbf{l}$ both vanish while the body is rotating is that $\mathbf{u}\times\varphi(\mathbf{u}) = \mathbf{u}\times\mathbf{p}_0 = 0$, which amounts to saying that the axis of rotation is a principal axis of inertia for the centroid. Since the constraints vanish under these conditions the body becomes in effect free and we have the theorem,

If a free rigid body is acted upon by forces constituting a null system, a necessary and sufficient condition that the body continue to rotate about any axis is that the axis pass through the centroid and be a principal axis of inertia for its points.

For this reason the principal axes of inertia for the centroid are sometimes called the *spontaneous axes of rotation* of the body.

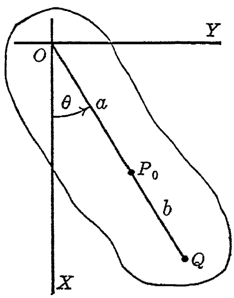

FIG. 84

As an application of the above theory we may discuss the *compound pendulum.* A rigid body with a fixed horizontal axis and acted upon only by the force of gravity is called a compound pendulum. To discuss the motion let us take a fixed coördinate system with the Z-axis on the fixed axis of the body and the XY-plane as the vertical plane through the centroid P_0 of the body and perpendicular to the fixed axis. Let us represent by a the length of the vector OP_0 and by θ the angle which OP_0 makes with the X-axis. The applied forces are equivalent to a single force $\mathbf{S}$ of amount m_0g directed downward

and acting at the centroid P_0. With the X-axis directed downward we then have the coördinates,

$$\mathbf{u} \equiv (0, 0, 1), \qquad \mathbf{S} \equiv (m_0 g, 0, 0), \qquad \mathbf{p}_0 \equiv (a \cos \theta, a \sin \theta, 0).$$

Consequently $\mathbf{u} \cdot \mathbf{M} = \mathbf{u} \cdot \mathbf{p}_0 \times \mathbf{S} = - agm_0 \sin \theta$ and equation (3″) becomes,

$$(8) \qquad r_0^2 \theta'' + ag \sin \theta = 0,$$

r_0 being the radius of gyration of the body about the fixed axis. On comparing this equation with equation § 39, (3) $l\theta'' + g \sin \theta = 0$ for the motion of the simple pendulum of length l, we see that the motion of the compound pendulum is the same as that of the simple pendulum of length $l = r_0^2/a$. If we lay off from O in the direction of P_0 this distance l, we locate a point Q known as the *center of oscillation* of the pendulum. The line through Q parallel to the fixed axis is called the *axis of oscillation* and the particles on this axis move about the fixed axis as if each were a simple pendulum. If we let ρ be the radius of gyration of the body about a centroidal line parallel to the fixed axis we have by Theorem 7 of § 82, $m_0 r_0^2 = m_0 \rho^2 + m_0 a^2$ and consequently,

$$l = \frac{r_0^2}{a} = \frac{\rho^2 + a^2}{a} = a + \frac{\rho^2}{a}.$$

If now we call $\rho^2/a = b$ it appears that we likewise have,

$$l = b + \frac{\rho^2}{b}$$

and thus l bears the same relation to b that it does to a. In other words, if the rigid body were set to rotating about the axis of oscillation with the original initial conditions the entire circumstances of the motion would be unaltered. The fixed axis and the axis of oscillation may be interchanged without affecting the motion. In particular if the body oscillates through a very small angle about the fixed axis the period of oscillation is given by § 39, Problem 7 as approximately,

$$T = 2\pi \sqrt{\frac{l}{g}} = 2\pi \sqrt{\frac{a^2 + \rho^2}{ag}} = 2\pi \sqrt{\frac{b^2 + \rho^2}{bg}}.$$

The fact that the fixed axis and the axis of oscillation are

interchangeable without affecting the motion and that the distance between them is the length of the synchronous simple pendulum are among the results obtained by Huygens (The Hague, 1629–1695) in a famous paper on the subject.

EXERCISES

1. A uniform rod slides with its ends on a smooth vertical circle. If the rod subtends an angle of 120° at the center of the circle, show that the simple pendulum with the same motion has a length equal to the radius of the circle.

2. The period of a clock pendulum is controlled by a rider adjustable up and down the axis of the pendulum. Show that when the rider is so placed that the period is as small as possible the distance from the fixed axis to the centroid of the rider is half the length of the simple pendulum synchronous with the clock pendulum with the rider in that position.

3. The hinges of an automobile door are toward the front of the door and the door stands open at right angles. If the automobile starts up with a constant acceleration f and the door has a radius of gyration r_0 about its axis of rotation and its centroid is at a distance a from the axis, show that the door will close in the time,

$$t = \frac{r_0}{\sqrt{2af}} \int^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \frac{r_0}{\sqrt{af}} 1.8541 \cdots.$$

4. A pendulum consists of a uniform rod of length a and mass m and a circular disc of radius r and mass M with its center attached at the lower end of the rod and free to rotate without friction in the plane in which the pendulum swings which is also the plane of the disc. Find the ratio of the length of the equivalent simple pendulum to what it would be if the disc were rigidly attached to the rod.

   Ans. $\dfrac{2ma^2 + 6Ma^2}{2ma^2 + 6Ma^2 + 3Mr^2}$

5. A circular disc rests flat on a smooth horizontal table. How may a force be applied to it so that it will start to rotate about a point on the edge of the disc?

   Ans. The line of action of the force should be horizontal and divide the diameter perpendicular to it in the ratio 3 : 1

6. A uniform triangular area is at rest and has a force applied to it perpendicular to its plane at the midpoint of one side. Show that it will start to rotate about a line passing through the midpoints of the other two sides. If the force is applied perpendicular to its

plane and at one vertex, show that it will start to rotate about a line dividing the two sides meeting at that vertex in the ratio 3 : 1.

7. About what axis in its plane will a uniform elliptic lamina oscillate as a pendulum with the shortest possible period?

Ans. Parallel to the major axis and bisecting one semiminor axis

8. A metronome used to mark time in music consists of a compound pendulum driven by a clockwork mechanism. The pendulum has a short heavy portion below the axis and a long light rod above the axis on which slides a small rider to control the period. If the distance of the rider above the axis is r, show that the period T for small oscillations is related to r by the formula,

$$gT^2 = \frac{A + r^2}{B - r}$$

in which g is the constant of gravity and A and B are positive constants of the instrument.

9. A lamina of given area A is to be cut from a sheet of given homogeneous material and set oscillating in a vertical plane about a horizontal axis perpendicular to it. What should be the form of the lamina and where should the axis be placed in order that the period be as small as possible?

Ans. A circular disc with the axis passing through the midpoint of a side of the inscribed square

10. A homogeneous sphere of radius a and mass m oscillates under the action of gravity about a horizontal tangent as axis. Show that the constraint of the axis will be horizontal when the sphere is in its lowest position provided its angular velocity ω is then $\sqrt{10g/7a}$.

11. Consider a compound pendulum in which the fixed axis is a principal axis of inertia for the foot O of the perpendicular let fall from the centroid P_0. Show that the only constraint needed to fix the axis is a force $\mathbf{k}$ applied at O with a measure along OP_0,

$$-\frac{m_0 g}{r_0^2}\{2a^2c + (3a^2 + \rho^2)\cos\theta\}$$

and a measure perpendicular to OP_0 in the plane of the motion,

$$-\frac{m_0 g}{r_0^2}\rho^2 \sin\theta,$$

where c is the constant $c = (r_0^2\theta'^2/2ag) - \cos\theta$ and the other letters have the meanings employed in the text.

12. Two uniform rods of the same length and mass are connected end to end by two light cords of equal length. One of the rods is free to rotate about its midpoint and the whole system swings in a

vertical plane. Discuss the motion under gravity, showing that as long as the cords are taut the upper rod rotates with a constant angular velocity and the midpoint of the lower rod has a simple pendulum motion.

13. A rigid body of mass m rotates without friction about a fixed vertical axis and its moment of inertia about this axis is mr^2. A particle of the same mass m slides without friction in a straight slender tube cut from the body, the tube intersecting the fixed axis at a point O. Discuss the motion of the system under the action of gravity and in particular the case in which the particle is initially at rest relative to the body and determine the conditions under which it will remain so.

CHAPTER XII

GENERAL PRINCIPLES OF MECHANICS

89. Introduction.

We observed in Chapter I that Theoretical Mechanics, like other branches of mathematics, proceeds in general by defining each new term introduced in terms of previously defined terms and by proving each new theorem on the basis of these definitions and of previously proven theorems. But it is necessary at the beginning to leave certain terms undefined and certain propositions unproven. These unproven propositions we call *postulates.* Frequently the postulates concerning an undefined term will to a certain extent serve to take the place of a definition of it. Theoretical Mechanics, like its simplest branch, Geometry, and indeed most other branches of mathematics, is susceptible of an interpretation in the physical world. This physical interpretation furnishes us with a valuable guide in choosing the postulates on which to base our study and after the study has progressed a ways we may test the desirability of the postulates chosen by seeing if our conclusions conform to the physical facts.

In the study of Theoretical Mechanics thus far here carried out the principal postulates assumed in addition to the usual ones of geometry and analysis have been Newton's three laws of motion. These, together with certain minor auxiliary hypotheses, have been found sufficient to develop the subject in such a way that the conclusions conform remarkably well with the facts of the physical interpretation, although, to be sure, the conformity can be shown to be somewhat imperfect. In the course of this study we occasionally encounter propositions of a noteworthy generality. If these propositions can readily be formulated into sentences and if they appeal to us as true from the point of view of their physical interpretation it has become the custom to call them *principles.* In fact Newton's three laws of motion might properly be called principles of Theoretical Mechanics.

In the present chapter we shall discuss six such propositions, although two of them, Lagrange's Equations and Hamilton's

Canonical Equations, are not readily formulated into sentences and thus are known as *equations* rather than principles. The remaining four principles are all of such generality that each could be used instead of Newton's second law as one of the postulates on which the entire subject is founded.

In the study of the motion of one or more particles by the direct application of Newton's laws of motion it is necessary to take specifically into account every force acting on every particle. This turns out to be very inconvenient if the motions of the particles are subject to imposed conditions, for the forces introduced by these conditions are often very difficult to determine. From the time of the publication of Newton's laws, *Principia mathematica*, 1687 up to the time of the publication of d'Alembert's Principle, *Traité de Dynamique*, 1743, a vast number of mechanical problems was solved and in most cases some special ingenious device was employed to avoid or surmount the difficulty of the forces introduced by the imposed conditions, or the "lost forces" as they were called. Such problems constituted a sort of trial of strength of the mathematicians of the time. The publication of d'Alembert's Principle put an end to this period because it provided a perfectly general device for eliminating the lost forces in all such problems.

In stating his principle d'Alembert says in part, "Decompose the movements a, b, c etc. imparted to each body each one into two others $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., α, β, γ, etc., which are such that if one had imparted to the bodies only the movements $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc. they would have been able to conserve the movements without reciprocally hindering each other; and if one had imparted to them only the movements α, β, γ, etc., the system would have remained in repose; it is clear that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., will be the movements which these bodies will take in virtue of their action." Interpreting this in modern terms we should call the, "imparted movements" a, b, c, etc., the applied forces $\mathbf{f}_i$ while $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, etc., would be called the effective forces $m_i\,\mathbf{p}_i''$ and α, β, γ, etc., would be the negatives of the constraints $\mathbf{k}_i$. Thus in effect d'Alembert says,

$$\mathbf{f}_i = m_i\,\mathbf{p}_i'' - \mathbf{k}_i$$

and that the forces $-\mathbf{k}_i$ would at each instant produce equilibrium of the particles under the imposed conditions. Now the

condition for equilibrium of the forces $-\mathbf{k}_i$ under the imposed conditions is given by the Principle of Virtual Work, Chapter IX, as,

$$\Sigma - \mathbf{k}_i \cdot \delta\mathbf{p}_i = 0,$$

where the $\delta\mathbf{p}_i$ are virtual displacements of the particles P_i permitted by the imposed conditions. Substituting from the previous equation this becomes,

$$\Sigma\, (\mathbf{f}_i - \mathbf{m}_i\, \mathbf{p}_i'') \cdot \delta\mathbf{p}_i = 0$$

and this is *d'Alembert's Principle as Formulated by Lagrange.*

In 1788 Lagrange published his famous *Mécanique Analytique* in which d'Alembert's Principle as formulated above is made the single basic principle on which the subject is developed. The object of the work is to show that the entire subject is implicitly included in this single principle and to give general formulas from which any particular result can be obtained. A striking feature of the treatment is the introduction of generalized coördinates which are a set of parameters whose expression as functions of the time completely determines the motion of the particles. In this way Lagrange arrives at equations of motion in which the individual particles are completely lost sight of. These equations for the conservative holonomic motion of a set of particles which are now written, § 92, (27),

$$\left(\frac{\partial L}{\partial q_j'}\right)' - \frac{\partial L}{\partial q_j} = 0$$

might properly be called *Lagrange's Principle.* The method of generalized coördinates and the method of Lagrange's multipliers and of the calculus of variations all of which he here invented have become classic and of the widest application. The analysis employed is so elegant that Sir William Hamilton said that the work can only be described as a kind of scientific poem.

In 1835 Sir William Hamilton, following a suggestion due to Poisson, published in the *Philosophical Transactions of the Royal Society of London* a transformation of Lagrange's equations for the conservative holonomic motion of a set of n particles which changed these equations from n differential equations of the second order to $2n$ differential equations of the first order. As so transformed the equations possess such a striking symmetry that

they have become known as *Hamilton's Canonical Equations.* They constitute the basis of much of the advanced theory of dynamics as developed by Hamilton, Jacobi, Lie, Koenigs and others.

The principles now known as *Hamilton's Principle*, the *Principle of Least Action* and *Jacobi's Form of the Principle of Least Action* all arose historically from a principle first developed by Maupertuis. He wrote papers appearing in 1740 and 1744 in the *Mémoires de l'Académie Royale des Sciences de Paris* and in 1745 in the *Mémoires de l'Académie Royale de Berlin* in which the idea of the principle gradually took form. He made no attempt at a formal proof of the principle, contenting himself with showing that it leads to conclusions consistent with the physical facts in a few simple cases. Although Maupertuis' treatment is neither clear nor correct, still there can be no doubt that he deserves full credit for originating the idea. As he states it the principle is to the effect that there is a kind of economy in nature by virtue of which natural processes are conducted with the least possible "expenditure." Mechanically he interpreted this to mean that in the actual motion of a single particle the "action"

$$\int mv\, ds$$

has a smaller value than for any other motion between the same two end points. Here m is the mass of the particle, v its scalar velocity and ds the element of arc, the integral being taken along the path between the given end points.

The publication of the principle aroused a considerable controversy and in 1751 Euler published a defense of Maupertuis in the *Mémoires de l'Académie Royale de Berlin.* Euler later derived the principle on theological grounds and showed that it holds for central motion of a particle. In 1788 Lagrange in writing his *Mécanique Analytique* started to employ the Principle of Least Action as the basic principle of dynamics, but soon turned aside to d'Alembert's Principle. The Principle of Least Action thus occupies only a secondary place in Lagrange's great work and his treatment of it is not sufficiently explicit on certain points. In 1815 Rodrigues published in the *Correspondance sur l'École Polytechnique, T.3,* a brief but altogether clear and correct treat-

ment of the principle and derived the equations of motion of the particles from it. Jacobi in writing his *Vorlesungen über Dynamik* complained that he could not understand Lagrange's discussion of the principle and, apparently unaware of the work of Rodrigues, he set himself to formulate a new statement and proof. He succeeded in establishing a principle but it was not exactly the one sought. His result is known as *Jacobi's Form of the Principle of Least Action.* Sir William Hamilton also seems to have started out in his paper of 1834 in the *Philosophical Transactions of the Royal Society of London* to prove the Principle of Least Action but succeeded in establishing the associated but distinct principle now known as *Hamilton's Principle.* The final clarification of these principles may be largely credited to Hölder who discussed them in the *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen* in 1896. They are also established by an entirely independent direct analysis by Whittaker in his *Analytical Dynamics*, 1904.

In 1829 Gauss published in the *Journal für Reine und Angewandete Mathematik* an entirely new principle stating that at each instant a certain quantity called the constraint or Gaussian curvature is less for the actual motion than for any varied motion in which the particles have the actual positions and velocities at the instant considered. The proof given by Gauss, although intuitively quite convincing, does not satisfy modern requirements as to rigor. The proof is reproduced in Appell's *Mécanique Rationnelle.*

90. Virtual Displacements; Degrees of Freedom.

In the statement and derivation of the general principles of mechanics an important concept is that of the virtual displacements $\delta\mathbf{p}_i$ of a set of particles P_i when the motion of the set is subject to imposed conditions. We employed this concept in Chapter IX in the discussion of the Principle of Virtual Work where we were concerned only with conditions of equilibrium for the set of particles, and where consequently the time did not enter explicitly into the consideration. But in employing virtual displacements in the discussion of the motion of a set of particles it will be necessary to be specific as to the exact manner in which these displacements are limited by the conditions imposed upon the motion.

If a set of particles P_i is subject to certain imposed conditions, then at any given instant any set of arbitrarily small displacements $\delta\mathbf{p}_i$ of the particles permitted by the imposed conditions *as they exist at that instant* is said to be a set of *virtual displacements* of the particles for the instant.

The virtual displacements at any instant thus depend only on the conditions imposed at that instant and not on the manner in which these conditions may vary with the time. In this respect the virtual displacements for a given instant differ fundamentally from any actual displacement of the particles, for the latter will depend not only upon the conditions imposed at the moment but also upon the manner in which they vary. Thus in the case of a single particle constrained to remain on a moving surface the virtual displacements at any instant will be tangent at the particle to the position of the surface at the instant, whereas the actual displacement will be due not only to the motion of the particle on the surface but also to the motion of the surface itself and may result in a displacement nearly normal to the surface. If, however, the imposed conditions depend only on the positions of the particles of the set and do not otherwise vary with the time, then the actual displacements during an arbitrarily short interval of time will by the above definition constitute one set of virtual displacements for the instant at the beginning of this interval.

We often speak of displacements, both virtual and real, as being *arbitrarily small* or *infinitesimal.* This means that we are concerned only with the limiting directions of these displacements and the limiting values of the ratios of their lengths as they approach zero. Thus any two systems of virtual displacements are for our purposes identical if they have the same limiting directions and length ratios as they approach zero.

Any system of virtual displacements $\delta\mathbf{p}_i$ of a set of n particles P_i which is permitted by the imposed conditions at a given instant can, as a consequence of the above remark, be represented by the equations,

$$(1) \quad \delta\mathbf{p}_i = \mathbf{c}_{i1}\,\delta q_1 + \mathbf{c}_{i2}\,\delta q_2 + \cdots + \mathbf{c}_{im}\,\delta q_m, \qquad i = 1, 2, \cdots, n,$$

where the $\mathbf{c}_{ij}$ are mn vector constants and the δq_j are m scalar infinitesimals and where $m \leqq 3n$. For even if the n particles were under no imposed conditions we could for $m = 3n$ by a proper choice of the $\mathbf{c}_{ij}$ and the δq_j cause the $\delta\mathbf{p}_i$ to have arbitrarily

given limiting directions and length ratios. And if the motion of the particles is subject to conditions it will in general be possible to take $m < 3n$ and still so choose the $\mathbf{c}_{ij}$ and the δq_j as to give the $\delta \mathbf{p}_i$ all the limiting directions and length ratios permitted by these conditions. The least value of m for which this can be done at any instant is called the *number of degrees of freedom* of the set of particles at the instant.

If we consider the virtual displacements of the set, not merely at an individual instant, but during a certain interval of time, we may represent by m the maximum number of degrees of freedom of the set of particles during this interval. The virtual displacements can then be represented at each instant in the form (1) and the vectors $\mathbf{c}_{ij}$ can be thought of as varying from instant to instant and thus forming a set of mn vector functions γ_{ij} dependent on the positions of the particles and on the time. Thus we have throughout the interval considered,

$$(2) \qquad \delta \mathbf{p}_i = \gamma_{i1}\,\delta q_1 + \gamma_{i2}\,\delta q_2 + \cdots + \gamma_{im}\,\delta q_m, \qquad i = 1, 2, \cdots, n.$$

To apply the above definitions in the discussion of the motion of a set of particles we proceed as follows. The radius vector $\mathbf{p}_i$ of each particle P_i can always be regarded as a function of its three Cartesian coördinates or any other three suitably chosen scalar parameters. Hence we may always regard the n vectors $\mathbf{p}_i$ as n vector functions of $3n$ scalar parameters $q_1, q_2, \cdots, q_{3n}$,

$$(3) \qquad \mathbf{p}_i = \psi_i(q_1, q_2, \cdots, q_{3n})$$

and consequently we may write,

$$(4) \qquad \delta \mathbf{p}_i = \frac{\partial \psi_i}{\partial q_1}\,\delta q_1 + \frac{\partial \psi_i}{\partial q_2}\,\delta q_2 + \cdots + \frac{\partial \psi_i}{\partial q_{3n}}\,\delta q_{3n},$$

where the δq_j are the increments in the q_j corresponding to the virtual displacement $\delta \mathbf{p}_i$. The conditions imposed upon the motion of the set may be restrictions upon the *positions* of the particles P_i and thus take the form of equations in the q_j and t,

$$(5) \qquad \varphi_k(q_1, q_2, \cdots, q_{3n}, t) = 0, \qquad k = 1, 2, \cdots, r$$

or they may be conditions directly limiting the virtual displacements $\delta \mathbf{p}_i$ and thus take the form of homogeneous equations in the δq_j,

$$(6) \qquad \varphi_{k1}\,\delta q_1 + \varphi_{k2}\,\delta q_2 + \cdots + \varphi_{k3n}\,\delta q_{3n} = 0, \qquad k = 1, 2, \cdots, r,$$

where the φ_{kj} are functions of the q_j and t. Conditions which can not be expressed in either of these forms, e.g. conditions depending on the velocities or expressible only in the form of inequalities, will not be discussed here. Equations (5) are assumed to be independent in the q_j and equations (6) in the δq_j.

Evidently conditions given in the form (5) can always be written in the form (6), for the δq_j being the infinitesimal increments of the q_j corresponding to the virtual displacements $\delta \mathbf{p}_i$ subject to conditions (5), will be related by equations obtained by differentiating equations (5) holding the time fixed, since the virtual displacements correspond to infinitesimal increments of the q_j at a given instant. Thus we obtain the equations,

$$(7) \quad \frac{\partial \varphi_k}{\partial q_1} \delta q_1 + \frac{\partial \varphi_k}{\partial q_2} \delta q_2 + \cdots + \frac{\partial \varphi_k}{\partial q_{3n}} \delta q_{3n} = 0, k = 1, 2, \cdots, r,$$

which are equivalent to equations (5) and in the form (6). On the other hand, equations (6) can not in general be put in the form (5), this being possible only when the differential system (6) is integrable. When the imposed conditions are given in the form (5) or can be put in this form then the motion of the set is said to be *holonomic*, and otherwise the motion is *nonholonomic*. These terms are due to the German physicist Hertz.

In both the holonomic and nonholonomic cases we may solve the r independent equations (6) or (7) for r of the increments δq_j as linear forms in the remaining $3n - r$ of them and substitute the results into equations (4), thus reducing equations (4) to the form (2) with $m = 3n - r$. Since these remaining δq_j are independent it follows that the set has $m = 3n - r$ degrees of freedom. In the holonomic case we may solve equations (5) for r of the q_j in terms of the remaining $3n - r$ of them and t and substitute the results into equations (3), thus reducing them to the form,

$$(8) \qquad \mathbf{p}_i = \mathbf{p}_i(q_1, q_2, \cdots, q_m, t), \qquad i = 1, 2, \cdots, n,$$

in which $m = 3n - r$ and the q_j are now independent. Equations (8) constitute a standard form of expression for the $\mathbf{p}_i$ in the case of holonomic motion with m degrees of freedom. If the equations of condition (5) are free from t then evidently equations (8) will also be free from t. The virtual displacements $\delta \mathbf{p}_i$ in the holonomic case may be obtained by differentiating equations (8)

holding t fixed, giving,

$$(9) \quad \delta \mathbf{p}_i = \frac{\partial \mathbf{p}_i}{\partial q_1} \delta q_1 + \frac{\partial \mathbf{p}_i}{\partial q_2} \delta q_2 + \qquad + \frac{\partial \mathbf{p}_i}{\partial q_m} \delta q_m, \; i = 1, 2, \quad , n.$$

The m independent parameters q_j appearing in equations (8) resemble the Cartesian coördinates of the particles P_i in the sense that at any given instant the location of these particles is determined by them. For this reason they are known as the *generalized coördinates* of Lagrange. They are *position coördinates* since they tell only the positions of the particles and not their velocities nor the instant at which the positions are occupied.

The following examples will illustrate the formation of equations (1) and (2) and the degrees of freedom of a set of particles. A single particle P required to remain on a surface has two degrees of freedom at all non-singular points of the surface, for if $\mathbf{c}_1$ and $\mathbf{c}_2$ are any two non-parallel vectors tangent to the surface at P, then the formula, $\delta \mathbf{p} = \mathbf{c}_1 \, \delta q_1 + \mathbf{c}_2 \, \delta q_2$, which is a special case of equations (1) gives us for every choice of the infinitesimals δq_1 and δq_2 a virtual displacement tangent to the surface at P and these constitute all of the displacements compatible with the constraints at the instant. If the condition imposed upon a set of particles P_i is that they shall be rigidly connected, the number of degrees of freedom is six if three or more of the particles are non-collinear. For by equation § 48, (2) we have every permissible displacement $\delta \mathbf{p}_i$ at any instant given by,

$$\delta \mathbf{p}_i = \delta \mathbf{a} + \mathbf{w} \times (2\, \mathbf{p}_i + \delta \mathbf{p}_i - 2\mathbf{a} - \delta \mathbf{a}),$$

where $\delta \mathbf{a}$, $\mathbf{w}$, $\mathbf{a}$ are the same for all the particles P_i. Here $\mathbf{w}$, $\delta \mathbf{a}$, $\delta \mathbf{p}_i$ are infinitesimals and consequently $\mathbf{w} \times \delta \mathbf{a}$ and $\mathbf{w} \times \delta \mathbf{p}_i$ are infinitesimals of higher order and may be dropped from the equation without affecting the limiting directions and length ratios of the $\delta \mathbf{p}_i$. Thus we have,

$$\delta \mathbf{p}_i = \delta \mathbf{a} + 2\, \mathbf{w} \times (\mathbf{p}_i - \mathbf{a}).$$

Since each of the infinitesimal vectors $\delta \mathbf{a}$ and $\mathbf{w} \times (\mathbf{p}_i - \mathbf{a})$ is entirely arbitrary each is expressed as the sum of three constant vectors $\mathbf{c}_j$ with three infinitesimal scalar multipliers δq_j and we have at every instant,

$$\delta \mathbf{p}_i = \mathbf{c}_1 \, \delta q_1 + \mathbf{c}_2 \, \delta q_2 + \mathbf{c}_3 \, \delta q_3 + \mathbf{c}_4 \, \delta q_4 + \mathbf{c}_5 \, \delta q_5 + \mathbf{c}_6 \, \delta q_6.$$

Obviously the $\mathbf{c}_j$ may be so chosen for each particle P_i that the δq_j are the same for all particles and the set has six degrees of freedom, as stated. If the set of particles P_i constitutes a rigid sphere constrained to roll, pivot and slide on a fixed plane, the number of degrees of freedom is reduced to five for in this case the point A of the sphere having the radius vector $\mathbf{a}$ may be chosen as the point of contact of the sphere with the plane, and $\delta\mathbf{a}$ being confined to the plane is expressed with two instead of three terms of the form $\mathbf{c}_j\,\delta q_j$. If the sphere is constrained to roll and pivot without slipping on the plane, the number of degrees of freedom is reduced to three because the virtual displacement $\delta\mathbf{a}$ of the point of contact must be zero.

The above case of the sphere constrained to roll and pivot without slipping on a plane is the classical example of nonholonomic motion, for the condition $\delta\mathbf{a} = 0$ is of type (6); that is a condition directly imposed upon the displacements $\delta\mathbf{p}_i$ which can not be integrated into the holonomic form (5). The distinction between the holonomic and nonholonomic cases will be clarified by the following two examples. Let a particle P be required to move in such a way that its path is everywhere normal to that one of the family of helices,

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad z = a^2\theta + b,$$

passing through its position. Here a and b are constants for each individual helix and θ is a parameter varying from point to point along it. Any vector tangent to the helix at the point $P \equiv (x, y, z)$ has evidently coördinates proportional to $(-y, x, x^2 + y^2)$ and so the virtual displacements of P written in the form (4) $\delta\mathbf{p} = \mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z$, being normal to the helix, must satisfy the relation of the form (6),

$$-y\,\delta x + x\,\delta y + (x^2 + y^2)\,\delta z = 0.$$

By this we reduce $\delta\mathbf{p}$ to the form (2),

$$\delta\mathbf{p} \equiv \left(1, 0, \frac{y}{x^2 + y^2}\right)\delta x + \left(0, 1, \frac{-x}{x^2 + y^2}\right)\delta y,$$

in which δx and δy have been chosen as the infinitesimals δq_j. In this case the equation of condition given above in form (6) is easily integrated to form (5),

$$\arctan\frac{y}{x} + (z - c) = 0 \qquad (c\text{ const.})$$

and the motion is consequently holonomic. Choosing x and y as the parameters q_j we express $\mathbf{p}$ in the typical holonomic form (8),

$$\mathbf{p} \equiv (x, y, c - \arctan y/x).$$

It thus appears that the condition originally imposed upon the particle was equivalent to requiring that it remain on that one of the family of surfaces,

$$\arctan \frac{y}{x} + (z - c) = 0$$

upon which it lay originally.

On the other hand if the particle be required to move orthogonally to the family of helices,

$$x = a \cos \theta, \qquad = a \sin \theta, \qquad = \theta + b,$$

the conditions would seem to be but slightly changed, but actually the change is fundamental. For the equation of condition in the form (6) now becomes,

$$- y\, \delta x + x\, \delta y + \delta z = 0$$

and by its aid $\delta\mathbf{p}$ may be written in the form (2)

$$\delta\mathbf{p} \equiv (1, 0, y)\, \delta x + (0, 1, -x)\, \delta y.$$

However in this case the equation of condition in form (6) is not integrable and $\mathbf{p}$ can not be written in form (8) and this motion is nonholonomic.

EXERCISES

1. *a.* A circular disc moves with its edge tangent to a fixed plane and the plane of the disc remains perpendicular to the fixed plane. Show that the motion of the disc has four degrees of freedom and express $OP = \mathbf{p}$ as a function of four generalized coordinates, q_1, q_2, q_3, q_4, where O is a fixed point and P is any point fixed in the disc.
   b. Determine the number of degrees of freedom and express $\mathbf{p}$ as a function of that number of generalized coördinates when the above disc is further constrained to remain parallel to its original position.
   c. When the same point of the edge remains in contact with the fixed plane.
   d. Can $\mathbf{p}$ be so expressed when the disc is sharp on the edge and thus can not slide perpendicularly to its plane?
   e. When the disc can roll but not slide in its plane?
   f. When both conditions (*b*) and (*e*) are imposed?

2. Determine the number of degrees of freedom of:
   (a) A rigidly connected set of collinear particles.
   (b) Two rigid bodies attached to each other at one point.
   (c) Two rigid bodies where a point of one slides on a curve of the other.
   (d) Two rigid bodies where each of two points of one slides on a curve of the other.

3. Prove that if the virtual displacements $\delta\mathbf{p}$ of a particle P are restricted only by the condition,

$$\mathbf{r}\cdot\delta\mathbf{p} = 0,$$

where $\mathbf{r}$ is a vector function of $\mathbf{p}$, then the motion is holonomic or nonholonomic according as the above equation is integrable or not. (Hint: Write $\delta\mathbf{p}$ in the form,

$$\delta\mathbf{p} = \mathbf{U}(u, v)\delta u + \mathbf{V}(u, v)\delta v,$$

where $\mathbf{U}$ and $\mathbf{V}$ are constantly perpendicular to $\mathbf{r}$. Then show that functions $\mathbf{F}(u, v)$ and $\lambda(u, v)$ such that,

$$\frac{\partial \mathbf{F}}{\partial u} = \lambda\mathbf{U}, \qquad \frac{\partial \mathbf{F}}{\partial v} = \lambda\mathbf{V}$$

will exist only when the given condition is integrable.)

91. D'Alembert's Principle.

Let us consider a set of n particles P_i each acted upon by a total applied force $\mathbf{f}_i$, with the motion of the set also subject to certain imposed conditions. We find it convenient to think of these conditions upon the motion as being maintained by means of certain additional forces $\mathbf{k}_i$ acting upon the particles. These forces are called constraints and any motion of the particles satisfying the imposed conditions is said to be compatible with the constraints. Since the constraints are assumed to act only in maintaining the imposed conditions, it follows that any motion satisfying these conditions is not affected by the constraints. In § 73 this fact was discussed and put in explicit form as the following principle, which may be regarded in effect as a definition of the constraints.

If at any given instant the particles P_i are given any virtual displacements $\delta\mathbf{p}_i$ satisfying the imposed conditions at that instant then the total virtual work of the constraints is zero,

$$\text{(1)} \qquad \Sigma\, \mathbf{k}_i\cdot \delta\mathbf{p}_i = 0.$$

Since the total force acting upon each particle P_i is now

$\mathbf{f}_i + \mathbf{k}_i$ its equation of motion is,

$$(2) \qquad m_i \mathbf{p}_i'' = \mathbf{f}_i + \mathbf{k}_i$$

and equation (1) takes the form,

$$(3) \qquad \Sigma (\mathbf{f}_i - m_i \mathbf{p}_i'') \cdot \delta \mathbf{p}_i = 0,$$

which is to hold for all sets of virtual displacements $\delta \mathbf{p}_i$ compatible with the imposed conditions *as they exist at the instant considered.*

This important formula is known as *d'Alembert's Principle as Formulated by Lagrange.* As d'Alembert originally conceived of his principle it consisted simply of the observation that any problem in kinetics can be formulated in terms of an associated problem in statics. For the equation of motion of a free particle acted upon by a total force $\mathbf{f}$ may be written, $\mathbf{f} - m\,\mathbf{p}'' = 0$ which is exactly the condition for equilibrium of a free particle of the same mass acted upon by a total force $\mathbf{f} - m\,\mathbf{p}''$. And in general if we are given any set of particles P_i each with a total applied force $\mathbf{f}_i$, we consider an associated set in which the masses, positions, and constraints are those of the given set at the instant considered, but in which the total force applied to each particle is the value of $\mathbf{f}_i - m_i \mathbf{p}_i''$ for the corresponding particle of the given set at the same instant. By d'Alembert's observation the *equations of motion* for the given set are the *conditions of equilibrium* for the associated set and by the Principle of Virtual Work (§ 73) these are exactly given by equations (3). Since, however, in the Principle of Virtual Work the imposed conditions are assumed to remain constant, so in the extension of this principle to the moving set of particles we must consider the imposed conditions as fixed at the instant under consideration.

We have just seen that d'Alembert's Principle is a consequence of Newton's Second Law of Motion as given by equation (2). We may, however, equally well follow the method of Lagrange and assume d'Alembert's Principle and from it deduce Newton's Second Law. For if we apply equation (3) to a single free particle, then since there are no constraints the virtual displacement $\delta \mathbf{p}$ becomes entirely arbitrary and the equation can only be satisfied for every choice of $\delta \mathbf{p}$ if we have, $\mathbf{f} - m\,\mathbf{p}'' = 0$. This is Newton's Second Law of Motion and the first follows at once by

integration. For if $\mathbf{f} = 0$ we have $m\,\mathbf{p}'' = 0$ and $\mathbf{p}'$ is a constant, as stated in the First Law. Lagrange also shows that we may deduce from d'Alembert's Principle the equally important implication in Newton's Second Law that forces are additive vectorially. Lagrange tacitly assumes Newton's Third Law in the form of the transmissibility of forces along a string or rod.

We shall see in the following articles that the basic principles of mechanics can be derived from d'Alembert's Principle. An interesting special theorem which is an immediate consequence of d'Alembert's Principle is the,

Theorem of Kinetic Energy for Applied Forces. If the conditions imposed upon the motion of a set of particles do not vary with the time, then the derivative of the kinetic energy of the set with respect to the time equals the power of the applied forces.

If the imposed conditions do not vary with the time it follows as explained in § 90 that the actual infinitesimal displacements $\mathbf{p}_i'\,dt$ of the particles form one permitted set of virtual displacements $\delta\mathbf{p}_i$. Equation (3) may therefore in particular be written after a mere transposition and dividing out dt as,

$$\Sigma\, m_i\, \mathbf{p}_i' \cdot \mathbf{p}_i'' = \Sigma\, \mathbf{f}_i \cdot \mathbf{p}_i'$$

or,

$$T' = W', \tag{4}$$

where T is the kinetic energy of the set and W' is the power of the applied forces. The theorem is thus easily established. It will be observed that there is here no implication that the motion is holonomic. A special proof for the holonomic case appears in the next article. If we compare this theorem with the general theorem of kinetic energy, § 79, we reach at once the conclusion that when the imposed conditions do not vary with the time the constraints do no work during the actual motion of the set; a conclusion obvious from the postulate on constraints stated at the beginning of this article.

EXERCISES

1. Show from d'Alembert's Principle that,

If the conditions imposed upon the motion of a set of particles always permit a translation of the set parallel to a given axis, then the derivative with respect to the time of the measure of the

momentum of the set upon this axis is equal to the measure of the applied forces upon this axis.

2. Show from d'Alembert's Principle that,

If the conditions imposed upon the motion of a set of particles always permit a rotation of the set about a given axis, then the derivative with respect to the time of the moment with respect to this axis of the momentum of the set is equal to the moment of the applied forces with respect to this axis.

92. Lagrange's Equations.

With the aid of d'Alembert's Principle the equations of motion of a set of particles may be put into certain forms noteworthy for their symmetry and very convenient in application, especially in theoretical investigations. Those in this article are due in principle to Lagrange. We observed in § 90 that the n radius vectors $\mathbf{p}_i$ of the n particles P_i of a set may always be regarded as functions of $3n$ scalar parameters, $q_1, q_2, \cdots, q_{3n}$. In the present discussion for greater generality we shall assume that certain holonomic conditions may already have been imposed upon the motion and the $\mathbf{p}_i$ thus expressed as functions of $m \leq 3n$ parameters $q_1, q_2, \cdots, q_m$ and possibly the time t, as in equations § 90, (8),

$$(1) \qquad \mathbf{p}_i = \mathbf{p}_i(q_1, q_2, \cdots, q_m, t), \qquad i = 1, 2, \cdots, n.$$

Then the virtual displacements $\delta\mathbf{p}_i$ of the particles take the form,

$$(2) \quad \delta\mathbf{p}_i = \frac{\partial \mathbf{p}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{p}_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \mathbf{p}_i}{\partial q_m}\delta q_m, \qquad i = 1, 2, \cdots, n$$

as previously explained. If now additional conditions, either holonomic or nonholonomic, be imposed upon the motion of the set, they may be put in the form of homogeneous linear equations in the quantities δq_j,

$$(3) \quad \varphi_{k1}\delta q_1 + \varphi_{k2}\delta q_2 + \cdots + \varphi_{km}\delta q_m = 0, \qquad k = 1, 2, \cdots, r,$$

where the φ_{kj} are functions of the q_j and possibly of t. These r equations are assumed to be independent and they thus reduce the number of degrees of freedom of the set to $m - r$. We may solve equations (3) for r of the δq_j as functions of the remaining $m - r$ of them and substitute the values into equations (2) which then take the form,

$$(4) \quad \delta\mathbf{p}_i = \gamma_{i1}\delta q_1 + \gamma_{i2}\delta q_2 + \cdots + \gamma_{is}\delta q_s, \qquad s = m - r,$$

where the γ_{ij} are vector functions of the m q_j and of t. If these values of the $\delta \mathbf{p}_i$ are substituted in the statement of d'Alembert's Principle,

$$\Sigma\,(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\delta\,\mathbf{p}_i = 0 \tag{5}$$

we have,

$$(Q_1 - P_1)\delta q_1 + (Q_2 - P_2)\delta q_2 + \cdots + (Q_s - P_s)\delta q_s = 0 \tag{6}$$

where,

$$Q_j = \sum_i \mathbf{f}_i\cdot\gamma_{ij}, \qquad P_j = \sum_i m_i\,\mathbf{p}_i''\cdot\gamma_{ij}, \qquad j = 1, 2, \cdots, s.$$

Since the m $\delta \mathbf{q}_j$ appearing in this equation are now independent infinitesimals their coefficients must vanish individually and we have the following equations of motion of the set of particles,

$$Q_j - P_j = 0 \qquad \text{or} \qquad \sum_i (\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\gamma_{ij} = 0, \tag{7}$$

$$j = 1, 2, \cdots, s.$$

The number of these equations is equal to the number s of degrees of freedom of the set and together with the r differential equations (3) they determine the m parameters q_j and thus by equations (1) determine the $\mathbf{p}_i$ as functions of t.

Lagrange has developed a method known as the method of *Lagrange's Multipliers* by which we may avoid solving equations (3) for r of the δq_j in terms of the others. If the values of the $\delta \mathbf{p}_i$ from equations (2) are substituted directly into the equation (5) of d'Alembert's Principle we have,

$$(Q_1 - P_1)\,\delta q_1 + (Q_2 - P_2)\,\delta q_2 + \cdots + (Q_m - P_m)\,\delta q_m = 0, \tag{8}$$

where,

$$Q_j = \sum_i \mathbf{f}_i\cdot\frac{\partial \mathbf{p}_i}{\partial q_j}, \qquad P_j = \sum_i m_i\,\mathbf{p}_i''\cdot\frac{\partial \mathbf{p}_i}{\partial q_j},$$

which is to hold for all values of the δq_j satisfying equations (3). Equations (3) are a set of r linear homogeneous equations in the m variables δq_j and equation (8) is a single equation of the same type. It is shown in algebra that a necessary and sufficient condition that one such equation be satisfied by all the sets of solutions of a set of such equations is that the coefficients in the one equation be linearly dependent on those of the set. This amounts in our case to saying that for equation (8) to be satisfied

for every set of values of the δq_j satisfying equations (3) it is necessary and sufficient that there exist r scalar quantities λ_1, $\lambda_2, \cdots, \lambda_r$ such that,

$$(9)\qquad \begin{gathered} Q_j - P_j + \lambda_1\varphi_{1j} + \lambda_2\varphi_{2j} + \cdots + \lambda_r\varphi_{rj} = 0, \\ j = 1, 2, \cdots, m \end{gathered}$$

or the equivalent,

$$\sum_i (\mathbf{f}_i - m_i\,\mathbf{p}_i'') \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} + \sum_k \lambda_k\varphi_{kj} = 0, \qquad j = 1, 2, \cdots, m.$$

These m equations together with the r differential equations (3) determine the m parameters q_j and the r Lagrange's Multipliers λ_k as functions of t. From the q_j the motion of the particles is determined by equations (1).

If the additional conditions (3) imposed upon the motion of the set are holonomic, the procedure outlined above may be somewhat modified. For the imposed conditions may then be written in the form of r equations in the q_j and t,

$$(10)\qquad \varphi_k(q_1, q_2, \cdots, q_m, t) = 0, \qquad k = 1, 2, \cdots, r.$$

Then as explained in § 90 the increments δq_j corresponding to the virtual displacements permitted by these conditions must satisfy relations obtained by differentiating equations (10) with the time held constant. Thus we have,

$$(11)\qquad \frac{\partial \varphi_k}{\partial q_1}\delta q_1 + \frac{\partial \varphi_k}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \varphi_k}{\partial q_m}\delta q_m = 0, \qquad k = 1, 2, \cdots, r.$$

As in the general case discussed above, we have now two methods by which we may employ d'Alembert's Principle to set up the equations of motion. We may solve equations (10) for r of the parameters q_j in terms of the remaining $m - r$ of them and of t and then substitute the values obtained into equations (1), reducing them to the form,

$$(12)\qquad \mathbf{p}_i = \mathbf{p}_i(q_1, q_2, \cdots, q_s, t), \qquad i = 1, 2, \cdots, n, \qquad s = m - r,$$

in which the q_j are now independent. As in equations (2) we have for the virtual displacements,

$$(13)\qquad \delta\mathbf{p}_i = \frac{\partial \mathbf{p}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{p}_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \mathbf{p}_i}{\partial q_s}\delta q_s, \qquad i = 1, 2, \cdots, n$$

and if these values are substituted into the formula (5) of d'Alembert's Principle we have,

$$(14)\qquad (Q_1 - P_1)\,\delta q_1 + (Q_2 - P_2)\,\delta q_2 + \cdots + (Q_s - P_s)\,\delta q_s = 0,$$

where,

$$Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{p}_i}{\partial q_j}, \qquad P_j = \sum_i m_i \mathbf{p}_i'' \cdot \frac{\partial \mathbf{p}_i}{\partial q_j}, \qquad j = 1, 2, \cdots, s.$$

Since the quantities $\delta q_1, \delta q_2, \cdots, \delta q_s$ are independent, their coefficients must vanish individually and we have,

$$(15)\qquad Q_j - P_j = 0 \qquad \text{or} \qquad \sum_i (\mathbf{f}_i - m_i\,\mathbf{p}_i'') \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} = 0,$$
$$j = 1, 2, \cdots, s.$$

The number s of these equations is equal to the number of degrees of freedom of the set. In this holonomic case, unlike the general case of equations (7), only s of the q_j are involved and they may consequently be determined by equations (15) alone. Thus equations (15) serve to determine completely the motion of the set by substitution of the values of the q_j into equations (1).

The method of Lagrange's Multipliers may likewise be applied to the holonomic conditions (10). The argument goes through as before except that equations (11) are used instead of equations (3). In other words φ_{kj} now becomes $\partial\varphi_k/\partial q_j$. Equation (9) then becomes,

$$(16)\qquad Q_j - P_j + \lambda_1 \frac{\partial \varphi_1}{\partial q_j} + \lambda_2 \frac{\partial \varphi_2}{\partial q_j} + \cdots + \lambda_r \frac{\partial \varphi_r}{\partial q_j} = 0,$$
$$j = 1, 2, \cdots, m$$

or,

$$\sum_i (\mathbf{f}_i - m_i\,\mathbf{p}_i'') \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} + \sum_k \lambda_k \frac{\partial \varphi_k}{\partial q_j} = 0, \qquad j = 1, 2, \cdots, m.$$

These m equations together with the r equations (10) serve to determine the m parameters q_j and the r multipliers λ_k. The motion of the set is then known from equations (1).

Equations (16) take an especially simple form in case the parameters q_j of equations (1) are simply the $3n$ Cartesian coordinates (x_i, y_i, z_i) of the particles P_i. In this case the motion

will be restricted only by the r conditions (10) and the number of degrees of freedom will be $3n - r$. Equations (1) now take the form,

$$\mathbf{p}_i = x_i\,\mathbf{i} + y_i\,\mathbf{j} + z_i\,\mathbf{k}, \qquad i = 1, 2, \cdots, n.$$

Writing out equations (16) for q_j equal respectively to x_i, y_i, z_i we have,

$$(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\mathbf{i} + \lambda_1 \frac{\partial \varphi_1}{\partial x_i} + \lambda_2 \frac{\partial \varphi_2}{\partial x_i} + \cdots + \lambda_r \frac{\partial \varphi_r}{\partial x_i} = 0,$$

$$(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\mathbf{j} + \lambda_1 \frac{\partial \varphi_1}{\partial y_i} + \lambda_2 \frac{\partial \varphi_2}{\partial y_i} + \cdots + \lambda_r \frac{\partial \varphi_r}{\partial y_i} = 0,$$

$$(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\mathbf{k} + \lambda_1 \frac{\partial \varphi_1}{\partial z_i} + \lambda_2 \frac{\partial \varphi_2}{\partial z_i} + \cdots + \lambda_r \frac{\partial \varphi_r}{\partial z_i} = 0.$$

Let us now define the vectors $\nabla_i\varphi_k$ by the coördinates,

$$\nabla_i\varphi_k \equiv \left(\frac{\partial \varphi_k}{\partial x_i}, \frac{\partial \varphi_k}{\partial y_i}, \frac{\partial \varphi_k}{\partial z_i}\right).$$

The symbol ∇ is read "del" and the vector $\nabla_i\varphi_k$ is called the *partial gradient* of φ_k with respect to $\mathbf{p}_i$ (§ 99). With this notation the above three scalar equations may be written in the vector form,

$$(17) \quad \mathbf{f}_i - m_i\,\mathbf{p}_i'' + \lambda_1\nabla_i\varphi_1 + \lambda_2\nabla_i\varphi_2 + \cdots + \lambda_r\nabla_i\varphi_r = 0.$$

These are *Lagrange's Equations of Motion in the First Form.* These n vector equations together with the r scalar equations (10) serve to determine the n vectors $\mathbf{p}_i''$ and the r scalars λ_k when the $\mathbf{f}_i$ and φ_k are known. It is evident that $\lambda_k\nabla_i\varphi_k$ is exactly the constraint imposed upon the particle P_i by the condition $\varphi_k = 0$. For if this condition were removed and each force $\mathbf{f}_i$ increased by the amount $\lambda_k\nabla_i\varphi_k$ the same values of $\mathbf{p}_i''$ would still satisfy equations (17) as so modified. Equations (16) and (17), although of great theoretical interest, are not very convenient in practical application unless the number of particles and the number of conditions are small, for otherwise the large number of unknown quantities to be determined makes the treatment very cumbersome.

Our two equations (7) and (9) together with their specializations (14) and (16) for the case of holonomic conditions may all be modified in form by an extremely ingenious device due to

Lagrange for the computation of the quantities P_j appearing in them. In all these cases we have,

$$P_j = \sum_i m_i \mathbf{p}_i'' \cdot \frac{\partial \mathbf{p}_i}{\partial q_j},$$

in which the parameters q_j may be independent or subject to any conditions, holonomic or nonholonomic. Starting with equations (1) we differentiate with respect to the time obtaining,

$$\mathbf{p}_i' = \frac{\partial \mathbf{p}_i}{\partial q_1} q_1' + \frac{\partial \mathbf{p}_i}{\partial q_2} q_2' + \cdots + \frac{\partial \mathbf{p}_i}{\partial q_m} q_m' + \frac{\partial \mathbf{p}_i}{\partial t}. \tag{18}$$

Inspection of this equation shows that,

$$\frac{\partial \mathbf{p}_i'}{\partial q_j'} = \frac{\partial \mathbf{p}_i}{\partial q_j} \tag{19}$$

and,

$$\begin{aligned}\left(\frac{\partial \mathbf{p}_i}{\partial q_j}\right)' = \frac{\partial^2 \mathbf{p}_i}{\partial q_j\, \partial q_1} q_1' + \frac{\partial^2 \mathbf{p}_i}{\partial q_j\, \partial q_2} q_2' + \cdots \\ + \frac{\partial^2 \mathbf{p}_i}{\partial q_j\, \partial q_m} q_m' + \frac{\partial^2 \mathbf{p}_i}{\partial q_j\, \partial t} = \frac{\partial \mathbf{p}_i'}{\partial q_j}.\end{aligned} \tag{20}$$

With the aid of these two results we may write,

$$\begin{aligned} P_j &= \sum_i m_i \mathbf{p}_i'' \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} = \left(\sum_i m_i \mathbf{p}_i' \cdot \frac{\partial \mathbf{p}_i}{\partial q_j}\right)' - \sum_i m_i \mathbf{p}_i' \cdot \left(\frac{\partial \mathbf{p}_i}{\partial q_j}\right)' \\ &= \left(\sum_i m_i \mathbf{p}_i' \cdot \frac{\partial \mathbf{p}_i'}{\partial q_j'}\right)' - \sum_i m_i \mathbf{p}_i' \cdot \frac{\partial \mathbf{p}_i'}{\partial q_j}. \end{aligned} \tag{21}$$

We now recall that the kinetic energy T of the set of particles is by definition,

$$T = \sum \frac{m_i}{2} \mathbf{p}_i'^2$$

and hence we have,

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \mathbf{p}_i' \cdot \frac{\partial \mathbf{p}_i'}{\partial q_j}, \qquad \frac{\partial T}{\partial q_j'} = \sum_i m_i \mathbf{p}_i' \cdot \frac{\partial \mathbf{p}_i'}{\partial q_j'}. \tag{22}$$

Substituting from equations (22) into (21) gives us the very important *Lagrangian Form*,

$$P_j = \left(\frac{\partial T}{\partial q_j'}\right)' - \frac{\partial T}{\partial q_j}. \tag{23}$$

This Lagrangian Form for P_j is extremely convenient in application as we may employ it as soon as we have T expressed in

terms of the q_j, q'_j, and t; the individual particles of the set being lost sight of. We see from equations (18) that T will never involve the q'_j to higher than the second degree and that if the $\partial \mathbf{p}_i/\partial t$ are identically zero, T will be a quadratic form in the q'_j. This will in particular occur when all the conditions imposed upon the motion are holonomic and independent of the time; which is the only case discussed by Lagrange, Hamilton and others of the early investigators.

On substituting the Lagrangian Form for P_j into equations (15) we find,

$$(24) \qquad \left(\frac{\partial T}{\partial q'_j}\right)' - \frac{\partial T}{\partial q_j} = Q_j \qquad \text{where} \qquad Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{p}_i}{\partial q_j},$$

$$j = 1, 2, \cdots, s.$$

These are *Lagrange's Equations of Motion in the Second Form* and apply to the case in which the parameters q_j have been rendered independent. Similarly the Lagrangian Form for P_j reduces equations (16) to the general form for holonomic motion,

$$(25) \quad \left(\frac{\partial T}{\partial q'_j}\right)' - \frac{\partial T}{\partial q_j} = Q_j + \sum_k \lambda_k \frac{\partial \varphi_k}{\partial q_j}, \qquad j = 1, 2, \cdots, m.$$

These are all differential equations of the second order in the q_j.

Not only may the quantities P_j appearing in equations (15) and (16) be thus easily evaluated, but the quantities Q_j also take on an important form in certain cases. Thus if there exists a scalar function U,

$$U = U(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, t)$$

such that at each instant each applied force $\mathbf{f}_i$ is the partial gradient (§ 99) of U with respect to the corresponding $\mathbf{p}_i$, then we have,

$$\mathbf{f}_i = \nabla_i U \equiv \left(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}, \frac{\partial U}{\partial z_i}\right) \qquad \text{where} \qquad \mathbf{p}_i \equiv (x_i, y_i, z_i)$$

and consequently,

$$Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} = \sum_i \nabla_i U \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} = \frac{\partial U}{\partial q_j}.$$

We may therefore write Lagrange's Equations (24) in the form,

$$(26) \qquad \left(\frac{\partial T}{\partial q'_j}\right)' - \frac{\partial T}{\partial q_j} = \frac{\partial U}{\partial q_j}, \qquad j = 1, 2, \cdots, s$$

and we have a corresponding form for equations (25). We now introduce the *Lagrangian Function*,

$$L = T + U.$$

Since U is by definition a function only of the q_j and t it does not involve the q'_j and evidently,

$$\frac{\partial L}{\partial q'_j} = \frac{\partial T}{\partial q'_j}$$

and equations (26) may be written in the compact form,

$$\left(\frac{\partial L}{\partial q'_j}\right)' - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, \cdots, s. \tag{27}$$

One of the chief advantages of writing Lagrange's Equations in this last form is that certain integrals are often immediately obvious. Thus if any particular parameter q_j does not appear in the expression for L, although its derivative q'_j may appear, we call such a q_j a *cyclic* or *ignorable coördinate*. Every ignorable coördinate yields at once an integral of equations (27) for we obviously have,

$$\left(\frac{\partial L}{\partial q'_j}\right)' = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial q'_j} = A_j \qquad (A_j \text{ const.})$$

or what amounts to the same thing,

$$\left(\frac{\partial T}{\partial q'_j}\right)' = 0 \qquad \text{and} \qquad \frac{\partial T}{\partial q'_j} = A_j \qquad (A_j \text{ const.}).$$

Furthermore if the time t does not appear explicitly in L we have an additional integral of equations (27). For in this case we have,

$$L' = \Sigma \left(\frac{\partial L}{\partial q_j} q_j + \frac{\partial L}{\partial q'_j} q''_j\right)$$

and by equations (27) this becomes,

$$L' = \Sigma \left\{\left(\frac{\partial L}{\partial q'_j}\right)' q'_j + \frac{\partial L}{\partial q'_j} q''_j\right\} = \Sigma \left(\frac{\partial L}{\partial q'_j} q'_j\right)'.$$

Thus,

$$\left(\Sigma \frac{\partial L}{\partial q'_j} q'_j\right)' - L' = 0,$$

and we have the integral,

$$(28) \qquad \sum \frac{\partial L}{\partial q_j'} q_j' - L = H \qquad (H \text{ const.}).$$

One important case in which this integral occurs and in which it may be further reduced is that in which the expressions (12) for the $\mathbf{p}_i$ are free from t and in which the function U exists and is also free from t. When U is thus free from t it is known as the *force function* and the system of applied forces $\mathbf{f}_i$ is said to be *conservative*. In this case we have,

$$T = \sum \frac{m_i}{2} \mathbf{p}_i'^2 = \sum_i \frac{m_i}{2} \left(\sum_j \frac{\partial \mathbf{p}_i}{\partial q_j} q_j' \right)^2$$

and T is evidently a quadratic form in the q_j'. By Euler's theorem for homogeneous functions we have,

$$\sum_j \frac{\partial T}{\partial q_j'} q_j' = 2T,$$

and equation (28) becomes the integral,

$$(29) \quad 2T - L = 2T - (T + U) = H \qquad \text{or} \qquad T - U = H.$$

The quantity T is the kinetic energy of the set of particles and $-U$ is known as the *potential energy*, so that equation (29) states that the total energy, kinetic plus potential, remains constant. It is because the motion thus *conserves* the total energy of the set that the forces $\mathbf{f}_i$ are in this case said to be *conservative*.

We conclude this section with some applications of the Lagrangian equations developed in it. As an example of the use of Lagrange's equations of motion in the first form (17) we may derive the *Theorem of Kinetic Energy for Applied Forces* already stated and proved in § 91. The present proof, however, only holds for holonomic motion. Starting with equations (17) we dot-multiply each equation by the corresponding $\mathbf{p}_i'$ and add the resulting equations term for term obtaining,

$$\sum \mathbf{f}_i \cdot \mathbf{p}_i' - \sum m_i \mathbf{p}_i' \cdot \mathbf{p}_i'' + \lambda_1 \sum \nabla_i \varphi_1 \cdot \mathbf{p}_i' + \lambda_2 \sum \nabla_i \varphi_2 \cdot \mathbf{p}_i' + \cdots + \lambda_r \sum \nabla_i \varphi_r \cdot \mathbf{p}_i' = 0.$$

By hypothesis the imposed conditions do not vary with the time and so the φ_k are functions of the coördinates x_i, y_i, z_i only and

do not contain t explicitly. Consequently on differentiating equations (10) with respect to the time we find,

$$\varphi_k' = \sum_i \nabla_i\varphi_k\cdot\mathbf{p}_i' = 0,$$

which is exactly the coefficient of λ_k in the above equation. Consequently that equation reduces to,

$$\Sigma\, m_i\,\mathbf{p}_i'\cdot\mathbf{p}_i'' = \Sigma\,\mathbf{f}_i\cdot\mathbf{p}_i',$$

which being the same as equation § 91, (4), the theorem follows.

As an application of Lagrange's equations in the second form we may deduce Euler's equations for the motion of a rigid body with one fixed point, § 85, (7). It will be recalled that Euler here employs a moving reference frame fixed in the moving body and having for axes the three principal axes of inertia of the body for its fixed point O. Relative to this frame the angular velocity $\boldsymbol{\omega}$ has the coördinates, $\boldsymbol{\omega} \equiv (\omega_i, \omega_j, \omega_k)$ and, as observed in § 86, (5) the kinetic energy of the body is,

$$T = \tfrac{1}{2}(I_1\omega_i^2 + I_2\omega_j^2 + I_3\omega_k^2),$$

where I_1, I_2, I_3 are the principal moments of inertia of the body for O. We may conveniently employ for the three parameters q_j determining the motion of the body Euler's angles ψ, φ, θ (§ 46) giving the position of the moving reference frame relative to a fixed frame with the same origin, O. We shall first write Lagrange's Equation (24) for the parameter φ,

$$\left(\frac{\partial T}{\partial\varphi'}\right)' - \frac{\partial T}{\partial\varphi} = \sum \mathbf{f}_i\cdot\frac{\partial\mathbf{p}_i}{\partial\varphi}.$$

We find by the aid of equations § 85, (9) that,

$$\begin{aligned}\frac{\partial T}{\partial\varphi} &= I_1\omega_i\frac{\partial\omega_i}{\partial\varphi} + I_2\omega_j\frac{\partial\omega_j}{\partial\varphi}\\ &= I_1\omega_i\omega_j - I_2\omega_j\omega_i = (I_1 - I_2)\omega_i\omega_j,\\ \frac{\partial T}{\partial\varphi'} &= I_3\omega_k\frac{\partial\omega_k}{\partial\varphi'} = I_3\omega_k.\end{aligned}$$

To compute the second member of Lagrange's equation we observe that in forming the partial derivatives of the $\mathbf{p}_i$ with respect to φ we are in effect restricting the body to a rotation about a fixed axis. If $\mathbf{u}$ is a unit vector along this axis we have

in such a motion,

$$\frac{\partial \mathbf{p}_i}{\partial \varphi} = \frac{\mathbf{p}_i'}{\varphi'} = \frac{\varphi' \, \mathbf{u} \times \mathbf{p}_i}{\varphi'} = \mathbf{u} \times \mathbf{p}_i$$

and thus,

$$\sum \mathbf{f}_i \cdot \frac{\partial \mathbf{p}_i}{\partial \varphi} = \sum \mathbf{f}_i \cdot \mathbf{u} \times \mathbf{p}_i = \mathbf{u} \cdot \sum \mathbf{p}_i \times \mathbf{f}_i = \mathbf{u} \cdot \mathbf{M} = M_k,$$

where $\mathbf{M}$ is the moment of the applied forces with respect to the fixed point O. Substituting these values into Lagrange's equation we have at once,

$$I_3 \omega_k' + (I_2 - I_1)\omega_i \omega_j = M_k,$$

which is the last of Euler's Equations, and the other two follow by symmetry.

As a final example we may consider the application of Lagrange's Equations (24) and (27) to a simple problem in which the imposed conditions vary with the time.

A bead P slides on a smooth straight wire which is rotating with a constant angular velocity ω about a fixed axis which it intersects at right angles at a fixed point O. The bead is acted upon by an applied force $\mathbf{f}$ whose measure upon the wire is a given function $mf(r)$ where r is the distance from O to P along the wire counted in a certain sense as positive. Set up the Lagrangian equation of motion for the bead and show how it may be integrated.

Let us take the plane of rotation of the wire as the XY-plane with the origin at the point O. Then we have,

$$\begin{aligned} \mathbf{p} &\equiv (r \cos \omega t, r \sin \omega t, 0), \\ \mathbf{p}' &\equiv (r' \cos \omega t - \omega r \sin \omega t, r' \sin \omega t + \omega r \cos \omega t, 0), \\ \frac{\partial \mathbf{p}}{\partial r} &\equiv (\cos \omega t, \sin \omega t, 0). \end{aligned}$$

Since $\partial \mathbf{p}/\partial r$ is a unit vector in the positive sense along the wire we have,

$$Q = \mathbf{f} \cdot \frac{\partial \mathbf{p}}{\partial r} = mf(r).$$

Also since $2T = m\,\mathbf{p}'^2 = m(r'^2 + \omega^2 r^2)$, it follows that,

$$\frac{\partial T}{\partial r} = m\omega^2 r, \qquad \frac{\partial T}{\partial r'} = mr', \qquad \left(\frac{\partial T}{\partial r'}\right)' = mr''.$$

With these values Lagrange's equation (24) becomes,

$$r'' - \omega^2 r = f(r).$$

To determine r we multiply through by $2r'$ and integrate, obtaining for a first integral,

$$r'^2 - \omega^2 r^2 = g(r) + a \qquad \text{where} \qquad g(r) = \int_{r_0}^{r} 2f(r)\,dr$$

and a is the constant $r_0'^2 - \omega^2 r_0^2$. Consequently we have finally,

$$t = \int_{r_0}^{r} \frac{dr}{h(r)} \qquad \text{where} \qquad h(r) = \sqrt{g(r) + \omega^2 r^2 + a.}$$

If in particular the given function $f(r)$ is odd so that $f(-r) \equiv -f(r)$, then $f(r)/r$ is a function of r^2 and we may set $F(r^2) = \int f(r)/r\,(dr^2)$ and consequently have,

$$Q = mf(r) = \frac{m}{2}\frac{dF(r^2)}{dr}.$$

Then,

$$U = \frac{m}{2}F(r^2), \qquad L = \frac{m}{2}\{r'^2 + \omega^2 r^2 + F(r^2)\}$$

and with these values Lagrange's equation (27) becomes,

$$r'' - \omega^2 r = r\frac{dF}{d(r^2)}.$$

Since L does not involve t explicitly we may write out at once the integral (28),

$$r'^2 - \omega^2 r^2 = F(r^2) + \frac{2H}{m}.$$

The completion of the solution is now obvious.

EXERCISES

1. Two ends of a rod of length 6 and mass 11 slide on a smooth horizontal circle of radius 5. An insect of mass 1 starts from the midpoint of the rod and walks at a uniform rate to the end of the rod. If the system was initially at rest, through what angle will the rod turn?

   Ans. $\frac{4}{15}\arctan\frac{1}{5} = 3°\,1'$

2. A cylindrical shell of radius b and mass m rolls without slipping inside another cylindrical shell of radius a and mass M, the latter shell being free to rotate about its axis which is fixed and horizontal. If θ is the angle which the plane of the two axes makes with the vertical, show from Lagrange's equations that in the motion under gravity,

$$(2M + m)(a - b)\theta'' + (M + n)g \sin \theta = 0.$$

Describe the motion.

3. Employ Lagrange's equations to solve the exercises of § 79 on the motions of discrete particles.

4. Employ Lagrange's equations to solve Exercises 2–16 of § 84.

5. Employ Lagrange's equations to discuss the plane motion of a rigid body, deriving equations (17), (18), (19) of § 84.

6. Employ Lagrange's equations to solve Exercises 1, 2 of § 85.

7. Employ Lagrange's equations to prove that: If the conditions imposed upon the motion of a set of particles permit a translation of the set parallel to a given axis and if the applied forces are the partial gradients of a function U unaffected by such a translation, then the measure of the momentum of the set of particles on this axis is a constant.

8. Employ Lagrange's equations to prove that: If the conditions imposed upon the motion of a set of particles permit a rotation of the set about a given axis and if the applied forces are the partial gradients of a function U unaffected by such a rotation, then the moment with respect to this axis of the momentum of the set is a constant.

9. Employ Lagrange's equations to deduce the theorems on the problem of n bodies given in § 81.

10. Employ Lagrange's equations to deduce the three integrals § 87, (3), § 87, (6), § 87, (7) of the motion of the top.

11. Employ Lagrange's equations to deduce the equation § 88, (3″) of the motion of a body with a fixed axis.

12. Specialize equations (7), (9), (15), (16) of this article to deduce conditions upon the applied forces under which they will produce equilibrium in sets of points satisfying certain imposed conditions.

13. If the kinetic energy of a set of particles is expressible in the form,

$$T = \Sigma\, v_j(q_j)\, q_j'^2$$

and if the applied forces $\mathbf{f}_i$ possess a force function U expressible in the form,

$$U = \Sigma\, u_j(q_j),$$

show that Lagrange's equations take the form,

$$\frac{d(v_j q_j'^2 - u_j)}{dq_j} = 0$$

and that consequently the motion of the set is determined by the quadratures,

$$t = \int \sqrt{\frac{v_j}{A_j + u_j}}\, dq_j + B_j,$$

where the A_j and the B_j are constants.

14. If, in the motion of a rigid body with one fixed point, the angular velocity $\boldsymbol{\omega}$ and the moment of momentum $\mathbf{L}$ are referred to a frame of reference fixed in the body, show that the moment of momentum is the gradient of the kinetic energy (§ 99) with respect to the angular velocity so that referred to this reference frame Euler's equations become, § 85, (6),

$$(\nabla_\omega T)' + \boldsymbol{\omega} \times \nabla_\omega T = \mathbf{M},$$

or in relative coordinates,

$$\left(\frac{\partial T}{\partial \omega_i}\right)' + \omega_j \frac{\partial T}{\partial \omega_k} - \omega_k \frac{\partial T}{\partial \omega_j} = M_i$$

and two similar equations. Deduce this result from Lagrange's equations.

93. Hamilton's Canonical Equations.

We observed in § 90 that when all the conditions imposed upon the motion of a set of particles are holonomic, certain of the parameters q_j can be eliminated by the aid of the equations of condition and the radius vectors $\mathbf{p}_i$ thus expressed in terms of the remaining q_j and the time t. This gives,

$$(1) \qquad \mathbf{p}_i = \mathbf{p}_i(q_1, q_2, \cdots, q_m, t), \qquad i = 1, 2, \cdots, n,$$

in which the q_j appearing are independent parameters and m is the number of degrees of freedom of the motion. The equations of motion of the set in Lagrange's Second Form are,

$$(2) \qquad \left(\frac{\partial T}{\partial q_j'}\right)' - \frac{\partial T}{\partial q_j} = Q_j \qquad \text{where} \qquad Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{p}_i}{\partial q_j},$$
$$j = 1, 2, \cdots, m.$$

These m equations are evidently of the second order in the q_j and the problem of their determination is thus of order $2m$.

Poisson originally made the suggestion that for this holonomic case equations (2) be transformed from a set of m differential equations of the second order to $2m$ differential equations of the first order by introducing, in addition to the m dependent variables q_j already appearing, the additional m dependent variables,

$$(3) \qquad p_j = \frac{\partial T}{\partial q_j'}, \qquad j = 1, 2, \cdots, m$$

in which T is the kinetic energy of the set. The complete development of this transformation is, however, due to Hamilton (1835). The m Lagrangian coördinates q_j are known as the *position coördinates* of the set P_i since at any given instant they will by equations (1) determine the *positions* of the particles. The $2m$ quantities q_j and p_j taken together are called the *phase coördinates* of the set because, as we shall see, at any given instant they will determine both the positions and the velocities of the particles; and the positions and velocities together constitute the *phase*. The $2m + 1$ quantities q_j, p_j, t are called the *state coördinates* of the set since they determine the time as well as the positions and velocities of the particles; and these taken all together constitute the *state* of the set.

To carry out *Hamilton's Transformation* of equations (2) we first observe that from equations (1),

$$(4) \qquad \mathbf{p}_i' = \sum_j \frac{\partial \mathbf{p}_i}{\partial q_j} q_j' + \frac{\partial \mathbf{p}_i}{\partial t}, \qquad i = 1, 2, \cdots, n$$

and that consequently we have,

$$T = \sum \frac{m_i}{2} \mathbf{p}_i'^2 = \sum_i \frac{m_i}{2} \left(\sum_j \frac{\partial \mathbf{p}_i}{\partial q_j} q_j' + \frac{\partial \mathbf{p}_i}{\partial t} \right)^2,$$

which may be expanded in the form,

$$(5) \qquad T = a + \sum_j b_j q_j' + \tfrac{1}{2} \sum_j \sum_k c_{jk} q_j' q_k', \qquad c_{jk} = c_{kj},$$

where a, the b_j and the c_{jk} are functions of the q_j and t but not of the q_j'. We now have the new variables p_j given by expressions linear in the q_j',

$$(6) \qquad p_j = \frac{\partial T}{\partial q_j'} = b_j + \sum_k c_{jk} q_k', \qquad j = 1, 2, \cdots, m,$$

and we shall show that these linear equations can always be

solved for the q'_j. Inspection of equations (4) shows that the last summation in equation (5) has the value,

$$\sum_j \sum_k c_{jk} q'_j q'_k = \sum_i m_i \left(\sum_j \frac{\partial \mathbf{p}_i}{\partial q_j} q'_j \right)^2$$

and that it is consequently a *positive definite* quadratic form in the q'_j. In fact this expression could only vanish if we had every,

$$\sum_j \frac{\partial \mathbf{p}_i}{\partial q_j} q'_j = 0, \qquad i = 1, 2, \cdots, n.$$

If this were so for values of the q'_j not all zero we could by giving to the q_j a set of virtual increments δq_j proportional to the q'_j, vary the q_j without moving any particle of the set, thus showing that the set would have less than m degrees of freedom, contrary to hypothesis. Since this quadratic form is definite it follows that the determinant of the c_{jk} can not vanish and this is exactly the condition that equations (6) be solvable for the q'_j. Having solved equations (6) for the q'_j, their values in terms of the q_j, p_j, t may be substituted into Lagrange's equations (2) after the partial differentiations have been performed and thus yield the following $2m$ first order differential equations in the q_j and p_j.

$$(7) \qquad p_j = \frac{\partial T}{\partial q'_j}, \qquad p'_j - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1, 2, \cdots, m.$$

The transformation suggested by Poisson is now complete, but Hamilton has given these equations a much more symmetric form by introducing the function,

$$(8) \qquad K = \Sigma\, p_j q'_j - T.$$

If now, holding t fixed, we give to the q_j and q'_j arbitrary infinitesimal increments δq_j and $\delta q'_j$, we may indicate by δK and δp_j the corresponding variations in K and the p_j. Then on the assumption that T is expressed in terms of the q_j, q'_j, and t we have,

$$\delta K = \sum p_j\, \delta q'_j + \sum q'_j\, \delta p_j - \sum \frac{\partial T}{\partial q_j}\, \delta q_j - \sum \frac{\partial T}{\partial q'_j}\, \delta q'_j,$$

which by the first of equations (7) becomes,

$$\delta K = \sum q'_j\, \delta p_j - \sum \frac{\partial T}{\partial q_j}\, \delta q_j.$$

On the other hand, if we express K in terms of the q_j, p_j, and t we have,

$$\delta K = \sum \frac{\partial K}{\partial p_j} \delta p_j + \sum \frac{\partial K}{\partial q_j} \delta q_j.$$

These two expressions for δK can differ only by infinitesimals of higher order and since the δq_j and δp_j may be chosen arbitrarily, we may equate their coefficients in the two expressions. Thus we have,

$$\frac{\partial K}{\partial p_j} = q_j', \qquad \frac{\partial K}{\partial q_j} = -\frac{\partial T}{\partial q_j}, \qquad j = 1, 2, \cdots, m.$$

With the aid of the latter of equations (7) these become the important,

Hamilton's Canonical Equations.

$$(9) \qquad q_j' = \frac{\partial K}{\partial p_j}, \qquad p_j' = Q_j - \frac{\partial K}{\partial q_j}, \qquad j = 1, 2, \cdots, m.$$

In setting up these canonical equations in any particular case it is essential to follow carefully the procedure by which they were obtained. Thus we first express T in terms of the q_j, q_j', t and then write down equations (3) which we solve for the q_j' in terms of the p_j, q_j, t. These values for the q_j' are then substituted into equations (8) and the resulting expression for K differentiated with respect to the p_j and q_j to obtain the values for $\partial K/\partial p_j$ and $\partial K/\partial q_j$ employed in equations (9). When the values of the q_j' have also been substituted into the Q_j, equations (9) take the form of $2m$ differential equations of the first order for the determination of the $2m$ phase coördinates p_j and q_j in terms of the time.

As in the case of Lagrange's equations, by specialization of the form of Q_j Hamilton's Canonical Equations take on particularly symmetric and convenient forms applicable to certain cases. Thus it may happen that each force $\mathbf{f}_i$ is the partial gradient (§ 99) with respect to $\mathbf{p}_i$ of a certain scalar function,

$$U = U(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, t).$$

In this case we have,

$$\mathbf{f}_i = \nabla_i U \equiv \left(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}, \frac{\partial U}{\partial z_i}\right) \qquad \text{where} \qquad \mathbf{p}_i \equiv (x_i, y_i, z_i)$$

and consequently,

$$Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} = \sum_i \nabla_i U \cdot \frac{\partial \mathbf{p}_i}{\partial q_j} = \frac{\partial U}{\partial q_j}, \qquad j = 1, 2, \cdots, m.$$

If we now define *Hamilton's Function* H by the equation,

$$H = K - U = \Sigma\, p_j q_j' - L,$$

where L is Lagrange's Function, then,

$$\frac{\partial K}{\partial p_j} = \frac{\partial H}{\partial p_j}, \qquad \frac{\partial K}{\partial q_j} - Q_j = \frac{\partial H}{\partial q_j}$$

and Hamilton's Equations (9) take the extremely symmetric form,

$$(10) \qquad q_j' = \frac{\partial H}{\partial p_j}, \qquad p_j' = -\frac{\partial H}{\partial q_j}, \qquad j = 1, 2, \cdots, m.$$

If the conditions imposed upon the motion of the set of particles do not vary with the time then the expressions (1) for the $\mathbf{p}_i$ will not contain t explicitly and in equation (5) a and every b_j will be zero, T being in this case a quadratic form in the q_j'. Employing Euler's theorem for homogeneous functions we may then write,

$$K = \sum p_j q_j' - T = \sum \frac{\partial T}{\partial q_j'} q_j' - T = 2T - T = T.$$

There is thus in this case no necessity of introducing the function K since it is identical with T, while H has now the value, $H = T - U$. This is the case which Hamilton originally discussed.

Let us now assume not only that the imposed conditions do not vary with the time, but also that each force $\mathbf{f}_i$ is the partial gradient with respect to $\mathbf{p}_i$ of the scalar function $U = U(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n)$ which in this case does not contain the time explicitly. When U is thus free from t it is called the *force function* and the system of forces $\mathbf{f}_i$ is said to be *conservative*. Since the hypotheses of the preceding paragraph are satisfied, we have $K = T$ and since the hypotheses of the paragraph before that are also satisfied Hamilton's Equations take the form (10) where H now stands for $T - U$. Since neither T nor U contains the time

explicitly we have,

$$H' = \sum \left(\frac{\partial H}{\partial q_i} q_i' + \frac{\partial H}{\partial p_i} p_i' \right).$$

By equations (10) every term in this summation vanishes and we have,

$$(11) \qquad H' = 0, \qquad H = H_0, \qquad (H_0 \text{ const.}).$$

This is the *energy integral* of equations (10).

We conclude this discussion of Hamilton's Canonical Equations by employing them to discuss the central motion of a particle P of mass m repelled from a point O with a force of amount $mf(\rho)$ where f is a given scalar function of the distance $OP = \rho$. We know from the first theorem of § 40 that P remains in a fixed plane through O and we may therefore take as the two parameters q_1 and q_2 respectively the two polar coördinates ρ and θ of P in this plane, the point O being the pole of the coördinate system. Since there are in this case no imposed conditions, we have $K = T$ and,

$$T = \tfrac{1}{2} m\,\mathbf{p}'^2 = \frac{m}{2}(\rho'^2 + \rho^2\theta'^2),$$

which is a quadratic form in ρ' and θ', as previously noted. The force $\mathbf{f}$ acting on the particle is by the statement of the problem,

$$\mathbf{f} = \frac{m}{\rho} f(\rho)\,\mathbf{p}, \qquad \mathbf{p} = OP$$

and we shall show that in this case there is a force function U,

$$U = \int_{\rho_0}^{\rho} mf(\rho)\,d\rho = F(\rho).$$

For with U so defined we have,

$$\nabla U = \frac{dF}{d\rho} \nabla\rho = mf(\rho)\,\nabla\rho$$

and since,

$$\rho = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \qquad \nabla\rho \equiv \left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho} \right) = \frac{\mathbf{p}}{\rho},$$

it follows that,

$$\nabla U = \frac{m}{\rho} f(\rho)\, \mathbf{p} = \mathbf{f}.$$

Following the indicated procedure for setting up Hamilton's Equations, we now compute,

$$p_1 = \frac{\partial T}{\partial \rho'} = m\rho', \qquad p_2 = \frac{\partial T}{\partial \theta'} = m\rho^2\theta'$$

and solve these for ρ' and θ' obtaining,

$$\rho' = \frac{p_1}{m}, \qquad \theta' = \frac{p_2}{m\rho^2}.$$

In terms of p_1 and p_2 we now have,

$$H = T - U = \frac{1}{2m}\left(p_1^2 + \frac{p_2^2}{\rho^2}\right) - F(\rho)$$

and Hamilton's Equations (10) become,

$$\rho' = \frac{\partial H}{\partial p_1} = \frac{p_1}{m}, \qquad \theta' = \frac{\partial H}{\partial p_2} = \frac{p_2}{m\rho^2},$$

$$p_1' = -\frac{\partial H}{\partial \rho} = \frac{p_2^2}{m\rho^3} + F'(\rho), \qquad p_2' = -\frac{\partial H}{\partial \theta} = 0.$$

The solution of these four differential equations for ρ, θ, p_1, p_2 in terms of t and the constants of integration will completely determine the motion. To do this we observe first that from the last equation p_2 is a constant which we may call Cm. The second equation then yields the integral of areas,

$$\rho^2\theta' = C.$$

As observed in equation (11), we also have the energy integral $H = H_0$ which is here,

$$\frac{m}{2}(\rho'^2 + \rho^2\theta'^2) - F(\rho) = H_0.$$

Eliminating p_1 between the first and third equations gives us with the aid of the integral of areas, the equation,

$$m\rho'' - \frac{C^2 m}{\rho^3} - F(\rho) = 0$$

for the determination of ρ in terms of t. The solution of this is readily reduced to quadratures by multiplying through by the integrating factor ρ' and when we thus have ρ in terms of t the integral of areas will give θ in terms of the time by an additional quadrature. The motion is thus completely determined.

EXERCISES

1. Show that the quantity K when expressed in terms of the q_j, p_j, t as in equations (9) has the form, here written for $m = 2$,

$$K = -a - \frac{1}{2C}\begin{vmatrix} 0 & p_1 - b_1 & p_2 - b_2 \\ p_1 - b_1 & c_{11} & c_{12} \\ p_2 - b_2 & c_{21} & c_{22} \end{vmatrix},$$

where

$$C = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

and where a, the b_j and the c_{jk} have the meanings employed in equation (5). Show that if the imposed conditions are independent of the time this reduces to,

$$K = T = -\frac{1}{2C}\begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & c_{11} & c_{12} \\ p_2 & c_{21} & c_{22} \end{vmatrix}.$$

2. *Reciprocal Functions.* Let $F(x_1, x_2, \cdots, x_m, u_1, u_2, \cdots, u_m)$ be a function of the $2m$ variables x_j, u_j such that the equations,

$$y_j = \frac{\partial F}{\partial x_j},$$

defining the new variables $y_1, y_2, \cdots, y_m$ are independent in the x_j. If then the new function,

$$G = \Sigma\, x_j y_j - F$$

is expressed in terms of the variables y_j, u_j, show that,

$$\frac{\partial G}{\partial y_j} = x_j, \qquad \frac{\partial G}{\partial u_j} = -\frac{\partial F}{\partial u_j}.$$

I.e., show that the hypotheses,

$$F + G = \Sigma\, x_j y_j, \qquad \frac{\partial F}{\partial x_j} = y_j, \qquad j = 1, 2, \cdots, m$$

entail,

$$G + F = \Sigma\, y_j x_j, \qquad \frac{\partial G}{\partial y_j} = x_j, \qquad j = 1, 2, \cdots, m$$

and vice versa, and that either set of hypotheses yields,

$$\frac{\partial F}{\partial u_j} + \frac{\partial G}{\partial u_j} = 0, \qquad j = 1, 2, \quad , m.$$

The functions F and G are said to be *reciprocal.*

3. Show that if the function F of Problem 2 is homogeneous of degree two in the x_j, then it is self-reciprocal.
4. Show that the functions T and K of this article are reciprocal. Show that the functions H and L of this article are reciprocal.
5. Using the properties of reciprocal functions derive Lagrange's Equations (2) from Hamilton's Equations (9) and similarly Lagrange's Equations § 92, (27) from Hamilton's Equations (10).
6. Employ Hamilton's Equations to deduce Euler's Equations § 85, (7) for the motion of a rigid body with one fixed point.
7. Solve the exercises of § 93 using Hamilton's instead of Lagrange's Equations.
8. Show that if Hamilton's Function H be expressed in terms of the q_j, p_j and t as in equations (10), then the total derivative of H with respect to the time is obtained by differentiating partially with respect to the time, i.e.,

$$H' = \frac{\partial H}{\partial t}.$$

94. The Variation of a Motion.

Closely associated with the concept of the virtual displacements of a set of particles is the concept of the *variation* of its motion and the variation of quantities that depend on the motion. This is a more involved concept than that of virtual displacement and the rigorous analytic treatment of it would require a development of the branch of mathematics known as the Calculus of Variations. However by the following descriptive treatment an idea of the variation of a motion and of associated quantities may be gained which will be sufficient for our purposes and which has possibly a stronger intuitive appeal than the more modern attack.

A set of particles with given initial positions and velocities, acted upon by given forces, and subject to given imposed conditions will in a certain interval of time pass by a uniquely determined motion from the initial to the terminal positions of the particles. We shall call this the *unvaried motion* of the set. We have here a continuous succession of positions of the set and we may assign to each position a new slightly different position; that

is we may assign to each radius vector $\mathbf{p}_i$ a variation $\delta\mathbf{p}_i$ which we regard as arbitrarily small. The succession of these new positions forms the *varied paths* of the particles. We assume not only that the positions on the varied paths are arbitrarily near the corresponding positions on the unvaried paths, but also that they form a continuous succession and that the directions of motion of the varied paths are arbitrarily near to the directions of motion at the corresponding points of the unvaried motion. We do not assume that the varied motions satisfy the imposed conditions nor need they even be physically possible. We do not assume that the positions on the varied motion are occupied at the same moment as the corresponding positions on the unvaried motion, but we do assume that the speed with which the varied paths are traced out is such that the time interval δt from the instant when the particles pass through the positions on the unvaried paths to the instant when the corresponding positions on the varied paths are traced out shall be arbitrarily small.

Thus in setting up the varied motion we first choose a continuous succession of values of the $\delta\mathbf{p}_i$ for the successive positions of the particles. Next a continuous succession of values of δt is chosen for these positions so that if t is the instant at which the positions $\mathbf{p}_i$ are occupied in the unvaried motion then $t + \delta t$ is to be the instant at which the positions $\mathbf{p}_i + \delta\mathbf{p}_i$ are occupied in the varied motion. The entire circumstances of the varied motion are thus determined.

By the variation $\delta\varphi$ of any quantity φ dependent on the motion we mean an approximation to the difference between its value for the varied motion and its value for the unvaried motion, this approximation to be correct within an infinitesimal of higher order than the above principal infinitesimals $\delta\mathbf{p}_i$ and δt. Thus for the variation $\delta(\mathbf{p}_i')$ of the velocity $\mathbf{p}_i'$ of any particle P_i we have by this definition,

$$\begin{aligned}\delta(\mathbf{p}_i') = \frac{d(\mathbf{p}_i + \delta\mathbf{p}_i)}{d(t + \delta t)} - \frac{d\mathbf{p}_i}{dt} &= \frac{\mathbf{p}_i' - (\delta\mathbf{p}_i)'}{1 + (\delta t)'} - \mathbf{p}_i' \\ &= \{\mathbf{p}_i' - (\delta\mathbf{p}_i)'\}\{1 + (\delta t)'\}^{-1} - \mathbf{p}_i'.\end{aligned}$$

Expanding this last expression and dropping infinitesimals of higher order, we have the formula,

$$(1) \qquad \delta(\mathbf{p}_i') = (\delta\mathbf{p}_i)' - \mathbf{p}_i'(\delta t)'.$$

In case δt remains constant during the motion this reduces to,

$$(1') \qquad \delta(\mathbf{p}_i') = (\delta \mathbf{p}_i)'.$$

Also if $f(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, \mathbf{p}_1', \mathbf{p}_2', \cdots, \mathbf{p}_n', t)$ is a scalar function of the state of the motion, its variation may be derived in a fashion entirely analogous to that employed in deriving the total differential df and we have,

$$(2) \qquad \delta f = \sum \nabla_i f \cdot \delta \mathbf{p}_i + \sum \nabla_i' f \cdot \delta \mathbf{p}_i' + \frac{\partial f}{\partial t} \delta t,$$

where $\nabla_i f$ is the partial gradient (§ 99) of f with respect to $\mathbf{p}_i$ and $\nabla_i' f$ is the partial gradient of f with respect to $\mathbf{p}_i'$. For a varied motion in which δt is chosen constantly zero or in case f does not involve t, this reduces to,

$$(2') \qquad \delta f = \Sigma \nabla_i f \cdot \delta \mathbf{p}_i + \Sigma \nabla_i' f \cdot \delta \mathbf{p}_i'.$$

Or again if F is a scalar *functional* of the motion, i.e. a quantity dependent on the entire motion such as,

$$F = \int_{t_1}^{t_2} f(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, \mathbf{p}_1', \mathbf{p}_2', \cdots, \mathbf{p}_n', t)\, dt,$$

we may derive its variation δF. By definition f and $f + \delta f$ are within an infinitesimal of higher order the values assumed by the function f at corresponding positions of the unvaried and varied motions, while t and $t + \delta t$ are exactly the corresponding values of the time. Thus we have for δF,

$$\delta F = \int_{t_1+\delta t_1}^{t_2+\delta t_2} (f + \delta f)\, d(t + \delta t) - \int_{t_1}^{t_2} f\, dt,$$

the first integration being over the varied motion and the second over the unvaried motion. If we introduce the time t of the unvaried motion as the variable of integration in the first integral, we have,

$$\delta F = \int_{t_1}^{t_2} \{f + \delta f\}\{1 + (\delta t)'\}\, dt - \int_{t_1}^{t_2} f\, dt,$$

where now δf may be taken from equation (2). Expanding and dropping infinitesimals of higher order we have the formula,

$$(3) \qquad \delta F = \int_{t_1}^{t_2} \{\delta f + f(\delta t)'\}\, dt.$$

In case δt remains constant during the motion this reduces to,

$$(3') \qquad \delta F = \int_{t_1}^{t_2} \delta f\, dt.$$

After the independent variations $\delta \mathbf{p}_i$ have been chosen, there are two particular methods of choosing δt which are of especial importance. As the first method we may vary the speed along the varied paths in such a way that corresponding positions of the varied and unvaried paths are traced out simultaneously. In other words we may choose δt identically zero. This readily visualized type of variation of the motion is that employed in *Hamilton's Principle* explained in the following article. Equations (1), (2), (3) of the present article here take on the simpler forms (1′), (2′), (3′). As a second type of variation of the motion we may so vary the speed along the varied paths that the kinetic energy of the set of particles has a variation δT constantly equal to the elementary work which would be done by the applied forces if the particles were moved from their positions on the unvaried paths to the corresponding positions on the varied paths. This elementary work is,

$$(4) \qquad \delta W = \Sigma\, \mathbf{f}_i \cdot \delta \mathbf{p}_i,$$

where $\mathbf{f}_i$ is the applied force acting on the particle P_i. For this type of variation we then have constantly,

$$(5) \qquad \delta T = \delta W.$$

When the variations $\delta \mathbf{p}_i$ have been chosen equation (5) may be used to determine δt and by equations (1), (2), (3) all the circumstances of the varied motion are then fixed. This is the type of variation employed in the *Principle of Least Action* explained in the second following article.

If it happens that there exists a force function, $U(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n)$ for the applied forces $\mathbf{f}_i$, we have, $\mathbf{f}_i = \nabla_i U$ and applying formula

(2) to U we find,

$$\delta U = \Sigma \nabla_i U \cdot \delta \mathbf{p}_i = \Sigma \mathbf{f}_i \cdot \delta \mathbf{p}_i = \delta W,$$

and equation (5) becomes,

$$\delta(T - U) = 0. \tag{6}$$

Since T is the kinetic energy of the set and $-U$ is the potential energy it thus appears that in this case this second type of variation of a motion may be expressed by the statement that δt is so chosen that the total energy $T - U$ has the same value at corresponding positions of the varied and unvaried motions.

Modern writers on the *Calculus of Variations* have found it desirable to employ a definite analytic form for the variation δf of a function f. Thus they write,

$$\delta f = \alpha \omega(t),$$

where α is an infinitesimal parameter and $\omega(t)$ is an arbitrary function of t subject only to certain conditions of continuity, differentiability, etc. Thus δf becomes a function of two variables α and t and our formulas (1), (2), (3) are derived as consequences of the usual rules for operating on a function of two variables. We can not here enter upon the further developments of this very extensive theory.

EXERCISES

1. In the holonomic motion of a set of particles the motion is determined by expressing m independent parameters q_j in terms of the time, the positions of the particles being then given by equations,

$$\mathbf{p}_i = \mathbf{p}_i(q_1, q_2, \cdots, q_m, t), \qquad i = 1, 2, \cdots, n.$$

Discuss in full the variation of the motion when independent variations δq_j and δt are assigned to the q_j and t. Show that,

$$\delta(q_j') = (\delta q_j)' - q_j'(\delta t)',$$

$$\delta \mathbf{p}_i = \sum_j \frac{\partial \mathbf{p}_i}{\partial q_j}\delta q_j + \frac{\partial \mathbf{p}_i}{\partial t}\delta t,$$

$$\delta(\mathbf{p}_i') = (\delta \mathbf{p}_i)' - \mathbf{p}_i'(\delta t)'.$$

If $f(q_1, q_2, \cdots, q_m, q_1', q_2', \cdots, q_m', t)$ is any function of the motion, show that,

$$\delta f = \sum \frac{\partial f}{\partial q_j}\delta q_j + \sum \frac{\partial f}{\partial q_j'}\delta q_j' + \frac{\partial f}{\partial t}\delta t.$$

If F is the functional,

$$F = \int_{t_1}^{t_2} f\, dt,$$

show that,

$$\delta F = \int_{t_1}^{t_2} \{\delta f + f(\delta t)'\}\, dt$$

and that by integration by parts this reduces to,

$$\delta F = \int_{t_1}^{t_2} \Sigma \left\{ \frac{\partial f}{\partial q_j} - \left(\frac{\partial f}{\partial q_j'} \right)' \right\} (\delta q_j - q_j'\, \delta t)\, dt$$
$$+ \left[\Sigma \frac{\partial f}{\partial q_j'} (\delta q_j - q_j'\, \delta t) \right]_{t_1}^{t_2} + \left[f\, dt \right]_{t_1}^{t_2}.$$

95. Hamilton's Principle.

Let us consider a set of n particles P_i subject to certain imposed conditions and acted upon by certain applied forces $\mathbf{f}_i$ and moving in a definite fashion from its position A at the instant t_1 to its position B at the instant t_2. As in the previous article, we apply variations $\delta\mathbf{p}_i$ to the intermediate positions of the particles and a variation δt to the time, but now *the initial and terminal positions A and B are left unvaried.* By formulas § 94, (1) and (2) we compute the corresponding variation of the kinetic energy T of the set finding,

$$\begin{aligned} \delta T &= \delta\, \Sigma\, \tfrac{1}{2} m_i\, \mathbf{p}_i'^2 = \Sigma\, m_i\, \mathbf{p}_i' \cdot (\delta\mathbf{p}_i)' - (\Sigma\, m_i\, \mathbf{p}_i'^2)(\delta t)' \\ &= \Sigma\, m_i\, \mathbf{p}_i' \cdot (\delta\mathbf{p}_i)' - 2T(\delta t)'. \end{aligned}$$

Evidently we may write this equation in the form,

$$(1) \qquad 2T(\delta t)' + \delta T = -\, \Sigma\, m_i\, \mathbf{p}_i'' \cdot \delta\mathbf{p}_i + (\Sigma\, m_i\, \mathbf{p}_i' \cdot \delta\mathbf{p}_i)'.$$

We recall also that if the particles were moved from their positions on the unvaried motion to the corresponding positions on the varied motion the elementary work of the applied forces would be,

$$(2) \qquad \delta W = \Sigma\, \mathbf{f}_i \cdot \delta\mathbf{p}_i.$$

If we now add equations (1) and (2) member for member and integrate with respect to the time from t_1 to t_2, we find the highly

important identity,

$$(3)\quad \int_{t_1}^{t_2} \{2T(\delta t)' + \delta T + \delta W\}\, dt = \int_{t_1}^{t_2} \Sigma\, (\mathbf{f}_i - m_i\, \mathbf{p}_i'') \cdot \delta \mathbf{p}_i\, dt,$$

where the integral of the exact derivative appearing at the end of equation (1) vanishes due to the fact that the end positions A and B are unvaried and that consequently every $\delta \mathbf{p}_i$ is zero for $t = t_1$ and $t = t_2$.

If now the variations $\delta \mathbf{p}_i$ are at each instant submitted to the conditions imposed upon the virtual displacements of the unvaried motion at that instant, then we may apply d'Alembert's Principle (§ 91) to the right member of the above identity and find the important relation,

$$(4)\qquad \int_{t_1}^{t_2} \{2T(\delta t)' + \delta T + \delta W\}\, dt = 0.$$

Actually, of course, this equation merely amounts to a statement that the first member is an infinitesimal of higher order than our principal infinitesimals $\delta \mathbf{p}_i$ and δt, because in the derivation of the equation we have repeatedly dropped infinitesimals of higher order. The reader should carefully note that we have here required that the variations $\delta \mathbf{p}_i$ shall at each instant form a permitted set of virtual displacements of the unvaried motion. This is *not* in general equivalent to requiring that the varied motion satisfy the conditions imposed upon the unvaried motion, the two requirements being equivalent only when the imposed conditions are holonomic and do not vary with the time.

In deriving equation (4) we have limited the variations $\delta \mathbf{p}_i$ to being virtual displacements of the original motion, but this still leaves the variation δt of the time at our disposal and by choosing it in various ways we may derive various forms of the equation. One of the most important of these is that obtained when we choose *δt identically zero.* We have at once in that case from equation (4),

Hamilton's Principle (1835).

$$(5)\qquad \int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0.$$

Here δT is the variation of the kinetic energy from its value for the unvaried motion to its value at the same instant for the varied motion; δW is the virtual work of the applied forces when the particles are moved from their positions on the actual motion to their positions at the same instant on the varied motion; and the positions of the particles on the varied motion differ from their positions at the same instant on the unvaried motion by virtual displacements permitted by the conditions imposed upon the unvaried motion at that instant; while at the instants t_1 and t_2 the varied and unvaried positions coincide.

Hamilton's Principle as here stated is of extremely wide application, the only type of motion usually studied to which it is not applicable being that in which the imposed conditions are unilateral, i.e. expressed by inequalities. In fact we shall show that it may be used instead of Newton's Second Law of Motion as a basic principle upon which, with certain minor auxiliary hypotheses, the whole of classical mechanics may be made to depend.

In certain cases Hamilton's Principle takes on simpler forms more convenient in application and more readily visualized. Thus if there exists a scalar function U,

$$U = U(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n, t)$$

such that at each instant each applied force $\mathbf{f}_i$ is the partial gradient (§ 99) of U with respect to the corresponding $\mathbf{p}_i$, then the virtual work δW of the applied forces as the particles are moved from their positions on the unvaried motion to the corresponding positions on the varied motion becomes simply the variation of U. For applying formula § 94, (2′) to U we have,

$$\delta U = \Sigma\, \nabla_i U \cdot \delta \mathbf{p}_i = \Sigma\, \mathbf{f}_i \cdot \delta \mathbf{p}_i = \delta W$$

and equation (5) takes the form,

$$\int_{t_1}^{t_2} (\delta T + \delta U)\, dt = 0.$$

If we define the scalar functional I by the formula,

$$I = \int_{t_1}^{t_2} L\, dt$$

where L is Lagrange's Function $T + U$, then Hamilton's Principle by formula § 94, (3′) becomes simply,

$$\delta I = 0. \tag{6}$$

In the form (6) Hamilton's Principle has a readily visualized significance which we shall indicate without attempting a rigorous discussion. If the actual motion of the set of particles be varied in the manner described above, we may seek among all these motions that one for which the quantity I has the least value. Just as in the calculus a necessary condition for a relative minimum of a differentiable function is the vanishing of its total differential, so in this case a necessary condition for a minimum of I is that its variation δI be zero. But by equation (6) this condition is satisfied for the actual motion. It thus appears that,

The actual motion of the set of particles satisfies the necessary condition, $\delta I = 0$, that the functional I be less along it than for any of the varied motions above described.

This, of course, by no means establishes that I is really a relative minimum for the actual motion, but it has been shown that this will be the case under certain auxiliary conditions and always will be if the interval t_1, t_2 be taken sufficiently short.

In § 91 we remarked that d'Alembert's Principle can be used instead of Newton's Second Law as a basic principle on which the whole of classical mechanics may be made to depend, and in fact this is the procedure followed by Lagrange. In § 92 and § 93 we deduced Lagrange's and Hamilton's Equations from d'Alembert's Principle and in the present article Hamilton's Principle has been so deduced. It is equally true that Hamilton's Principle may form the basic postulate of classical mechanics, for we may deduce d'Alembert's Principle from that of Hamilton. To do this we assume equation (5) together with the hypotheses there made concerning the variations $\delta \mathbf{p}_i$ and δt and find from equations (1) and (2),

$$\int_{t_1}^{t_2} \{\Sigma\, (\mathbf{f}_i - m_i\, \mathbf{p}_i'') \cdot \delta \mathbf{p}_i\}\, dt = 0.$$

Now, whether the motion be holonomic or nonholonomic, the conditions imposed upon the variations $\delta \mathbf{p}_i$ requiring that they

be virtual displacements permitted by the imposed conditions at the instant are in the form § 90, (6) of linear homogeneous equations in the coördinates of the $\delta\mathbf{p}_i$. So if the variations $\delta\mathbf{p}_i$ form a permitted set of virtual displacements the same will be true of $h\,\delta\mathbf{p}_i$, where h may be taken as an arbitrary continuous function of t. Thus we also have,

$$\int_{t_1}^{t_2} h\{\Sigma\,(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\delta\mathbf{p}_i\}\,dt = 0.$$

Now if the quantity $\Sigma\,(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\delta\mathbf{p}_i$ were ever different from zero, we could so choose the function h as always to have the sign of this quantity and the above integrand would never be negative. Since the integral is zero the integrand must thus vanish identically and we have at each instant,

$$\Sigma\,(\mathbf{f}_i - m_i\,\mathbf{p}_i'')\cdot\delta\mathbf{p}_i = 0$$

for all variations $\delta\mathbf{p}_i$ compatible with the constraints. This is exactly d'Alembert's Principle, for since it applies only to individual instants the fact that the variations $\delta\mathbf{p}_i$ must be differentiable with respect to t and vanish for $t = t_1$ and $t = t_2$ is without effect.

Lagrange's Equations have already been derived from d'Alembert's Principle and so now in effect from Hamilton's Principle. The direct derivation of Lagrange's Equations from Hamilton's Principle is also readily accomplished by a modification of the "h" method just used in the proof of d'Alembert's Principle.

EXERCISES

1. Apply Hamilton's Principle to a set of n particles P_i whose motion is holonomic with m degrees of freedom, i.e.,

$$\mathbf{p}_i = \mathbf{p}_i(q_1, q_2, \cdots, q_m, t)$$

and derive Lagrange's Equations in the second form,

$$\left(\frac{\partial T}{\partial q_j'}\right)' - \frac{\partial T}{\partial q_j} = Q_j, \qquad j = 1, 2, \cdots, m,$$

where,

$$T = \sum \frac{1}{2}\,m_i\mathbf{p}_i'^2, \qquad Q_j = \sum_i \mathbf{f}_i\cdot\frac{\partial\mathbf{p}_i}{\partial q_j}.$$

2. *Hamilton's Differential Equation.* Employ the result of § 94, Prob. 1 and equations § 92, (27) to show that,

$$\delta I = \delta \int^{t_2} L\, dt = \Big[\sum p_j(\delta q_j - q'_j\, \delta t) + L\, \delta t \Big]_{t_1}^{t_2},$$

where,

$$p_j \quad \frac{\partial L}{\partial q'_j}.$$

For a variation of the motion in which δt is identically zero this reduces to,

$$\delta I = \Sigma\, (p_j\, \delta q_j - b_j\, \delta a_j),$$

in which the a_j and b_j are the values of the q_j and p_j at the instant t_1, and in which t_2 has been taken as the general instant t. Show that if the motion of the set of particles is known, we may express each q_j and p_j as a function of the a_j, b_j, and t. Assuming that we may solve these $2m$ equations for the p_j and b_j and thus write,

$$I = I(q_1, q_2, \cdots, q_m, a_1, a_2, \cdots, a_m, t),$$

show that,

$$\frac{\partial I}{\partial q_j} = p_j, \qquad \frac{\partial I}{\partial a_j} = -\, b_j,$$

and hence that,

$$L = \frac{\partial I}{\partial t} + \sum p_j q'_j.$$

If now H is Hamilton's Function, § 93, (10),

$$H = \Sigma\, p_j q'_j - L,$$

show that I satisfies *Hamilton's Differential Equation,*

$$\frac{\partial I}{\partial t} + H\left(q_1, q_2, \cdots, q_m, \frac{\partial I}{\partial q_1}, \frac{\partial I}{\partial q_2}, \cdots, \frac{\partial I}{\partial q_m}, t \right) = 0,$$

in which the form of the function H is determined by writing,

$$H = H(q_1, q_2, \cdots, q_m, p_1, p_2, \cdots, p_m, t).$$

96. The Principle of Least Action.

In the preceding article we obtained Hamilton's Principle from equation § 95, (4) by requiring that the variation δt of the time be identically zero, thus requiring that corresponding positions on the varied and unvaried paths be passed through simultaneously. Another method of choosing δt which yields an important result

is to vary t so that the variation δT of the kinetic energy remains constantly equal to what the virtual work of the applied forces would be if the particles were moved from their positions on the unvaried paths to the corresponding positions on the varied paths. We are still assuming that the variations $\delta \mathbf{p}_i$ have been so chosen as to constitute at each instant a permitted set of virtual displacements of the original motion and that the variations of the initial and terminal positions are zero. But it will in general no longer be true that corresponding positions on the unvaried and varied motions are passed through simultaneously. Under these hypotheses we may employ equation § 95, (4), setting $\delta W = \delta T$ and find at once,

$$\int_{t_1}^{t_2} \{2\,\delta T + 2T(\delta t)'\}\,dt = 0,$$

which by formula § 94, (3) reduces exactly to what is known as the,

Principle of Least Action.

$$\delta \int_{t_1}^{t_2} 2T\,dt = 0. \tag{1}$$

We define the *action* J of the set of particles during the time interval t_1, t_2 as the functional,

$$J = \int_{t_1}^{t_2} 2T\,dt$$

and have from equation (1), $\delta J = 0$, and the Principle of Least Action may be stated as follows,

The actual motion of a set of particles satisfies the necessary condition $\delta J = 0$ that the action J be less along it than for any of the varied motions above described.

As in the case of I in Hamilton's Principle, it may be shown that J will really be less along the actual motion than along these varied motions if certain auxiliary conditions are satisfied and always will be if the time interval t_1, t_2 be taken sufficiently short. The reader should not loose sight of the fact that the varied motions with which the actual motion is compared are in general

not traced out with the same speed in the Principle of Least Action as in Hamilton's Principle.

The Principle of Least Action takes on a more readily visualizable form if there exists a force function,

$$U = U(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n).$$

For by definition of the force function and with the aid of formula § 94, (2) we have,

$$\delta U = \Sigma \nabla_i U \cdot \delta \mathbf{p}_i = \Sigma \mathbf{f}_i \cdot \delta \mathbf{p}_i = \delta W$$

and our hypothesis $\delta W = \delta T$ is equivalent to,

$$(2) \qquad \delta(T - U) = 0.$$

This enables us to state the,

Restricted Principle of Least Action.

If the total energy $T - U$ of the set remains constant during the actual motion and if the motion is varied by virtual displacements satisfying the imposed conditions and with the variations of the end positions zero and the time variation such that the total energy retains the same constant value on the varied as on the unvaried motion, then the variation δJ of the action is zero.

For when U exists and $T - U$ has the same value throughout every varied motion that it has throughout the unvaried motion, we must have equation (2) holding and the hypothesis $\delta W = \delta T$ of our general principle therefore holds and the conclusion follows.

The above hypothesis that $T - U$ retains a constant value during the actual motion will in particular be satisfied if there exists a force function U and if the imposed conditions do not vary with the time. For recalling the Theorem of Kinetic Energy for Applied Forces (§ 91) we have

$$0 = T' - W' = T' - U' \qquad \text{or} \qquad T - U = H_0 \qquad (H_0 \text{ const.}).$$

The first hypothesis of the Restricted Principle of Least Action is thus established on the basis of these two new hypotheses. When the imposed conditions do not involve the time the requirement that the variations of the given motion be virtual displacements satisfying the imposed conditions is free from any consideration of the time. When there also exists a force function U, then all the hypotheses of the Restricted Principle are free from any consideration of the time, and in this case Jacobi has developed

the following method by which the action J may likewise be freed from t.

The unvaried motion of the set from the position A to the position B being given, we indicate by s_i the length of the path of the particle P_i from its position at some instant t_0. We also indicate by S a parameter taking on the value zero at the instant t_0 and having its time derivative S' given by the formula,

$$S' = \sqrt{\Sigma\, m_i s_i'^2}.$$

If now λ be any function of t with nonvanishing derivative in the interval considered, we evidently have likewise,

$$\frac{dS}{d\lambda} = \pm \sqrt{\Sigma m_i \left(\frac{ds_i}{d\lambda}\right)^2}.$$

Thus, although S is completely determined at each stage of the given motion, its value depends only on the succession of sets of simultaneous values assumed by the s_i up to that stage and not upon the times at which these sets of values are assumed. Under the above hypotheses we now have,

$$S'^2 = \Sigma\, m_i s_i'^2 = \Sigma\, m_i\, \mathrm{p}_i'^2 = 2T = 2U + 2H_0$$

and may therefore write,

$$(3) \qquad J = \int_{t_1}^{t_2} 2T\, dt = \int_{t_1}^{t_2} \sqrt{2T}\, S'\, dt = \int_{S_1}^{S_2} \sqrt{2U + 2H_0}\, dS,$$

where S_1 and S_2 are the values of S for the initial and terminal positions of the set. We thus have,

Jacobi's Form of the Principle of Least Action.

If there exists a force function U and if the imposed conditions do not vary with the time and if the motion be varied by virtual displacements satisfying the imposed conditions and with the variations of the end positions zero and the time variation such that the total energy $T - U$ retains the same constant value H_0 on the varied as on the unvaried motion, then the variation,

$$(4) \qquad \delta \int_{S_1}^{S_2} \sqrt{2U + 2H_0}\, dS$$

is zero.

The principle as thus stated involves only the geometric properties of the motion, the time being entirely eliminated. If it be further assumed that the imposed conditions are holonomic, the above varied motions will themselves satisfy the imposed conditions. It was this holonomic case which was discussed by Jacobi.

If no applied forces act on the particles of the set the above force function U is a constant and the paths of the particles are known as *geodesics*. In this case we have from equation (4),

$$0 = \delta \int_{S_1}^{S_2} dS = \delta(S_2 - S_1).$$

In the case of a single particle with no applied force and where the particle is constrained to remain on the surface of a fixed sphere, the Principle of Least Action becomes intuitively obvious. For in this case J is proportional to the length of the path of the particle during the interval. This path will be an arc of a great circle and it is clear that unless this path exceeds a semicircumference it will be shorter than any other curve on the surface of the sphere joining its two end points. This is a special case of the theorem previously quoted that the action is less along the actual motion than along any varied motion if the time interval is sufficiently short.

We have just derived the Principle of Least Action from d'Alembert's Principle, and by a method exactly like that employed in the previous article for Hamilton's Principle, we may derive d'Alembert's Principle from the Principle of Least Action. It thus appears that these three principles are equivalent, but nevertheless it is not possible to derive Hamilton's Principle and the Principle of Least Action *directly* from each other because the types of variation involved in them are not the same.

97. Gauss' Principle of Least Constraint.

In 1829 Gauss published a new mechanical principle, like Hamilton's Principle and the Principle of Least Action in that it states that a certain quantity is smaller for the actual motion of a set of particles than for a particular type of varied motion. It is, however, different from these principles in that the quantity stated to be a minimum is not in this case an integral over a por-

tion of the motion, but has a definite value at each instant of the motion; i.e. it is a *function* of the motion and not a *functional.*

If we consider a moving particle P and indicate by $\mathbf{p}_0, \mathbf{p}_0', \mathbf{p}_0'', \cdots$ its position, velocity, acceleration etc. at an instant t_0, then by Taylor's Theorem we have for its position $\mathbf{p}$ at the instant $t = t_0 + dt$,

$$(1) \qquad \mathbf{p} = \mathbf{p}_0 + \mathbf{p}_0'\, dt + \mathbf{p}_0'' \frac{(dt)^2}{2} + \varepsilon_2,$$

where ε_2 is used to indicate any quantity vanishing with dt to a higher order than $(dt)^2$. If at the instant t_0 an applied force $\mathbf{f}$ and a constraint $\mathbf{k}$ act on the particle, then the acceleration $\mathbf{p}_0''$ in the above formula has by Newton's Second Law the value,

$$\mathbf{p}_0'' = (\mathbf{f} + \mathbf{k})/m.$$

Let us now consider what the position $\mathbf{p} - \Delta\mathbf{p}$ of the particle P would have been at the instant $t_0 + dt$ if at the instant t_0 the particle had its former position $\mathbf{p}_0$ and velocity $\mathbf{p}_0'$ but was acted upon only by the applied force $\mathbf{f}$. As in equation (1) we would have,

$$(2) \qquad \mathbf{p} - \Delta\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_0'\, dt + \frac{\mathbf{f}}{m}\frac{(dt)^2}{2} + \varepsilon_2$$

with a different ε_2. Here $\Delta\mathbf{p}$ is the change in the position of the particle at the instant t due to the action of the constraint $\mathbf{k}$. Just as $\Delta\mathbf{p}$ measures the effect of $\mathbf{k}$ on the motion of a single particle, so Gauss suggested that in dealing with a set of particles P_i we measure the effect of all the constraints $\mathbf{k}_i$ by the sum $\Sigma\, m_i(\Delta\mathbf{p}_i)^2$ obtained by multiplying the square of each $\Delta\mathbf{p}_i$ by the mass of the corresponding particle and adding. Evaluating equations (1) and (2) for the particle P_i and subtracting we have,

$$\Delta\mathbf{p}_i = \frac{(dt)^2}{2}(\mathbf{p}_i'' - \mathbf{f}_i/m_i) + \varepsilon_2,$$

where $\mathbf{p}_i''$ and $\mathbf{f}_i$ are evaluated at the instant t_0. Then Gauss' measure of the effect of the constraints takes the form,

$$\sum m_i(\Delta\mathbf{p}_i)^2 = \frac{(dt)^4}{4}\sum m_i(\mathbf{p}_i'' - \mathbf{f}_i/m_i)^2 + \varepsilon_4,$$

where ε_4 vanishes with dt to a higher order than $(dt)^4$. Thus as

dt approaches zero, Gauss' measure has a principal part proportional to the quantity,

$$G = \Sigma\, m_i(\mathbf{p}_i'' - \mathbf{f}_i/m_i)^2$$

and it is this quantity G which we shall designate as *Gauss' Constraint.*

We now introduce a variation of the motion of the set of particles, the variations $\delta\mathbf{p}_i$ of the positions being so chosen as to satisfy the imposed conditions, and the variation δt being chosen identically zero. We also require that the variations $\delta\mathbf{p}_i$ of the positions and $\delta\mathbf{p}_i'$ of the velocities shall all be zero at the instant t_0. Since δt is identically zero we have by formula § 94, (1′),

$$(3) \qquad (\delta\mathbf{p}_i)' = \delta(\mathbf{p}_i'), \qquad (\delta\mathbf{p}_i)'' = \delta(\mathbf{p}_i'').$$

Since the variations $\delta\mathbf{p}_i$ satisfy the imposed conditions at each instant we have by d'Alembert's Principle, § 94, (1), $\Sigma\, \mathbf{k}_i \cdot \delta\mathbf{p}_i = 0$ holding identically, and on differentiating twice with respect to the time we find,

$$\Sigma\, \mathbf{k}_i'' \cdot \delta\mathbf{p}_i + 2\, \Sigma\, \mathbf{k}_i' \cdot (\delta\mathbf{p}_i)' + \Sigma\, \mathbf{k}_i \cdot (\delta\mathbf{p}_i)'' = 0.$$

Under our hypothesis that $\delta\mathbf{p}_i = \delta\mathbf{p}_i' = 0$ at the instant t_0, this equation takes the form, $\Sigma\, \mathbf{k}_i \cdot \delta\mathbf{p}_i'' = 0$ for $t = t_0$ and since $\mathbf{p}_i'' = (\mathbf{f}_i + \mathbf{k}_i)/m_i$ we may write this as,

$$(4) \qquad \Sigma\, m_i(\mathbf{p}_i'' - \mathbf{f}_i/m_i) \cdot \delta\mathbf{p}_i'' = 0 \qquad \text{for} \qquad t = t_0.$$

The variation δG of Gauss' Constraint for any varied motion will be,

$$\begin{aligned} \delta G &= \Sigma\, m_i(\mathbf{p}_i'' + \delta\mathbf{p}_i'' - \mathbf{f}_i/m_i)^2 - \Sigma\, m_i(\mathbf{p}_i'' - \mathbf{f}_i/m_i)^2 \\ &= \Sigma\, m_i(\delta\mathbf{p}_i'')^2 - 2\, \Sigma\, m_i(\mathbf{p}_i'' - \mathbf{f}_i/m_i) \cdot \delta\mathbf{p}_i'', \end{aligned}$$

and at the instant t_0 this reduces by equation (4) to,

$$(5) \qquad \delta G = \Sigma\, m_i(\delta\mathbf{p}_i'')^2.$$

Thus the variation δG is always positive at the instant t_0 considered and we have proven,

Gauss' Principle of Least Constraint.

The constraint, $G = \Sigma\, m_i(\mathbf{p}_i'' - \mathbf{f}_i/m_i)^2$, has at each instant a smaller value for the actual motion of the set of particles than for any varied motion above described.

The principle as stated by Gauss himself has a somewhat wider scope than as stated here, for Gauss seems to have felt that his proof had shown that the constraint G would be less for the actual motion than for *any* varied motion satisfying the imposed conditions. He says, *Journal für Mathematik*, Vol. 4, p. 232 (1829):

"The motion of a set of particles, connected in any manner and subject to any imposed conditions, takes place at each instant in the greatest possible agreement with the free motion, i.e., with the least possible constraint, if one adopts as the measure of the constraint undergone by the set of particles in any arbitrarily small time interval the sum of the products of the mass of each particle by the square of the amount by which it departs from its free motion."

As in the case of the other principles discussed in this chapter, we may with the assumption of Gauss' Principle and certain minor auxiliary hypotheses develop the entire subject of classical mechanics.

Closely associated with Gauss' Principle of Least Constraint is *Hertz' Principle of Least Curvature.* Hertz defined the *curvature* of the motion at each instant as $\sqrt{G}$ and of course the fact that it is a minimum for the actual as compared with the above varied motions follows as above. Hertz, however, confined his discussion to the case in which there is no applied force. In the case of a single particle confined to a fixed smooth surface and with no applied force, Hertz' Principle results in the well known fact that the ordinary curvature of the path is the least consistent with the particle remaining on the surface.

CHAPTER XIII

VECTOR CALCULUS

98. Vector and Scalar Functions.

Following the definition originally due to Dirichlet, we say that a variable y is a single valued function of another variable x when there is a value of y assigned to each of a certain set of values of x. If we let x and y be either scalar or vector independently we have then four types of functions to consider.

	$y = y(x)$	y	x
I.	Scalar-scalar functions	Scalar	Scalar
II.	Vector-scalar functions	Vector	Scalar
III.	Scalar-vector functions	Scalar	Vector
IV.	Vector-vector functions	Vector	Vector

Many important applications of the first type are developed in the ordinary calculus and we have employed this type in Chapter II in the study of the rectilinear motion of a particle. The somewhat less simple Type II we employed in Chapter IV in the study of the curvilinear motion of a particle where we let the radius vector $OP = \mathbf{p}$ as a function of the time t determine the motion of the particle P, where O is regarded as a fixed point.

We have also on a few occasions considered functions of Type III in which a scalar U is a function $U(\mathbf{p})$ of a vector $\mathbf{p}$. Thus the length of $\mathbf{p}$ is a scalar function of $\mathbf{p}$, as is also the scalar product of $\mathbf{p}$ with one or two constant vectors. Thus we have,

$$U = |\mathbf{p}|, \qquad U = \mathbf{a}\cdot\mathbf{p}, \qquad U = [\mathbf{a}\;\mathbf{b}\;\mathbf{p}].$$

If we here interpret the independent variable $\mathbf{p}$ as the radius vector OP running from a fixed point O to a variable point P then the scalar function $U(\mathbf{p})$ depends on the position of the point P and is consequently known as a *scalar-point function.* The temperature of the air at the different points of a room and the density of a solid at its various points are easily visualized examples of this type of function.

There remains Type IV, the *vector-vector* function. As familiar examples of this type we recall,

$$\mathbf{V} = k\,\mathbf{p}, \qquad \mathbf{V} = \mathbf{a} \times \mathbf{p}, \qquad \mathbf{V} = \mathbf{a} \times (\mathbf{b} \times \mathbf{p}),$$

where k, $\mathbf{a}$, $\mathbf{b}$ are constants. Here again we often interpret the independent variable $\mathbf{p}$ as the radius vector OP of a variable point P and consequently call such functions *vector-point functions.* The vector velocity of the air at the various points of a room or the vector force of gravity at a given instant at the different points of the solar system are obvious examples of this type of function.

We saw in § 24, to which the reader is referred, that the definition of a limit employed in the case of scalar-scalar functions was readily extended to vector-scalar functions and the same is true of scalar-vector and vector-vector functions. Thus for a scalar-vector function we may write,

$$\lim_{\mathbf{p} \to \mathbf{p}_0} U(\mathbf{p}) = U_0 \qquad \text{or} \qquad U \to U_0 \qquad \text{for} \qquad \cdot\ \mathbf{p}_0,$$

which is read, "The limit of $U(\mathbf{p})$ as $\mathbf{p}$ approaches $\mathbf{p}_0$ is U_0" and which means that being given any positive scalar ϵ, it is always possible to find a second positive scalar δ such that,

$$|U(\mathbf{p}) - U_0| < \epsilon \qquad \text{whenever} \qquad |\mathbf{p} - \mathbf{p}_0| < \delta.$$

A similar extension gives us the concept of limit for vector-vector functions. In every case we have merely to replace the *absolute value* of the difference of two scalars by the *length* of the difference of two vectors. The reader will have no difficulty in thus setting up the definition of the limit of a vector-vector function and in proving theorems such as that the limit of the sum of two such functions is the sum of their limits. By a similar extension the *continuity* of scalar-vector and vector-vector functions may be defined and the usual properties proven, in so far as they are applicable.

EXERCISES

1. If $U(\mathbf{p})$ and $W(\mathbf{p})$ are scalar functions of the vector $\mathbf{p}$ and if,

$$U \to U_0, \qquad W \to W_0 \qquad \text{for} \qquad \mathbf{p} \to \mathbf{p}_0,$$

show that,

$$U + W \to U_0 + W_0, \quad UW \to U_0W_0, \quad U/W \to U_0/W_0 \quad (W_0 \neq 0).$$

2. If $\mathbf{V}(\mathbf{p})$ and $\mathbf{W}(\mathbf{p})$ are vector functions of the vector $\mathbf{p}$ and if,

$$\mathbf{V} \to \mathbf{V}_0, \qquad \mathbf{W} \to \mathbf{W}_0 \qquad \text{for} \qquad \mathbf{p} \to \mathbf{p}_0,$$

show that,

$$\mathbf{V} + \mathbf{W} \to \mathbf{V}_0 + \mathbf{W}_0, \qquad \mathbf{V}\cdot\mathbf{W} \to \mathbf{V}_0\cdot\mathbf{W}_0, \qquad \mathbf{V}\times\mathbf{W} \to \mathbf{V}_0\times\mathbf{W}_0.$$

3. Define the continuity of scalar-vector and vector-vector functions of the vector $\mathbf{p}$ for a given value of $\mathbf{p}$ and state and prove theorems concerning the continuity of the sum, product, etc., of two continuous functions.

99. Gradient of a Scalar-Vector Function.

If we consider a scalar function $y(x)$ of the scalar variable x we recall from the scalar calculus that concepts of the greatest importance in the study of such a function are the derivative y' of y with respect to x and the differential dy of y for a given variation Δx in x. In this chapter we shall be largely concerned with the extension of these concepts to scalar-vector and vector-vector functions and in view of this extension we shall first slightly modify the *form* in which the definitions of the derivative and differential are expressed. Let x_0 be any particular value of x and let y_0 be the corresponding value of y and then let Δx be any variation of x from the value x_0 and Δy the corresponding variation of y. We then define the derivative y' of y with respect to x as a number independent of Δx such that there exists a scalar ϵ satisfying the equations,

$$(1') \qquad (y' + \epsilon)\,\Delta x = \Delta y, \qquad \lim_{\Delta x \to 0} \epsilon = 0.$$

It is of course not implied that any such number y' always exists, but if any one does exist it is easily proven to be unique and to be identical with the derivative as usually defined. If the derivative y' exists we define the differential of y as the quantity,

$$(2') \qquad dy = y'\,\Delta x$$

and easily show that,

$$(3') \qquad \Delta y = dy + \epsilon\,\Delta x.$$

This last equation states the fundamental property of the differential which may be roughly expressed by saying that for sufficiently small values of Δx the variation Δy is approximately given by the differential dy.

Similarly if we consider a vector function $\mathbf{p}(t)$ of the scalar variable t and define Δt and $\Delta\mathbf{p}$ as above, then the derivative $\mathbf{p}'$ of $\mathbf{p}$ with respect to t is defined as a vector independent of Δt such that there exists a vector $\boldsymbol{\varepsilon}$ satisfying the equations,

$$(1'') \qquad (\mathbf{p}' + \boldsymbol{\varepsilon})\,\Delta t = \Delta\mathbf{p}, \qquad \lim_{\Delta t \to 0} \boldsymbol{\varepsilon} = 0.$$

If any such vector $\mathbf{p}'$ exists it may easily be proven to be unique and to be identical with the derivative as previously defined. If the derivative $\mathbf{p}'$ exists we define the differential of $\mathbf{p}$ as the quantity,

$$(2'') \qquad d\mathbf{p} = \mathbf{p}'\,\Delta t$$

and easily show that,

$$(3'') \qquad \Delta\mathbf{p} = d\mathbf{p} + \boldsymbol{\varepsilon}\,\Delta t.$$

This equation states the fundamental property of the differential which may be roughly expressed by saying that for sufficiently small values of Δt the variation $\Delta\mathbf{p}$ is approximately given by the differential $d\mathbf{p}$.

Let us now consider the corresponding situation in the case of a scalar function $U(\mathbf{p})$ of the vector variable $\mathbf{p}$. As the analog of the derivative we here define the *gradient* ∇U * of U with respect to $\mathbf{p}$ as a vector independent of $\Delta\mathbf{p}$ such that there exists a vector $\boldsymbol{\varepsilon}$ satisfying the equations,

$$(1) \qquad (\nabla U + \boldsymbol{\varepsilon})\cdot\Delta\mathbf{p} = \Delta U, \qquad \lim_{\Delta\mathbf{p} \to 0} \boldsymbol{\varepsilon} = 0.$$

If the gradient ∇U exists then we define the differential of U as the quantity,

$$(2) \qquad dU = \nabla U\cdot\Delta\mathbf{p}$$

and easily show that,

$$(3) \qquad \Delta U = dU + \boldsymbol{\varepsilon}\cdot\Delta\mathbf{p}.$$

We may roughly express this fundamental property of the differential by saying that for sufficiently small values of $\Delta\mathbf{p}$ the variation ΔU is approximately given by the differential dU.

As a first step in the study of the gradient ∇U we observe that if a vector ∇U exists satisfying conditions (1) it must be unique. For suppose there were two vectors $\nabla_1 U$ and $\nabla_2 U$ satisfying these

* The symbol ∇ is pronounced "del," a name due to J. W. Gibbs. The concept and symbol are due to Sir William Hamilton, 1847.

conditions. We would then have,

$$(\nabla_1 U + \boldsymbol{\varepsilon}_1)\cdot\Delta\mathbf{p} = \Delta U, \qquad (\nabla_2 U + \boldsymbol{\varepsilon}_2)\cdot\Delta\mathbf{p} = \Delta U$$

and subtracting find,

$$(\nabla_2 U - \nabla_1 U)\cdot\Delta\mathbf{p} = (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)\cdot\Delta\mathbf{p}.$$

If we now let $\Delta\mathbf{p}$ approach zero while retaining the direction and sense of a constant unit vector $\mathbf{u}$, then during this limiting process we would have,

$$(\nabla_2 U - \nabla_1 U)\cdot\mathbf{u} = (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)\cdot\mathbf{u},$$

and since the left member is a constant while the right member approaches zero, it follows that the left member must *be* zero. Thus we have $(\nabla_2 U - \nabla_1 U)\cdot\mathbf{u} = 0$ for *every choice* of the unit vector $\mathbf{u}$. Consequently $\nabla_1 U = \nabla_2 U$ and the gradient must be unique if it exists.

If we now choose a Cartesian coördinate system, the scalar U which is a function of the vector $\mathbf{p}$ may be regarded as a function of the three coördinates (x, y, z) of $\mathbf{p}$. To find a vector ∇U satisfying conditions (1) is equivalent to finding its three coördinates (L, M, N) as scalar functions of (x, y, z) satisfying the conditions,

$$(4) \qquad (L + \epsilon_1)\,\Delta x + (M + \epsilon_2)\,\Delta y + (N + \epsilon_3)\,\Delta z = \Delta U,$$
$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \epsilon_i = 0 \qquad (i = 1, 2, 3).$$

We obtain at once a necessary condition for this by allowing $\Delta\mathbf{p}$ to approach zero while remaining parallel to the X-axis; that is we set $\Delta y = \Delta z = 0$ and then allow Δx to approach zero. On comparing equation (4) under these conditions with equation (1′) and recalling the definition of the partial derivative, we see at once that,

$$L = \frac{\partial U}{\partial x}; \qquad \text{and similarly} \qquad M = \frac{\partial U}{\partial y}, \qquad N = \frac{\partial U}{\partial z}.$$

Thus when ∇U exists it is given by the formula,

$$(5) \qquad \nabla U \equiv \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right),$$

no matter what coördinate system is employed. Conversely we can

show that if the partial derivatives of formula (5) exist and are continuous, then condition (4) is satisfied by formula (5). In fact this is a well known theorem of the scalar calculus.

The important general properties of the gradient follow at once from formula (5) and the familiar facts of the scalar calculus. Thus if $U_1, U_2, \cdots, U_n$ are scalar functions of the vector $\mathbf{p}$ and $F(U_1, U_2, \cdots, U_n)$ is a scalar function of $U_1, U_2, \cdots, U_n$, then we have,

$$\nabla F = \frac{\partial F}{\partial U_1} \nabla U_1 + \frac{\partial F}{\partial U_2} \nabla U_2 + \cdots + \frac{\partial F}{\partial U_n} \nabla U_n. \tag{6}$$

As particular cases of this might be mentioned,

$$\begin{aligned} \nabla(U_1 + U_2) &= \nabla U_1 + \nabla U_2, \\ \nabla(U_1 U_2) &= U_1 \nabla U_2 + U_2 \nabla U_1, \\ \nabla(U_1/U_2) &= (U_2 \nabla U_1 - U_1 \nabla U_2)/U_2^2. \end{aligned} \tag{6}$$

In certain cases we may readily derive the gradient of $U(\mathbf{p})$ directly from the definition (1). Thus for,

$$U(\mathbf{p}) = \mathbf{a} \cdot \mathbf{p} \qquad (\mathbf{a} \text{ const.})$$

we have,

$$U + \Delta U = \mathbf{a} \cdot (\mathbf{p} + \Delta \mathbf{p}) \qquad \text{and hence} \qquad \Delta U = \mathbf{a} \cdot \Delta \mathbf{p}.$$

It follows by inspection that equations (1) are satisfied for $\nabla U = \mathbf{a}$, $\boldsymbol{\varepsilon} = 0$ and consequently we may write,

$$\nabla(\mathbf{a} \cdot \mathbf{p}) = \mathbf{a}. \tag{7}$$

Similarly for $U = \mathbf{p}^2$ we have $U + \Delta U = (\mathbf{p} + \Delta \mathbf{p})^2$ and hence $\Delta U = (2\,\mathbf{p} + \Delta \mathbf{p}) \cdot \Delta \mathbf{p}$. It follows by inspection that equations (1) are satisfied for $\nabla U = 2\mathbf{p}$ and $\boldsymbol{\varepsilon} = \Delta \mathbf{p}$ and consequently we may write,

$$\nabla(\mathbf{p}^2) = 2\,\mathbf{p}. \tag{8}$$

The reader will have no difficulty in deriving from formula (8) by means of the general formula (6) the result that,

$$\nabla |\mathbf{p}| = \frac{\mathbf{p}}{|\mathbf{p}|} \qquad (\mathbf{p} \neq 0). \tag{9}$$

Thus,

The gradient of the length of a vector with respect to that vector is a unit vector having its direction and sense.

If $y(x)$ is a scalar function of the scalar x the derivative y' of y with respect to x has a geometric interpretation as the slope of the tangent to the graph of this function on the Cartesian coördinate system. Also if $\mathbf{p}(t)$ is a vector function of the scalar t, the derivative $\mathbf{p}'$ of $\mathbf{p}$ with respect to t has a kinematical interpretation as the velocity of the point P if we let $OP = \mathbf{p}$, where O is a fixed point, and let t be the time. These interpretations of the derivatives of the scalar-scalar and vector-scalar functions are a great aid in clearly understanding them as well as a first step in their application. Similarly the gradient of a scalar-vector function has a geometric interpretation of great value in its understanding and application. This will be discussed in the following article.

EXERCISES

1. If $\mathbf{a}$ and $\mathbf{b}$ are any constant vectors show that,

(a) $\nabla\{(\mathbf{a}\cdot\mathbf{p})^n\} = n(\mathbf{a}\cdot\mathbf{p})^{n-1}\,\mathbf{a}$,
(b) $\nabla\{|\mathbf{a}\times\mathbf{p}|^n\} = n|\mathbf{a}\times\mathbf{p}|^{n-2}\,\mathbf{a}\times(\mathbf{p}\times\mathbf{a})$,
(c) $\nabla\{(\mathbf{a}\times\mathbf{p})\cdot(\mathbf{b}\times\mathbf{p})\} = \mathbf{a}\times(\mathbf{p}\times\mathbf{b}) + \mathbf{b}\times(\mathbf{p}\times\mathbf{a})$.

2. If $p = |\mathbf{p}|$ show that,

(a) $\nabla p^n = np^{n-2}\,\mathbf{p}$,

(b) $\nabla\{f(p)\} = \dfrac{df}{dp}\dfrac{\mathbf{p}}{p}$.

3. If r is the distance from a variable point P to a fixed point, line or plane and Q is either the fixed point or the foot of the perpendicular dropped from P upon the fixed line or plane, then if $QP = \mathbf{r}$ show that if r be regarded as a function of the vector $OP = \mathbf{p}$ where O is any fixed point,

$$\nabla r = \frac{\mathbf{r}}{r},$$

i.e. the gradient of r is a unit vector with the direction and sense of $\mathbf{r}$.

4. Show by transformation to new rectangular coördinates that ∇U is a *differential invariant* of the scalar function $U(x, y, z)$. That is, show that such a transformation carrying $U(x, y, z)$ into $U'(x', y', z')$ will likewise carry ∇U into $\nabla' U'$. Has this been proven in the text?

5. Show that a necessary and sufficient condition that two scalar-vector functions $U_1(\mathbf{p})$ and $U_2(\mathbf{p})$ be functionally related, i.e. that there exist a relation of the form,

$$F(U_1, U_2) = 0$$

is that $\nabla U_1 \times \nabla U_2 \equiv 0$. Extend this theorem to the case of three scalar-vector functions, $U_1(\mathbf{p})$, $U_2(\mathbf{p})$, $U_3(\mathbf{p})$.

100. Directional Derivatives.

If we consider a scalar-vector function $U(\mathbf{p})$ and interpret $\mathbf{p}$ as the radius vector from a fixed point O to a variable point P, then U depends on the position of the point P and is consequently known as a scalar-point function. We now pick a definite value of $\mathbf{p}$, calling it $\mathbf{p}_0$, and compute the corresponding value of U, calling it U_0, and then give to $\mathbf{p}$ a variable increment $\Delta\mathbf{p}$ *parallel to a given constant unit vector* $\mathbf{u}$. In other words we think of the variable point P as moving from its initial position P_0 in the direction but not necessarily in the sense of the vector $\mathbf{u}$. If we now indicate by Δs the measure of $\Delta\mathbf{p}$ on $\mathbf{u}$ then Δs is numerically equal to the length of $\Delta\mathbf{p}$ and we have,

$$(1) \qquad \Delta\mathbf{p} = \Delta s\,\mathbf{u}.$$

Under these conditions the scalar function U varies from its initial value U_0 by an amount ΔU which depends only on Δs and we define *the derivative of U in the direction of* $\mathbf{u}$ as a quantity $U'_{\mathbf{u}}$ independent of Δs such that there exists a scalar ϵ satisfying the equations,

$$(2) \qquad (U'_{\mathbf{u}} + \epsilon)\Delta s = \Delta U, \qquad \lim_{\Delta s \to 0} \epsilon = 0.$$

On comparison with equations (1′) of the last article it is apparent that the directional derivative $U'_{\mathbf{u}}$ as here defined is merely the usual derivative of the scalar U with respect to the scalar Δs and is consequently unique when it exists.

We now recall from the last article equations (1) which define the gradient ∇U,

$$(\nabla U + \boldsymbol{\varepsilon})\cdot\Delta\mathbf{p} = \Delta U, \qquad \lim_{\Delta\mathbf{p}\to 0} \boldsymbol{\varepsilon} = 0$$

and replacing $\Delta\mathbf{p}$ by its above value $\Delta s\,\mathbf{u}$, we have,

$$(\mathbf{u}\cdot\nabla U + \mathbf{u}\cdot\boldsymbol{\varepsilon})\Delta s = \Delta U, \qquad \lim_{\Delta s\to 0} \mathbf{u}\cdot\boldsymbol{\varepsilon} = 0,$$

which on comparison with equations (2) shows at once the important relation,

$$(3) \qquad U'_{\mathbf{u}} = \mathbf{u}\cdot\nabla U.$$

In Cartesian coördinates, calling $\mathbf{p} \equiv (x, y, z)$ and $\mathbf{u} \equiv (u_1, u_2, u_3)$, this becomes,

$$(3') \qquad U'_{\mathbf{u}} \equiv u_1\frac{\partial U}{\partial x} + u_2\frac{\partial U}{\partial y} + u_3\frac{\partial U}{\partial z}.$$

If we indicate the direction of differentiation of U by any non-zero constant vector $\mathbf{a}$, then $\mathbf{a} = |\mathbf{a}|\ \mathbf{u}$ and by equation (3) we have,

$$(4) \qquad |\mathbf{a}|\ U'_{\mathbf{a}} = \mathbf{a}\cdot\nabla U.$$

With the aid of the above concept of the directional derivative of the scalar-point function U we now find a ready geometric interpretation of the gradient ∇U. For if, while holding P_0 fixed, we allow the unit vector $\mathbf{u}$ of formula (3) to vary in direction making a variable angle θ with the gradient ∇U, we have from formula (3)

$$U'_{\mathbf{u}} = |\nabla U|\ \cos\theta$$

and it immediately appears that $U'_{\mathbf{u}}$ will be largest for $\theta = 0$ and will then equal $|\nabla U|$. That is,

The gradient ∇U *points in the direction and sense of the maximum directional derivative of* U.

The length $|\nabla U|$ *of the gradient equals the maximum directional derivative of* U.

All of the above fails in case $\nabla U = 0$ and a point P at which this occurs is said to be a *singular point* of the function U.

If U_0 is the value which a continuous scalar-point function U takes on at a point P_0, then the equation,

$$U = U_0,$$

will in general be satisfied only by points P lying on a certain surface passing through P_0. Since U retains the same value for all points of such a surface we call the surface a *level surface* for the function. Since there can be but one level surface through each point the level surfaces for a given function do not intersect each other and between any two adjacent level surfaces will lie a thin layer or *lamella* of space. Thus we say that the distribution of a continuous scalar-point function is *lamellar*. If $\Delta\mathbf{p}$ is an infinitesimal displacement of the point P along a level surface $U = U_0$, the corresponding change ΔU in U must be zero and by equations § 99, (1) we have,

$$(\nabla U + \boldsymbol{\varepsilon})\cdot\Delta\mathbf{p} = 0, \qquad \lim_{\Delta\mathbf{p}\to 0} \boldsymbol{\varepsilon} = 0.$$

If now $\mathbf{u}$ is the variable unit vector having the direction and sense of $\Delta\mathbf{p}$ the first of the above equations yields, $(\nabla U + \boldsymbol{\varepsilon})\cdot\mathbf{u} = 0$

and if $\mathbf{u}$ approaches the limit $\mathbf{u}_0$ as $\Delta\mathbf{p}$ approaches zero, we have from this,

$$\nabla U \cdot \mathbf{u}_0 = 0.$$

Thus we see that,

The gradient ∇U *at any point is perpendicular to the limiting direction of every infinitesimal displacement from that point along the level surface of* U *which passes through that point.*

Either unit vector $\mathbf{n}$ having the direction of ∇U is called the *normal vector* to the level surface at the point and the plane perpendicular to ∇U at the point is called the *tangent plane* to the surface there. Thus the equations of the normal line and tangent plane to the surface $U = U_0$ at the point P_0 are respectively,

$$(5)\quad (\mathbf{p} - \mathbf{p}_0) \times \nabla U]_0 = 0, \qquad \text{i.e.} \quad \frac{x - x_0}{\left.\frac{\partial U}{\partial x}\right]_0} = \frac{y - y_0}{\left.\frac{\partial U}{\partial y}\right]_0} = \frac{z - z_0}{\left.\frac{\partial U}{\partial z}\right]_0}$$

and,

$$(6)\quad (\mathbf{p} - \mathbf{p}_0) \cdot \nabla U]_0 = 0, \qquad \text{i.e.}$$

$$(x - x_0)\left.\frac{\partial U}{\partial x}\right]_0 + (y - y_0)\left.\frac{\partial U}{\partial y}\right]_0 + (z - z_0)\left.\frac{\partial U}{\partial z}\right]_0 = 0,$$

where in every case the subscript 0 means that the corresponding quantity is to be evaluated at P_0. All of the above fails at a point for which $\nabla U = 0$. Such points are called *singular points* of the surface $U = U_0$.

The concept of directional derivative is readily extended to vector-point functions, $\mathbf{V}(\mathbf{p})$. As before we pick a value of $\mathbf{p}$, as $\mathbf{p}_0$, and compute the corresponding value $\mathbf{V}_0$ of $\mathbf{V}$, and we indicate corresponding variations from these values by $\Delta\mathbf{p}$ and $\Delta\mathbf{V}$. As before we restrict the variation $\Delta\mathbf{p}$ by the condition,

$$\Delta\mathbf{p} = \Delta s\, \mathbf{u}$$

where $\mathbf{u}$ is a given unit vector, and we define *the derivative of* $\mathbf{V}$ *in the direction of* $\mathbf{u}$ as the vector $\mathbf{V}'_{\mathbf{u}}$ independent of Δs such that there exists a vector $\boldsymbol{\varepsilon}$ satisfying the equations,

$$(7)\qquad (\mathbf{V}'_{\mathbf{u}} + \boldsymbol{\varepsilon})\Delta s = \Delta\mathbf{V}, \qquad \lim_{\Delta s \to 0} \boldsymbol{\varepsilon} = 0.$$

On comparison with equations (1″) of the last article it is apparent that the directional derivative $\mathbf{V}'_{\mathbf{u}}$ as here defined is merely the usual derivative of the vector $\mathbf{V}$ with respect to the scalar Δs

and is consequently unique when it exists. If we now let $\mathbf{a}$ be any constant vector and dot-multiply the above equations through by $\mathbf{a}$, observing that $\mathbf{a}\cdot\Delta\mathbf{V} = \Delta(\mathbf{a}\cdot\mathbf{V})$, we find,

$$(\mathbf{a}\cdot\mathbf{V}'_{\mathrm{u}} + \mathbf{a}\cdot\boldsymbol{\varepsilon})\Delta s = \Delta(\mathbf{a}\cdot\mathbf{V}), \qquad \lim_{\Delta s\to 0} \mathbf{a}\cdot\boldsymbol{\varepsilon} = 0.$$

On comparison of these equations with equations (2) it is evident that,

$$(8) \qquad (\mathbf{a}\cdot\mathbf{V})'_{\mathrm{u}} = \mathbf{a}\cdot\mathbf{V}'_{\mathrm{u}}$$

which by equation (3) becomes,

$$(9) \qquad \mathbf{a}\cdot\mathbf{V}'_{\mathrm{u}} = \mathbf{u}\cdot\nabla(\mathbf{a}\cdot\mathbf{V}).$$

If we now call the Cartesian coördinates of $\mathbf{V} \equiv (V_1, V_2, V_3)$ and let $\mathbf{a}$ in the above equation take on successively the values $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the three unit coördinate vectors, we find,

$$\begin{aligned}
\mathbf{i}\cdot\mathbf{V}'_{\mathrm{u}} &= \mathbf{u}\cdot\nabla(\mathbf{i}\cdot\mathbf{V}) = \mathbf{u}\cdot\nabla V_1,\\
\mathbf{j}\cdot\mathbf{V}'_{\mathrm{u}} &= \mathbf{u}\cdot\nabla(\mathbf{j}\cdot\mathbf{V}) = \mathbf{u}\cdot\nabla V_2,\\
\mathbf{k}\cdot\mathbf{V}'_{\mathrm{u}} &= \mathbf{u}\cdot\nabla(\mathbf{k}\cdot\mathbf{V}) = \mathbf{u}\cdot\nabla V_3.
\end{aligned}$$

Thus the coördinates of $\mathbf{V}'_{\mathrm{u}}$ are

$$(10) \qquad \mathbf{V}'_{\mathrm{u}} \equiv (\mathbf{u}\cdot\nabla V_1, \mathbf{u}\cdot\nabla V_2, \mathbf{u}\cdot\nabla V_3),$$

being the derivatives of the coördinates of $\mathbf{V}$ in the direction of $\mathbf{u}$.

In dealing with directional derivatives in practice we find it convenient to introduce the symbol $\mathbf{a}\cdot\nabla$ which when placed before a scalar or vector point function indicates its derivative in the direction of the vector $\mathbf{a}$ multiplied by the length of $\mathbf{a}$. Thus we define,

$$(11) \qquad (\mathbf{a}\cdot\nabla)U = |\mathbf{a}|\,U'_{\mathrm{a}}, \qquad (\mathbf{a}\cdot\nabla)\mathbf{V} = |\mathbf{a}|\,\mathbf{V}'_{\mathrm{a}}.$$

On comparison with equations (4) and (10) it is evident that we have,

$$(12) \quad (\mathbf{a}\cdot\nabla)U = \mathbf{a}\cdot\nabla U, \quad (\mathbf{a}\cdot\nabla)\mathbf{V} \equiv (\mathbf{a}\cdot\nabla V_1, \mathbf{a}\cdot\nabla V_2, \mathbf{a}\cdot\nabla V_3).$$

The symbol $\mathbf{a}\cdot\nabla$ is said to represent a *scalar differential operator* because it acts like a scalar multiplier in that when applied to a scalar it yields a scalar and when applied to a vector it yields a vector.

In working with formulas involving the symbol ∇ it is a great convenience to think of ∇ as a sort of fictitious or *symbolic vector*

with the symbolic coördinates,

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

With this in mind we easily remember that,

$$\nabla U \equiv \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right).$$

Then $\mathbf{a} \cdot \nabla$ becomes the symbolic scalar product,

$$\mathbf{a} \cdot \nabla = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z},$$

which makes it easy to remember that,

$$(13) \qquad \mathbf{a} \cdot \nabla U = a_1 \frac{\partial U}{\partial x} + a_2 \frac{\partial U}{\partial y} + a_3 \frac{\partial U}{\partial z}$$

and

$$(14) \quad \mathbf{a} \cdot \nabla \mathbf{V} \equiv \left(a_1 \frac{\partial V_1}{\partial x} + a_2 \frac{\partial V_1}{\partial y} + a_3 \frac{\partial V_1}{\partial z}, \right.$$
$$\left. a_1 \frac{\partial V_2}{\partial x} + a_2 \frac{\partial V_2}{\partial y} + a_3 \frac{\partial V_2}{\partial z}, a_1 \frac{\partial V_3}{\partial x} + a_2 \frac{\partial V_3}{\partial y} + a_3 \frac{\partial V_3}{\partial z} \right).$$

This symbolic vector ∇ also resembles a true vector in that it remains invariant under a transformation of rectangular coördinates as observed in § 99, Problem 3. Further illustrations of the convenience of this concept of ∇ as a symbolic vector will be observed in the following article.

EXERCISES

1. Establish the following formulas,

(*a*) $(\mathbf{a} \cdot \nabla)(\mathbf{V} + \mathbf{W}) = (\mathbf{a} \cdot \nabla)\, \mathbf{V} + (\mathbf{a} \cdot \nabla)\, \mathbf{W}$,
(*b*) $(\mathbf{a} \cdot \nabla)(U\, \mathbf{V}) = \{(\mathbf{a} \cdot \nabla) U\}\, \mathbf{V} + U (\mathbf{a} \cdot \nabla)\, \mathbf{V}$,
(*c*) $(\mathbf{a} \cdot \nabla)(\mathbf{V} \cdot \mathbf{W}) = \{(\mathbf{a} \cdot \nabla)\, \mathbf{V}\} \cdot \mathbf{W} + \mathbf{V} \cdot (\mathbf{a} \cdot \nabla)\, \mathbf{W}$,
(*d*) $(\mathbf{a} \cdot \nabla)(\mathbf{V} \times \mathbf{W}) = \{(\mathbf{a} \cdot \nabla)\, \mathbf{V}\} \times \mathbf{W} + \mathbf{V} \times (\mathbf{a} \cdot \nabla)\, \mathbf{W}$,
(*e*) $(\mathbf{a} + \mathbf{b}) \cdot \nabla\, \mathbf{V} = (\mathbf{a} \cdot \nabla)\, \mathbf{V} + (\mathbf{b} \cdot \nabla)\, \mathbf{V}$.

2. Derive a formula for the expansion of $(\mathbf{a} \cdot \nabla)[\mathbf{V}_1\, \mathbf{V}_2\, \mathbf{V}_3]$.

3. Prove that the operator $(\mathbf{a} \cdot \nabla)$ is commutative with itself and with ∇. This is, prove,

(*a*) $(\mathbf{a} \cdot \nabla)\{(\mathbf{b} \cdot \nabla) U\} = (\mathbf{b} \cdot \nabla)\{(\mathbf{a} \cdot \nabla) U\}$,
(*b*) $(\mathbf{a} \cdot \nabla)\{(\mathbf{b} \cdot \nabla)\mathbf{V}\} = (\mathbf{b} \cdot \nabla)\{(\mathbf{a} \cdot \nabla)\mathbf{V}\}$,
(*c*) $(\mathbf{a} \cdot \nabla)(\nabla U) = \nabla\{(\mathbf{a} \cdot \nabla) U\}$.

4. Show that $(\mathbf{a}\cdot\nabla)\mathbf{p} = \mathbf{a}$.

5. Show that if a vector point function $\mathbf{V}$ is of constant length, then it is perpendicular to all of its directional derivatives.

6. Show that if r is the distance from any fixed point, line or plane to a moving point P, then the derivative r' of r with respect to the time t, equals the measure of the velocity of P on the perpendicular drawn from the fixed line or plane to P or on the line from the fixed point to P.

101. Divergence of a Vector-Vector Function.

We saw in the last article that the gradient ∇U gives us a complete picture of the behavior of the scalar-vector function $U(\mathbf{p})$ as far as first derivatives are concerned because if the gradient is known we may find the derivative of U in the direction of any given unit vector $\mathbf{u}$ by merely evaluating $\mathbf{u}\cdot\nabla U$. To obtain a similarly complete picture of the behavior of the vector-vector function $\mathbf{V}(\mathbf{p})$ we should need to know a quantity of some sort such that when properly combined with a given unit vector $\mathbf{u}$ it would yield the derivative of $\mathbf{V}$ in the direction of $\mathbf{u}$. However, a little consideration shows that such a quantity would need to be of a more complex nature than a vector, having in fact *nine* coördinates. Such quantities are known as *tensors* and their study would carry us beyond the limitations of this book. However, it happens that most of the important differential properties of a vector-vector function can be derived from a knowledge of the value of a certain scalar quantity called the *divergence* of the function $\mathbf{V}(\mathbf{p})$ and represented by the symbol $\nabla\cdot\mathbf{V}$ and of a certain vector quantity called the *curl* of $\mathbf{V}$ and represented by the symbol $\nabla\times\mathbf{V}$.

We shall consider here the divergence * $\nabla\cdot\mathbf{V}$ of the vector-vector function $\mathbf{V}(\mathbf{p})$. Let $\mathbf{p}_0$ be any particular value of $\mathbf{p}$ and $\mathbf{V}_0$ the corresponding value of $\mathbf{V}$ and let $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ be any three non-coplanar variations of $\mathbf{p}$ from the value $\mathbf{p}_0$ and let $\Delta_1\mathbf{V}$, $\Delta_2\mathbf{V}$, $\Delta_3\mathbf{V}$ be the corresponding variations of $\mathbf{V}$ from the value $\mathbf{V}_0$. We then follow the procedure previously used in the definitions of the derivatives of scalar-scalar and vector-scalar functions and of the gradient of a scalar-vector function and we define the *divergence* of the vector-vector function $\mathbf{V}(\mathbf{p})$ as a scalar quantity independent of the $\Delta\mathbf{p}$ such that there exists a scalar quantity ϵ

* The term "divergence" is due to Clifford, 1878.

satisfying the relations,

$$(1)\quad (\nabla\cdot\mathbf{V}+\epsilon)[\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}] = [\Delta_1\mathbf{V}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}] + [\Delta_1\mathbf{p}\;\Delta_2\mathbf{V}\;\Delta_3\mathbf{p}] + [\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{V}],$$
$$\lim_{\Delta_1\mathbf{p},\,\Delta_2\mathbf{p},\,\Delta_3\mathbf{p}\to 0} \epsilon = 0.$$

Here ϵ is to approach zero for every manner of approach of $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ to zero for which these vectors remain non-coplanar. By this we mean that there must exist some positive number δ such that $[\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3]$ remains in absolute value greater than δ, where $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are variable unit vectors having the direction and sense respectively of $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$. If the divergence $\nabla\cdot\mathbf{V}$ exists, then we define the *differential of the flux* as the quantity,

$$(2)\qquad df = \nabla\cdot\mathbf{V}[\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}]$$

and easily show that,

$$(3)\quad [\Delta_1\mathbf{V}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}] + [\Delta_1\mathbf{p}\;\Delta_2\mathbf{V}\;\Delta_3\mathbf{p}] + [\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{V}] = df + \epsilon[\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}].$$

The significance of the term "flux" will appear when we discuss the kinematical interpretation of the divergence.

As a first step in the study of the divergence $\nabla\cdot\mathbf{V}$ we observe that if a scalar $\nabla\cdot\mathbf{V}$ exists satisfying conditions (1), it must be unique. For suppose there were two values $\nabla_1\cdot\mathbf{V}$ and $\nabla_2\cdot\mathbf{V}$ with their corresponding ϵ_1 and ϵ_2 satisfying conditions (1). Substituting them in the first of equations (1) and subtracting the results, we have after dividing through by $[\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}]$,

$$\Delta_1\cdot\mathbf{V} - \Delta_2\cdot\mathbf{V} = \epsilon_2 - \epsilon_1.$$

As $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ approach zero the first member of this equation remains constant while the second member approaches zero and it follows that the first member must *be* zero. Consequently $\nabla_1\cdot\mathbf{V} = \nabla_2\cdot\mathbf{V}$ and the divergence is unique if it exists.

We next establish the important result that,

If $\mathbf{V}(\mathbf{p})$ *is of constant direction, its divergence is the derivative of its length in this direction.*

Thus if $\mathbf{V}(\mathbf{p})$ is of the form $\mathbf{V}(\mathbf{p}) = U(\mathbf{p})\,\mathbf{a}$ where U is a scalar function of $\mathbf{p}$ and $\mathbf{a}$ is a constant vector, the above theorem states in effect that,

$$(4)\qquad \nabla\cdot(U\,\mathbf{a}) = \mathbf{a}\cdot\nabla U.$$

To establish this we first observe that by equation § 99, (1),

$$(\nabla U + \boldsymbol{\varepsilon}_1)\cdot\Delta_1\mathbf{p} = \Delta_1 U, \qquad \lim_{\Delta_1\mathbf{p}\to 0} \boldsymbol{\varepsilon}_1 = 0,$$

$$(\nabla U + \boldsymbol{\varepsilon}_2)\cdot\Delta_2\mathbf{p} = \Delta_2 U, \qquad \lim_{\Delta_2\mathbf{p}\to 0} \boldsymbol{\varepsilon}_2 = 0,$$

$$(\nabla U + \boldsymbol{\varepsilon}_3)\cdot\Delta_3\mathbf{p} = \Delta_3 U, \qquad \lim_{\Delta_3\mathbf{p}\to 0} \boldsymbol{\varepsilon}_3 = 0.$$

Using these results and the obvious fact that $\Delta(U\,\mathbf{a}) = (\Delta U)\mathbf{a}$, we find that equation (1) becomes for $\mathbf{V} = U\,\mathbf{a}$,

$$\begin{aligned}\{\nabla\cdot(U\,\mathbf{a}) + \epsilon\}[\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}] = {} & (\nabla U + \boldsymbol{\varepsilon}_1)\cdot\Delta_1\mathbf{p}[\mathbf{a}\;\Delta_2\mathbf{p}\;\Delta_3\mathbf{p}] \\ & + (\nabla U + \boldsymbol{\varepsilon}_2)\cdot\Delta_2\mathbf{p}[\Delta_1\mathbf{p}\;\mathbf{a}\;\Delta_3\mathbf{p}] \\ & + (\nabla U + \boldsymbol{\varepsilon}_3)\cdot\Delta_3\mathbf{p}[\Delta_1\mathbf{p}\;\Delta_2\mathbf{p}\;\mathbf{a}].\end{aligned}$$

Employing the above variable unit vectors $\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3$ having the direction and sense of $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$, we may set,

$$\Delta_1\mathbf{p} = \mathbf{u}_1\,\Delta s_1, \qquad \Delta_2\mathbf{p} = \mathbf{u}_2\,\Delta s_2, \qquad \Delta_3\mathbf{p} = \mathbf{u}_3\,\Delta s_3,$$

and the above equation becomes after division by $\Delta s_1\,\Delta s_2\,\Delta s_3$,

$$\begin{aligned}\{\nabla\cdot(U\,\mathbf{a}) + \epsilon\}[\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3] = \mathbf{a}\cdot[\{(\nabla U + \boldsymbol{\varepsilon}_1)\cdot\mathbf{u}_1\}\mathbf{u}_2\times\mathbf{u}_3 \\ + \{(\nabla U + \boldsymbol{\varepsilon}_2)\cdot\mathbf{u}_2\}\mathbf{u}_3\times\mathbf{u}_1 + \{(\nabla U + \boldsymbol{\varepsilon}_3)\cdot\mathbf{u}_3\}\mathbf{u}_1\times\mathbf{u}_2].\end{aligned}$$

We now reduce the terms involving ∇U in the second member by means of the familiar identity § 21, Prob. 1, *f*,

$$(\mathbf{a}\cdot\mathbf{x})\mathbf{b}\times\mathbf{c} + (\mathbf{b}\cdot\mathbf{x})\mathbf{c}\times\mathbf{a} + (\mathbf{c}\cdot\mathbf{x})\mathbf{a}\times\mathbf{b} = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]\mathbf{x}$$

and find that our equation takes the form,

$$\begin{aligned}\{\nabla\cdot(U\,\mathbf{a}) + \epsilon\}[\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3] = \mathbf{a}\cdot\nabla U[\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3] \\ + \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2\times\mathbf{u}_3 + (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\,\mathbf{u}_3\times\mathbf{u}_1 + (\boldsymbol{\varepsilon}_3\cdot\mathbf{u}_3)\,\mathbf{u}_1\times\mathbf{u}_2\}.\end{aligned}$$

It now appears that both of conditions (1) will be satisfied by setting $\nabla\cdot(U\,\mathbf{a}) = \mathbf{a}\cdot\nabla U$ for this leaves,

$$\begin{aligned}\epsilon[\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3] = \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2\times\mathbf{u}_3 \\ + (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\mathbf{u}_3\times\mathbf{u}_1 + (\boldsymbol{\varepsilon}_3\cdot\mathbf{u}_3)\mathbf{u}_1\times\mathbf{u}_2\},\end{aligned}$$

and since the second member vanishes with $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ and since by hypothesis $[\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3]$ can not approach zero, it follows that ϵ as here defined must also vanish with $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$. Remembering that the divergence if it exists is unique, we find

theorem (4) established. The argument also makes it clear that if U possesses a gradient ∇U, then $U\mathbf{a}$ possesses a divergence for every choice of the constant vector $\mathbf{a}$.

It is now a simple matter to extend formula (4) to the general vector-vector function $\mathbf{V}(\mathbf{p})$. The reader may first readily establish directly from conditions (1) that the divergence is distributive with respect to addition. That is, if $\mathbf{V}$ and $\mathbf{W}$ possess divergences $\nabla\cdot\mathbf{V}$ and $\nabla\cdot\mathbf{W}$ then $\mathbf{V}+\mathbf{W}$ possesses the divergence $\nabla\cdot(\mathbf{V}+\mathbf{W}) = \nabla\cdot\mathbf{V}+\nabla\cdot\mathbf{W}$. Now if $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are any three non-coplanar constant vectors, it is a familiar fact that we may always write,

$$\mathbf{V}(\mathbf{p}) = U_1(\mathbf{p})\mathbf{a}_1 + U_2(\mathbf{p})\mathbf{a}_2 + U_3(\mathbf{p})\mathbf{a}_3,$$

and applying formula (4) to each of the terms of the second member yields with the above distributive law,

$$(5) \qquad \nabla\cdot\mathbf{V} = \mathbf{a}_1\cdot\nabla U_1 + \mathbf{a}_2\cdot\nabla U_2 + \mathbf{a}_3\cdot\nabla U_3.$$

In particular if $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be chosen as three mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ having a positive sense of rotation, we have $U_1 = \mathbf{V}\cdot\mathbf{i}$, $U_2 = \mathbf{V}\cdot\mathbf{j}$, $U_3 = \mathbf{V}\cdot\mathbf{k}$ and equation (5) becomes,

$$(6) \qquad \nabla\cdot\mathbf{V} = \mathbf{i}\cdot\nabla(\mathbf{V}\cdot\mathbf{i}) + \mathbf{j}\cdot\nabla(\mathbf{V}\cdot\mathbf{j}) + \mathbf{k}\cdot\nabla(\mathbf{V}\cdot\mathbf{k}).$$

By the aid of the formula of § 100, Prob. (1, c) this may also be written in the form,

$$(7) \qquad \nabla\cdot\mathbf{V} = \mathbf{i}\cdot(\mathbf{i}\cdot\nabla\mathbf{V}) + \mathbf{j}\cdot(\mathbf{j}\cdot\nabla\mathbf{V}) + \mathbf{k}\cdot(\mathbf{k}\cdot\nabla\mathbf{V}),$$

and finally if these vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the coördinate vectors and if $\mathbf{V}$ has the coördinates (V_1, V_2, V_3), either of the last two equations yields the simple formula,

$$(8) \qquad \nabla\cdot\mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

We may remember this as the symbolic scalar product of the symbolic vector $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and the vector function $\mathbf{V} \equiv (V_1\, V_2\, V_3)$. The argument makes it clear that if $\mathbf{V}(\mathbf{p})$ possesses derivatives in three non-coplanar directions it possesses a divergence. In this connection see Problem 1 at the end of this article.

In certain simple cases the divergence of a given vector-vector function can be derived directly from the definition (1) but in the more complicated cases formula (7) or (8) is usually more convenient. As an example of the first type we may derive the divergence of the independent variable $\mathbf{p}$ itself. In this case the first of equations (1) becomes,

$$(\nabla\cdot\mathbf{p} + \epsilon)[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}] = 3[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$$

and we have at once,

$$\nabla\cdot\mathbf{p} = 3, \tag{9}$$

the quantity ϵ being in this case identically zero. As an application of formula (7) or (8) we may prove the important formula,

$$\nabla\cdot(U\,\mathbf{V}) = U\,\nabla\cdot\mathbf{V} + (\nabla U)\cdot\mathbf{V}, \tag{10}$$

in which U and $\mathbf{V}$ are respectively scalar and vector functions of $\mathbf{p}$. By § 100, Problem (1, b) we have,

$$(\mathbf{a}\cdot\nabla)(U\,\mathbf{V}) = \{(\mathbf{a}\cdot\nabla)U\}\mathbf{V} + U(\mathbf{a}\cdot\nabla)\mathbf{V},$$

in which $\mathbf{a}$ is any constant vector. If we now let $\mathbf{a}$ take on in succession the values $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of the coördinate vectors and dot-multiply the resulting equations through by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively and add the results member for member, we see by formula (7) that the sum of the first members is exactly $\nabla\cdot(U\,\mathbf{V})$ and the sum of the last terms of the second members is exactly $U\,\nabla\cdot\mathbf{V}$, while the sum of the first terms of the second members is,

$$\begin{aligned}\{(\mathbf{i}\cdot\nabla)U\}\mathbf{V}\cdot\mathbf{i} + \{(\mathbf{j}\cdot\nabla)U\}\mathbf{V}\cdot\mathbf{j} + \{(\mathbf{k}\cdot\nabla)U\}\mathbf{V}\cdot\mathbf{k}\\ = \{\mathbf{i}\cdot(\nabla U)\mathbf{i} + \mathbf{j}\cdot(\nabla U)\mathbf{j} + \mathbf{k}\cdot(\nabla U)\mathbf{k}\}\cdot\mathbf{V} = (\nabla U)\cdot\mathbf{V}.\end{aligned}$$

Thus we find formula (10) established.

The divergence of a vector-point function $\mathbf{V}(\mathbf{p})$ has a kinematical interpretation which is very helpful in understanding and applying it. We shall now briefly describe this interpretation. We consider the region of space in which the vector-point function $\mathbf{V}(\mathbf{p})$ is defined as being occupied by some fluid such as air in a state of steady flow so that the velocity of the fluid at each point is a constant vector given by the value of $\mathbf{V}$ at that point. We let $\mathbf{p}_0$ be the radius vector of a point P_0 and $\mathbf{V}_0$ the vector velocity of the fluid at that point. Then $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ are three non-coplanar infinitesimal vectors which we may think of as

radiating from P_0 and which form three concurrent edges of a parallelopiped of volume $[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$ having P_0 at one vertex. Let us consider that face of this parallelopiped having P_0 at one vertex and having $\Delta\mathbf{p}_2$ and $\Delta\mathbf{p}_3$ as edges radiating from P_0. Since the vector $\Delta_2\mathbf{p} \times \Delta_3\mathbf{p}$ is perpendicular to this face and equal in length to the area of this face, it follows that in the quantity,

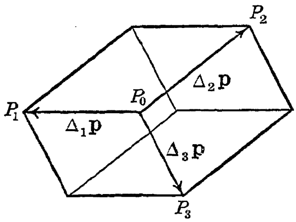

FIG. 85

$$\mathbf{V}_0 \cdot \Delta_2\mathbf{p} \times \Delta_3\mathbf{p} = [\mathbf{V}_0\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$$

we have the product of the area of the face by the velocity of the fluid in a direction normal to this face. Consequently this expression gives approximately the volume of fluid flowing through the face in unit time or, as it is called, the *flux* through the face. The flux will be into the parallelopiped or outward from it according as $[\mathbf{V}_0\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$ and $[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$ have like or unlike signs. If we now proceed from P_0 along the edge $\Delta_1\mathbf{p}$ to the vertex P_1 of the parallelopiped, the velocity of the fluid will change from $\mathbf{V}_0$ to $\mathbf{V}_0 + \Delta_1\mathbf{V}$ and consequently the flux through the face opposite to that previously discussed will be approximately,

$$(\mathbf{V}_0 + \Delta_1\mathbf{V}) \cdot \Delta_2\mathbf{p} \times \Delta_3\mathbf{p} = [(\mathbf{V}_0 + \Delta_1\mathbf{V})\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}],$$

but in this case the flux will be outward from the parallelopiped or into it according as $[(\mathbf{V}_0 + \Delta_1\mathbf{V})\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$ and $[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$ have like or unlike signs. If we subtract the first of these results from the second we have,

$$[\Delta_1\mathbf{V}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$$

as the approximate net flux through these two faces of the parallelopiped, the sign convention being that last stated. The flux through the other pairs of opposite faces of the parallelopiped are similarly expressed and it thus appears that we have in the second member of the first of equations (1) an approximate expression for the total flux of the fluid through the surface of the parallelopiped. As $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ approach zero this approximate expression for the flux becomes more nearly correct and differs from the actual flux by an infinitesimal of higher order than

$[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$, and it follows from equations (1) that the quantity,

$$df = \nabla \cdot \mathbf{V}[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}],$$

being the principal part of the flux, is properly called the differential of the flux. Remembering that $[\Delta_1\mathbf{p}\ \Delta_2\mathbf{p}\ \Delta_3\mathbf{p}]$ is the volume of the infinitesimal parallelopiped it follows from the last equation that in the limit as $\Delta_1\mathbf{p}$, $\Delta_2\mathbf{p}$, $\Delta_3\mathbf{p}$ approach zero we have,

The divergence $\nabla \cdot \mathbf{V}$ *is at each point the net outward flux of the fluid from that point per unit volume.*

Or in other words the divergence at a point is the volume of fluid leaving that point per unit volume per unit time.

In the case of the air in a room heated by a stove we should find that the divergence of the air is positive at points near the stove and negative at points near the cold windows due to the expanding effect of heat and the contracting effect of cold. If the fluid were practically incompressible as in the case of water at constant temperature, the divergence would be everywhere zero and we could write,

$$\nabla \cdot \mathbf{V} = 0,$$

an equation of the greatest importance in hydrodynamics

EXERCISES

1. Show that if a vector-point function $\mathbf{V}(\mathbf{p})$ possesses a derivative in the direction of each of three constant non-coplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, then $\mathbf{V}(\mathbf{p})$ has a divergence $\nabla \cdot \mathbf{V}$ given by the formula,

$$\nabla \cdot \mathbf{V} = \mathbf{b}_1 \cdot \{(\mathbf{a}_1 \cdot \nabla)\mathbf{V}\} + \mathbf{b}_2 \cdot \{\mathbf{a}_2 \cdot \nabla\mathbf{V}\} + \mathbf{b}_3 \cdot \{a_3 \cdot \nabla\mathbf{V}\},$$

where,

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]}, \qquad \mathbf{b}_2 - \frac{\mathbf{a}_3 \times \mathbf{a}_1}{[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]}, \qquad \mathbf{b}_3 \quad \frac{\mathbf{a}_1 \times \mathbf{a}_2}{[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]}.$$

2. If $\mathbf{a}$ and $\mathbf{b}$ are any constant vectors, show that,

(*a*) $\nabla \cdot \{(\mathbf{a} \cdot \mathbf{p})\mathbf{b}\} = \nabla \cdot \{(\mathbf{b} \cdot \mathbf{p})\mathbf{a}\} = \mathbf{a} \cdot \mathbf{b}$,
(*b*) $\nabla \cdot \{\mathbf{a} \times (\mathbf{p} \times \mathbf{b})\} = \nabla \cdot \{\mathbf{b} \times (\mathbf{p} \times \mathbf{a})\} = 2\,\mathbf{a} \cdot \mathbf{b}$,
(*c*) $\nabla \cdot \{f(|\mathbf{a} \times \mathbf{p}|)\,\mathbf{a} \times \mathbf{p}\} = 0$.

3. Show that if $\mathbf{a}$ is a constant vector,

(*a*) $\nabla \cdot (|\mathbf{p}|^n\,\mathbf{p}) = (n + 3)|\mathbf{p}|^n$,
(*b*) $\nabla \cdot \{(\mathbf{a} \cdot \mathbf{p})^n\,\mathbf{p}\} = (n + 3)(\mathbf{a} \cdot \mathbf{p})^n$,
(*c*) $\nabla \cdot \{|\mathbf{a} \times \mathbf{p}|^n\,\mathbf{p}\} = (n + 3)|\mathbf{a} \times \mathbf{p}|^n$.

4. If $\mathbf{V}$ has the direction and sense of $\mathbf{p}$ and the length of $\mathbf{V}$ varies inversely as the square of the length of $\mathbf{p}$, show that $\nabla \cdot \mathbf{V} \equiv 0$.

5. Show by application of the general transformation to new rectangular coördinates that $\nabla \cdot \mathbf{V}$ is a differential invariant of the vector-point function $\mathbf{V}(x, y, z)$. In other words, show that any such transformation carrying $\mathbf{V}(x, y, z)$ into $\mathbf{V}'(x', y', z')$ will likewise carry $\nabla \cdot \mathbf{V}$ into the divergence of $\mathbf{V}'$ with respect to (x', y', z'). Has this been proven in the text?

6. By the aid of Euler's theorem on homogeneous functions, generalize Problem (3) to show that,

$$\nabla \cdot (U\,\mathbf{p}) = (n + 3)\,U,$$

where U is homogeneous of degree n in the coördinates (x, y, z) of $\mathbf{p}$.

102. Curl of a Vector-Vector Function.

As remarked in the previous article, a large part, but not all, of the first order differential properties of a vector-vector function $\mathbf{V}(\mathbf{p})$ are determined by its divergence $\nabla \cdot \mathbf{V}$ and its curl $\nabla \times \mathbf{V}$. In this article we shall discuss the curl $\nabla \times \mathbf{V}$. The reader would do well to observe the striking analogy in definition and properties between the curl as here discussed and the divergence as discussed in the previous article. Being given a vector-vector function $\mathbf{V}(\mathbf{p})$, we let $\mathbf{p}_0$ be any particular value of $\mathbf{p}$ and $\mathbf{V}_0$ the corresponding value of $\mathbf{V}$. We let $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ be any two nonparallel variations of $\mathbf{p}$ from the value $\mathbf{p}_0$ and $\Delta_1\mathbf{V}$ and $\Delta_2\mathbf{V}$ be the corresponding variations of $\mathbf{V}$ from the value $\mathbf{V}_0$. We then define the *curl* * $\nabla \times \mathbf{V}$ as a vector quantity independent of $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ such that there exists a vector quantity $\boldsymbol{\varepsilon}$ satisfying the relations,

$$(1) \qquad (\nabla \times \mathbf{V} + \boldsymbol{\varepsilon}) \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p} = \Delta_1\mathbf{V} \cdot \Delta_2\mathbf{p} - \Delta_2\mathbf{V} \cdot \Delta_1\mathbf{p},$$
$$\lim_{\Delta_1\mathbf{p},\ \Delta_2\mathbf{p} \to 0} \boldsymbol{\varepsilon} = 0.$$

Here $\boldsymbol{\varepsilon}$ is to approach zero for every manner of approach of $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ to zero for which these vectors remain nonparallel. By this we mean that there must exist some positive number δ such that $\mathbf{u}_1 \times \mathbf{u}_2$ remains in length greater than δ, where $\mathbf{u}_1$, $\mathbf{u}_2$ are variable unit vectors having the direction and sense respectively of $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$. If the curl $\nabla \times \mathbf{V}$ exists then we define the *differ-*

* The term curl is due to Maxwell. 1890.

ential of the circulation as the quantity,

$$(2) \qquad dg = (\nabla \times \mathbf{V}) \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p}$$

and easily show that,

$$(3) \qquad \Delta_1\mathbf{V} \cdot \Delta_2\mathbf{p} - \Delta_2\mathbf{V} \cdot \Delta_1\mathbf{p} = dg + \boldsymbol{\varepsilon} \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p}.$$

The significance of the term *circulation* will appear when we discuss the kinematical interpretation of the curl.

We first observe that if a vector $\nabla \times \mathbf{V}$ exists satisfying conditions (1) it must be unique. For suppose there were two values $\nabla_1 \times \mathbf{V}$ and $\nabla_2 \times \mathbf{V}$ with their corresponding $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ satisfying conditions (1). Substituting them in the first of equations (1) and subtracting the results we have,

$$(\nabla_1 \times \mathbf{V} - \nabla_2 \times \mathbf{V}) \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p} = (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p}.$$

If we now let $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ approach zero in such a fashion that $\Delta_1\mathbf{p} \times \Delta_2\mathbf{p}$ retains the direction and sense of a constant unit vector $\mathbf{u}$, then during the limiting process we have,

$$(\nabla_1 \times \mathbf{V} - \nabla_2 \times \mathbf{V}) \cdot \mathbf{u} = (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) \cdot \mathbf{u},$$

and since the left member is a constant while the right member approaches zero it follows that the left member must *be* zero. Thus we have $(\nabla_1 \times \mathbf{V} - \nabla_2 \times \mathbf{V}) \cdot \mathbf{u} = 0$ for *every choice* of the unit vector $\mathbf{u}$. Consequently $\nabla_1 \times \mathbf{V} = \nabla_2 \times \mathbf{V}$ and the curl must be unique if it exists.

We next establish the important result that,

If $\mathbf{V}(\mathbf{p})$ *is of constant direction, its curl is the vector product of the gradient of its length and a unit vector having its direction and sense.*

Thus if $\mathbf{V}(\mathbf{p})$ is of the form $\mathbf{V}(\mathbf{p}) = U(\mathbf{p})\,\mathbf{a}$ where U is a scalar function of $\mathbf{p}$ and $\mathbf{a}$ is a constant vector, the above theorem states in effect that,

$$(4) \qquad \nabla \times (U\,\mathbf{a}) = (\nabla U) \times \mathbf{a}.$$

To establish this we first observe that by equation § 99, (1),

$$(\nabla U + \boldsymbol{\varepsilon}_1) \cdot \Delta_1\mathbf{p} = \Delta_1 U, \qquad \lim_{\Delta_1\mathbf{p} \to 0} \boldsymbol{\varepsilon}_1 = 0,$$

$$(\nabla U + \boldsymbol{\varepsilon}_2) \cdot \Delta_2\mathbf{p} = \Delta_2 U, \qquad \lim_{\Delta_2\mathbf{p} \to 0} \boldsymbol{\varepsilon}_2 = 0.$$

Using these results and the obvious fact that $\Delta(U\,\mathbf{a}) = (\Delta U)\,\mathbf{a}$ we find that equation (1) becomes for $\mathbf{V} = U\,\mathbf{a}$,

$$\{\nabla \times (U\,\mathbf{a}) + \boldsymbol{\varepsilon}\}\cdot\Delta_1\mathbf{p}\times\Delta_2\mathbf{p} = \{(\nabla U + \boldsymbol{\varepsilon}_1)\cdot\Delta_1\mathbf{p}\}\mathbf{a}\cdot\Delta_2\mathbf{p} - \{(\nabla U + \boldsymbol{\varepsilon}_2)\cdot\Delta_2\mathbf{p}\}\mathbf{a}\cdot\Delta_1\mathbf{p}.$$

Employing the above variable unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ having the direction and sense of $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$, we may set,

$$\Delta_1\mathbf{p} = \mathbf{u}_1\,\Delta s_1, \qquad \Delta_2\mathbf{p} = \mathbf{u}_2\,\Delta s_2$$

and the above equation becomes after division by $\Delta s_1\,\Delta s_2$,

$$\{\nabla \times (U\,\mathbf{a}) + \boldsymbol{\varepsilon}\}\cdot\mathbf{u}_1\times\mathbf{u}_2 = \mathbf{a}\cdot\{(\nabla U\cdot\mathbf{u}_1)\mathbf{u}_2 - (\nabla U\cdot\mathbf{u}_2)\mathbf{u}_1\} + \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2 - (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\mathbf{u}_1\}$$

or

$$\begin{aligned}\{\nabla \times (U\,\mathbf{a}) + \boldsymbol{\varepsilon}\}\cdot\mathbf{u}_1\times\mathbf{u}_2 &= \mathbf{a}\cdot\nabla U\times(\mathbf{u}_2\times\mathbf{u}_1) + \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2 - (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\mathbf{u}_1\}\\ &= (\nabla U\times\mathbf{a})\cdot(\mathbf{u}_1\times\mathbf{u}_2) + \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2 - (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\mathbf{u}_1\}.\end{aligned}$$

It now appears that both of conditions (1) will be satisfied by setting $\nabla \times (U\,\mathbf{a}) = (\nabla U)\times\mathbf{a}$ for this leaves,

$$\boldsymbol{\varepsilon}\cdot\mathbf{u}_1\times\mathbf{u}_2 = \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2 - (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\mathbf{u}_1\}.$$

A vector $\boldsymbol{\varepsilon}$ satisfying this equation and vanishing with $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ evidently exists, being in fact given by the equation,

$$(\mathbf{u}_1\times\mathbf{u}_2)^2\,\boldsymbol{\varepsilon} = \mathbf{a}\cdot\{(\boldsymbol{\varepsilon}_1\cdot\mathbf{u}_1)\mathbf{u}_2 - (\boldsymbol{\varepsilon}_2\cdot\mathbf{u}_2)\mathbf{u}_1\}\;\mathbf{u}_1\times\mathbf{u}_2,$$

for since the second member vanishes with $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ and since $(\mathbf{u}_1\times\mathbf{u}_2)^2$ can not approach zero, it follows that this $\boldsymbol{\varepsilon}$ must vanish with $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$. Remembering that the curl, if it exists, is unique, we find theorem (4) established. The argument also makes it clear that if U possesses a gradient then $U\,\mathbf{a}$ possesses a curl for every choice of the constant vector $\mathbf{a}$.

It is now possible to extend formula (4) to the general vector-vector function $\mathbf{V}(\mathbf{p})$. The reader may first readily establish directly from conditions (1) that the curl is distributive with respect to addition. That is, if $\mathbf{V}$ and $\mathbf{W}$ possess curls $\nabla\times\mathbf{V}$ and $\nabla\times\mathbf{W}$, then $\mathbf{V}+\mathbf{W}$ possesses the curl $\nabla\times(\mathbf{V}+\mathbf{W}) = \nabla\times\mathbf{V} + \nabla\times\mathbf{W}$. If now $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are any three non-coplanar constant vectors it is a familiar fact that we may always write,

$$\mathbf{V}(\mathbf{p}) = U_1(\mathbf{p})\mathbf{a}_1 + U_2(\mathbf{p})\mathbf{a}_2 + U_3(\mathbf{p})\mathbf{a}_3$$

and applying formula (4) to each term of the second member yields with the above distributive law,

$$(5) \qquad \nabla \times \mathbf{V} = (\nabla U_1) \times \mathbf{a}_1 + (\nabla U_2) \times \mathbf{a}_2 + (\nabla U_3) \times \mathbf{a}_3.$$

In particular if $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be chosen as three mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ having a positive sense of rotation, we have, $U_1 = \mathbf{V} \cdot \mathbf{i}$, $U_2 = \mathbf{V} \cdot \mathbf{j}$, $U_3 = \mathbf{V} \cdot \mathbf{k}$ and equation (5) becomes,

$$(6) \qquad \nabla \times \mathbf{V} = \nabla(\mathbf{V} \cdot \mathbf{i}) \times \mathbf{i} + \nabla(\mathbf{V} \cdot \mathbf{j}) \times \mathbf{j} + \nabla(\mathbf{V} \cdot \mathbf{k}) \times \mathbf{k}.$$

From this we may also derive another useful form of $\nabla \times \mathbf{V}$. We first observe that,

$$\begin{aligned} \nabla \times \mathbf{V} &= \nabla(\mathbf{V} \cdot \mathbf{i}) \times (\mathbf{j} \times \mathbf{k}) + \nabla(\mathbf{V} \cdot \mathbf{j}) \times (\mathbf{k} \times \mathbf{i}) + \nabla(\mathbf{V} \cdot \mathbf{k}) \times (\mathbf{i} \times \mathbf{j}) \\ &= \mathbf{k} \cdot \nabla(\mathbf{V} \cdot \mathbf{i})\mathbf{j} - \mathbf{j} \cdot \nabla(\mathbf{V} \cdot \mathbf{i})\mathbf{k} \\ &\quad + \mathbf{i} \cdot \nabla(\mathbf{V} \cdot \mathbf{j})\mathbf{k} - \mathbf{k} \cdot \nabla(\mathbf{V} \cdot \mathbf{j})\mathbf{i} \\ &\quad + \mathbf{j} \cdot \nabla(\mathbf{V} \cdot \mathbf{k})\mathbf{i} - \mathbf{i} \cdot \nabla(\mathbf{V} \cdot \mathbf{k})\mathbf{j}. \end{aligned}$$

Since $\mathbf{a} \cdot \nabla(\mathbf{V} \cdot \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \cdot \nabla \mathbf{V})$ the above becomes on regrouping the terms,

$$\begin{aligned} \nabla \times \mathbf{V} &= \mathbf{j} \cdot (\mathbf{i} \cdot \nabla \mathbf{V})\mathbf{k} - \mathbf{k} \cdot (\mathbf{i} \cdot \nabla \mathbf{V})\mathbf{j} \\ &\quad + \mathbf{k} \cdot (\mathbf{j} \cdot \nabla \mathbf{V})\mathbf{i} - \mathbf{i} \cdot (\mathbf{j} \cdot \nabla \mathbf{V})\mathbf{k} \\ &\quad + \mathbf{i} \cdot (\mathbf{k} \cdot \nabla \mathbf{V})\mathbf{j} - \mathbf{j} \cdot (\mathbf{k} \cdot \nabla \mathbf{V})\mathbf{i} \\ &= (\mathbf{i} \cdot \nabla \mathbf{V}) \times (\mathbf{k} \times \mathbf{j}) + (\mathbf{j} \cdot \nabla \mathbf{V}) \times (\mathbf{i} \times \mathbf{k}) + (\mathbf{k} \cdot \nabla \mathbf{V})(\mathbf{j} \times \mathbf{i}) \end{aligned}$$

and finally,

$$(7) \qquad \nabla \times \mathbf{V} = \mathbf{i} \times (\mathbf{i} \cdot \nabla \mathbf{V}) + \mathbf{j} \times (\mathbf{j} \cdot \nabla \mathbf{V}) + \mathbf{k} \times (\mathbf{k} \cdot \nabla \mathbf{V}).$$

If these vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the coördinate vectors and if $\mathbf{V}$ has the coördinates $\mathbf{V} \equiv (V_1, V_2, V_3)$, either (6) or (7) yields,

$$(8) \qquad \nabla \times \mathbf{V} \equiv \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right).$$

We may remember this as the symbolic vector product of the symbolic vector $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ and the vector function $\mathbf{V} \equiv (V_1, V_2, V_3)$. The argument makes it clear that if $\mathbf{V}(\mathbf{p})$ possesses derivatives in three non-coplanar directions it possesses a curl. In this connection see Problem 1 at the end of this article. The converse of this is however not always true.

In certain simple cases the curl of a given vector-vector function can be derived directly from the definition (1) but in the

more complicated cases formula (7) or (8) is usually more convenient. As an example of the first type we may derive the curl of the independent variable **p** itself. In this case the first of equations (1) becomes,

$$(\nabla \times \mathbf{p} + \boldsymbol{\varepsilon}) \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p} = \Delta_1\mathbf{p} \cdot \Delta_2\mathbf{p} - \Delta_2\mathbf{p} \cdot \Delta_1\mathbf{p} = 0,$$

and we have at once,

$$(9) \qquad \nabla \times \mathbf{p} = 0,$$

the quantity $\boldsymbol{\varepsilon}$ being in this case identically zero. As an application of formula (7) or (8) we may prove the important formula,

$$(10) \qquad \nabla \times (U\,\mathbf{V}) = U\,\nabla \times \mathbf{V} + (\nabla U) \times \mathbf{V},$$

in which U and **V** are respectively scalar and vector functions of **p**. By § 100, Problem 1, *b* we have,

$$(\mathbf{a} \cdot \nabla)(U\,\mathbf{V}) = (\mathbf{a} \cdot \nabla U)\mathbf{V} + U(\mathbf{a} \cdot \nabla)\mathbf{V},$$

in which **a** is any constant vector. If we now let **a** take on successively the values **i**, **j**, **k** of the coördinate vectors and cross-multiply the resulting equations through by **i**, **j**, **k** respectively and add the results member for member, we see by formula (7) that the sum of the first members is exactly $\nabla \times (U\mathbf{V})$ and the sum of the last terms of the second members is exactly $U\,\nabla \times \mathbf{V}$ while the sum of the first terms of the second members is,

$$\begin{aligned}(\mathbf{i} \cdot \nabla U)\mathbf{i} \times \mathbf{V} &+ (\mathbf{j} \cdot \nabla U)\mathbf{j} \times \mathbf{V} + (\mathbf{k} \cdot \nabla U)\mathbf{k} \times \mathbf{V} \\ &= \{\mathbf{i} \cdot (\nabla U)\mathbf{i} + \mathbf{j} \cdot (\nabla U)\mathbf{j} + \mathbf{k} \cdot (\nabla U)\mathbf{k}\} \times \mathbf{V} = (\nabla U) \times \mathbf{V}.\end{aligned}$$

Consequently we find formula (10) established.

We saw in the preceding article that the divergence of a vector-point function has an interpretation associated with the steady flow of a fluid. The same is true of the curl of such a function, although in this case the interpretation is less simple and possibly does not have such a strong intuitional appeal. We consider the region of space in which the vector-point function $\mathbf{V}(\mathbf{p})$ is defined as being occupied by some fluid such as air in a state of steady flow, so that the velocity of the fluid at each point is a constant vector given by the value of **V** at that point. We let $\mathbf{p}_0$ be the radius vector of a point P_0 and $\mathbf{V}_0$ the vector velocity of the fluid at that point. Then $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ are non-parallel infinitesimal vectors which we may think of as radiating from

P_0 and which form two adjacent edges of a parallelogram of area $|\Delta_1\mathbf{p} \times \Delta_2\mathbf{p}|$ having P_0 at one vertex. If we now consider that edge of this parallelogram running from P_0 along $\Delta_1\mathbf{p}$ to P_1, and form the scalar product,

$$\mathbf{V}_0 \cdot \Delta_1\mathbf{p},$$

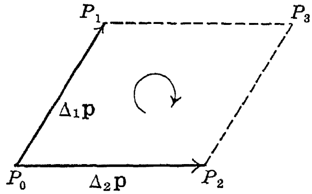

FIG. 86

we have a quantity which, being equal to the length of the vector $\Delta_1\mathbf{p}$ multiplied by the velocity of the fluid in its direction, constitutes a measure of the extent to which the fluid flows along this vector. This quantity is consequently known as the *elementary circulation* of $\mathbf{V}$ along $\Delta_1\mathbf{p}$. If we similarly set up the elementary circulation around the entire edge of the parallelogram in the positive sense of rotation relative to $\Delta_1\mathbf{p} \times \Delta_2\mathbf{p}$, we have,

$$\Delta g = \mathbf{V}_0 \cdot \Delta_1\mathbf{p} + (\mathbf{V}_0 + \Delta_1\mathbf{V}) \cdot \Delta_2\mathbf{p} - (\mathbf{V}_0 + \Delta_2\mathbf{V}) \cdot \Delta_1\mathbf{p} - \mathbf{V}_0 \cdot \Delta_2\mathbf{p},$$

which reduces to,

$$\Delta g = \Delta_1\mathbf{V} \cdot \Delta_2\mathbf{p} - \Delta_2\mathbf{V} \cdot \Delta_1\mathbf{p}.$$

Referring to equation (1) we see that the quantity,

$$dg = (\nabla \times \mathbf{V}) \cdot \Delta_1\mathbf{p} \times \Delta_2\mathbf{p}$$

differs from this elementary circulation by an infinitesimal of higher order than $|\Delta_1\mathbf{p} \times \Delta_2\mathbf{p}|$ and so is properly called the differential of the circulation.

Let us now represent by θ the angle which the curl $\nabla \times \mathbf{V}$ makes with the normal $\Delta_1\mathbf{p} \times \Delta_2\mathbf{p}$ to the plane of the above parallelogram and by $\Delta\sigma$ the area $|\Delta_1\mathbf{p} \times \Delta_2\mathbf{p}|$ of the parallelogram. Then the above equation becomes,

$$|\nabla \times \mathbf{V}| \cos\theta \, \Delta\sigma = dg,$$

and it immediately appears that dg will be largest for a given $\Delta\sigma$ when $\theta = 0$ and will then equal $|\nabla \times \mathbf{V}| \, \Delta\sigma$. Thus in the limit as $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ approach zero, we have,

The curl $\nabla \times \mathbf{V}$ is at each point normal to the plane of maximum circulation per unit area about that point.

The circulation about each point is positive in the positive sense of rotation relative to the curl at that point.

The length $|\nabla \times \mathbf{V}|$ of the curl at each point equals the maximum circulation per unit area about that point.

As an example of the intuitional application of the above kinematical interpretation of the curl, let us let $\mathbf{V}$ represent the vector velocity of the points of a rigid body in a state of steady rotation about an axis through the point considered. Let $\mathbf{u}$ be a unit vector having the direction and sense of this axis and let ω be the scalar angular velocity of the body about the axis. If we take a little circle of radius a surrounding the point considered, it is evident that the velocities of the points of the body will have the maximum circulation around this circle when the plane of the circle is perpendicular to $\mathbf{u}$. And in that case the length of $\mathbf{V}$ at each point of the circle will be ωa and since $\mathbf{V}$ is everywhere tangent to the circle the entire circulation around the circle in the sense of the rotation will be the circumference of the circle multiplied by the length of $\mathbf{V}$; i.e. the circulation will be, $(2\pi a)(\omega a) = 2\pi\omega a^2$. The curl of $\mathbf{V}$ will by the above theorems have the direction and sense of $\mathbf{u}$ and its length being the limit of the maximum circulation per unit area will be the above circulation divided by the area of the above circle, i.e. $2\pi\omega a^2/\pi a^2 = 2\omega$. We thus conclude that the curl is in this case equal to twice the vector angular velocity of the body, i.e.

$$\nabla \times \mathbf{V} = 2\omega\,\mathbf{u} = 2\,\boldsymbol{\omega}.$$

This intuitional result is of course readily checked by analytic methods.

EXERCISES

1. Show that if a vector-point function $\mathbf{V}(\mathbf{p})$ possesses a derivative in the direction of each of three constant non-coplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, then $\mathbf{V}(\mathbf{p})$ has a curl $\nabla \times \mathbf{V}$ given by the formula,

$$\nabla \times \mathbf{V} = \mathbf{b}_1 \times \{(\mathbf{a}_1 \cdot \nabla)\mathbf{V}\} + \mathbf{b}_2 \times \{(\mathbf{a}_2 \cdot \nabla)\mathbf{V}\} + \mathbf{b}_3 \times \{(\mathbf{a}_3 \cdot \nabla)\mathbf{V}\},$$

where,

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3]}, \qquad \mathbf{b}_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{[\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3]}, \qquad \mathbf{b}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{[\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3]}.$$

2. Show that,

(a) $\nabla(\mathbf{V} \cdot \mathbf{W}) = (\mathbf{V} \cdot \nabla)\mathbf{W} + \mathbf{V} \times (\nabla \times \mathbf{W}) + (\mathbf{W} \cdot \nabla)\mathbf{V} + \mathbf{W} \times (\nabla \times \mathbf{V})$,

(b) $\nabla \cdot (\mathbf{V} \times \mathbf{W}) = \mathbf{W} \cdot (\nabla \times \mathbf{V}) - \mathbf{V} \cdot (\nabla \times \mathbf{W})$,

(c) $\nabla \times (\mathbf{V} \times \mathbf{W}) = \mathbf{V}(\nabla \cdot \mathbf{W}) - (\mathbf{V} \cdot \nabla)\mathbf{W} - \mathbf{W}(\nabla \cdot \mathbf{V}) + (\mathbf{W} \cdot \nabla)\mathbf{V}$.

3. If $\mathbf{a}$ and $\mathbf{b}$ are any constant vectors, show that,

(a) $\nabla\times\{(\mathbf{a}\cdot\mathbf{p})\,\mathbf{b}\} = \mathbf{a}\times\mathbf{b}$,
(b) $\nabla\times\{\mathbf{a}\times(\mathbf{p}\times\mathbf{b})\} = \mathbf{b}\times\mathbf{a}$,
(c) $\nabla\times\{|\mathbf{a}\times\mathbf{p}|^n\mathbf{a}\times\mathbf{p}\} = (n+2)|\mathbf{a}\times\mathbf{p}|^n\,\mathbf{a}$,
(d) $\nabla\times\{|\mathbf{a}\times\mathbf{p}|^n\,\mathbf{p}\} = n\,\mathbf{a}\cdot\mathbf{p}\,|\mathbf{a}\times\mathbf{p}|^{n-2}(\mathbf{p}\times\mathbf{a})$,
(e) $\nabla\times\{\mathbf{p}\times(\mathbf{a}\times\mathbf{p})\} = 3\,\mathbf{p}\times\mathbf{a}$,
(f) $\nabla\times\left\{-\dfrac{\mathbf{a}\cdot\mathbf{p}}{|\mathbf{p}|}\dfrac{\mathbf{a}\times\mathbf{p}}{(\mathbf{a}\times\mathbf{p})^2}\right\} = \dfrac{\mathbf{p}}{|\mathbf{p}|^3}$.

4. If f is an arbitrary scalar-scalar function, show that,

(a) $\nabla\times\{f(|\mathbf{p}|)\,\mathbf{p}\} = 0$,
(b) $\nabla\times\{f(\mathbf{a}\cdot\mathbf{p})\,\mathbf{a}\} = 0$.

5. Show by application of the general transformation to new rectangular coördinates that $\nabla\times\mathbf{V}$ is a differential invariant of the vector-point function $\mathbf{V}(x, y, z)$. In other words, show that any such transformation carrying $\mathbf{V}(x, y, z)$ into $\mathbf{V}'(x', y', z')$ will likewise carry $\nabla\times\mathbf{V}$ into the curl of $\mathbf{V}'$ with respect to $(x'y'z')$. Has this been proven in the text?

6. If $\mathbf{a}$ and $\mathbf{b}$ are any two constant vectors and if $\mathbf{V}$ is a vector point function possessing a curl and derivatives in the directions of $\mathbf{a}$ and $\mathbf{b}$, show that,

$$[\mathbf{a}\,\mathbf{b}(\nabla\times\mathbf{V})] = \mathbf{b}\cdot(\mathbf{a}\cdot\nabla\,\mathbf{V}) - \mathbf{a}\cdot(\mathbf{b}\cdot\nabla\,\mathbf{V}).$$

7. Show that the cross-product of any two gradients is a curl, i.e., if U_1 and U_2 are given scalar-point functions, then there exists a vector-point function $\mathbf{V}$ such that

$$\nabla U_1\times\nabla U_2 = \nabla\times\mathbf{V}.$$

103. Repeated Operations with ∇.

Just as we may differentiate a scalar-scalar function $y(x)$ twice with respect to x, so in the case of a scalar-vector or vector-vector function we may apply successively two differential operations involving ∇. Thus in the case of the scalar-vector function $U(\mathbf{p})$ we have,

$$(a)\qquad \nabla\cdot(\nabla U),\qquad (b)\quad \nabla\times(\nabla U).$$

The second of these expressions is identically zero as may be seen by applying formula § 99, (5) to the result of formula § 102, (8), provided all the partial derivatives involved in the application are continuous. That is,

The curl of the gradient of a scalar-vector function is identically zero,

$$(1)\qquad \nabla\times(\nabla U) \equiv 0.$$

The first of the above expressions,

$$\nabla \cdot \nabla U = \nabla^2 U,$$

is of great importance in mathematical physics. The operator ∇^2 is called the *Laplacian operator* and the equation,

$$\nabla^2 U = 0, \tag{2}$$

is called *Laplace's Equation.* A scalar-vector function which satisfies Laplace's Equation is said to be *harmonic.* When expanded in Cartesian coördinates $\nabla^2 U$ becomes,

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}, \tag{3}$$

which we easily remember by regarding the operator ∇^2 as the symbolic scalar product of the symbolic vector ∇ by itself, i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{4}$$

The Laplacian $\nabla^2 U$ of a scalar-point function $U(\mathbf{p})$ is called the *dispersion* of U at the point considered. For if the point P be surrounded by a sphere of radius ρ and the *average value* of U on the surface of this sphere be represented by U_1, then it may be shown that,

$$\nabla^2 U = 6 \lim_{\rho \to 0} \frac{U_1 - U}{\rho^2},$$

where $\nabla^2 U$ and U are evaluated at P. This shows the reason for the term dispersion but an adequate understanding of the average value U_1 of U on the surface of the sphere requires the concept of a surface integral which we have not yet discussed.

If we now consider the vector-vector function $\mathbf{V}(\mathbf{p})$ we find four second order differential expressions involving ∇,

(*a*) $\nabla^2 \mathbf{V}$, (*b*) $\nabla(\nabla \cdot \mathbf{V})$, (*c*) $\nabla \cdot (\nabla \times \mathbf{V})$, (*d*) $\nabla \times (\nabla \times \mathbf{V})$.

Of these the third $\nabla \cdot (\nabla \times \mathbf{V})$ is identically zero as may be seen by applying formula § 101, (8) to the result of formula § 102, (8), provided all the partial derivatives involved in the application are continuous. That is,

The divergence of the curl of a vector-vector function is identically zero.

$$\nabla \cdot (\nabla \times \mathbf{V}) \equiv 0. \tag{5}$$

The Laplacian operator ∇^2 as applied to a vector-vector function is not the result of the successive application of two of the first order differential operators ∇, $\nabla\cdot$, $\nabla\times$ and it must consequently be defined independently. If however we assume that it obeys the two laws,

$$(6) \quad \nabla^2(U\,\mathbf{a}) = (\nabla^2 U)\mathbf{a} \quad (\mathbf{a}\text{ const.}), \qquad \nabla^2(\mathbf{V}+\mathbf{W}) = \nabla^2\mathbf{V} + \nabla^2\mathbf{W},$$

it follows that if $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are any three noncoplanar constant vectors and if we set,

$$\mathbf{V} = U_1\,\mathbf{a}_1 + U_2\,\mathbf{a}_2 + U_3\,\mathbf{a}_3,$$

then

$$(7) \qquad \nabla^2\mathbf{V} = (\nabla^2 U_1)\mathbf{a}_1 + (\nabla^2 U_2)\mathbf{a}_2 + (\nabla^2 U_3)\mathbf{a}_3.$$

In particular if $\mathbf{V}$ has the Cartesian coördinates $\mathbf{V} \equiv (V_1, V_2, V_3)$ then it follows from the above that,

$$(8) \qquad \nabla^2\mathbf{V} \equiv (\nabla^2 V_1, \nabla^2 V_2, \nabla^2 V_3).$$

The meaning and coördinate formulas of $\nabla(\nabla\cdot\mathbf{V})$ and $\nabla\times(\nabla\times\mathbf{V})$ are evident from those of the separate operators involved. The reader will find no difficulty in verifying the important relation,

$$(9) \qquad \nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}.$$

This may be regarded as a symbolic form of the familiar identity,

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\mathbf{a}\cdot\mathbf{b}) - \mathbf{a}^2\,\mathbf{b}.$$

The formula (9) is particularly useful when solved for $\nabla^2\mathbf{V}$ as it then expresses $\nabla^2\mathbf{V}$ in terms of the first order operators ∇, $\nabla\cdot$, $\nabla\times$.

EXERCISES

1. Show that for $\mathbf{a}$ and $\mathbf{b}$ constants,

(*a*) $\nabla^2(\mathbf{a}\cdot\mathbf{p}) = 0,$
(*b*) $\nabla^2(\mathbf{p}) = \nabla^2(\mathbf{a}\times\mathbf{p}) = \nabla^2\{\mathbf{a}\times(\mathbf{p}\times\mathbf{b})\} = \mathbf{0}.$

2. Show that for $p = |\mathbf{p}|$,

(*a*) $\nabla^2(p^n) = n(n+1)p^{n-2},$
(*b*) $\nabla^2\{f(p)\} = \dfrac{2}{p}\dfrac{df}{dp} + \dfrac{d^2f}{dp^2}.$

3. Prove the successively more general formulas,

(a) $\nabla^2(p^n\,\mathbf{p}) = n(n+3)p^{n-2}\,\mathbf{p}, \qquad p = |\mathbf{p}|,$

(b) $\nabla^2\{f(p)\,\mathbf{p}\} = \left(\frac{4}{p}\frac{df}{dp} + \frac{d^2f}{dp^2}\right)\mathbf{p},$

(c) $\nabla^2(U\,\mathbf{p}) = 2\nabla U + (\nabla^2 U)\mathbf{p},$

(d) $\nabla^2(U\,\mathbf{V}) = U\nabla^2\mathbf{V} + 2(\nabla U\cdot\nabla)\mathbf{V} + \mathbf{V}\,\nabla^2 U.$

4. Show that for $\mathbf{a}$ and $\mathbf{b}$ constants,

(a) $\nabla^2\{(\mathbf{a}\cdot\mathbf{p})\mathbf{p}\} = 2\,\mathbf{a},$

(b) $\nabla^2\{\mathbf{p}\times(\mathbf{a}\times\mathbf{p})\} = 4\,\mathbf{a},$

(c) $\nabla^2(\mathbf{p}^2\,\mathbf{a}) = 6\,\mathbf{a},$

(d) $\nabla^2\{(\mathbf{a}\times\mathbf{p})\cdot(\mathbf{b}\times\mathbf{p})\} = 4\,\mathbf{a}\cdot\mathbf{b},$

(e) $\nabla^2\{|\mathbf{a}\times\mathbf{p}|^n(\mathbf{a}\times\mathbf{p})\} = n(n+2)\mathbf{a}^2|\mathbf{a}\times\mathbf{p}|^{n-2}(\mathbf{a}\times\mathbf{p}),$

(f) $\nabla^2\{(\mathbf{a}\cdot\mathbf{p})^n\} = n(n-1)\mathbf{a}^2(\mathbf{a}\cdot\mathbf{p})^{n-2},$

(g) $\nabla^2(|\mathbf{a}\times\mathbf{p}|^n) = n^2\mathbf{a}^2|\mathbf{a}\times\mathbf{p}|^{n-2},$

(h) $\nabla^2\left(\frac{\mathbf{p}}{\mathbf{a}\cdot\mathbf{p}}\right) = \frac{2}{(\mathbf{a}\cdot\mathbf{p})^3}\,\mathbf{a}\times(\mathbf{p}\times\mathbf{a}).$

5. Show that the only functions $f(p)$ satisfying the partial differential equation,

$$\nabla^2\{f(p)\mathbf{p}\} = 0, \qquad p = |\mathbf{p}|,$$

are of the form,

$$f(p) = \frac{a}{p^3} + b,$$

where a and b are constants.

6. Show that if $\nabla\cdot\mathbf{V}$ and $\nabla\times\mathbf{V}$ are identically zero for all values of $\mathbf{p}$, then $\nabla^2\,\mathbf{V}$ is identically zero. Is the converse theorem true? (Hint: Consider the case $\mathbf{V} = \mathbf{a}\times\mathbf{p}$.)

7. If U is any scalar point function and $F(U)$ is an arbitrary scalar function of U, show that,

$$\nabla\times(F(U)\nabla U) = 0.$$

8. Show that for constant $\mathbf{a}$,

(a) $\nabla\cdot(\mathbf{a}\cdot\nabla\,\mathbf{V}) = \mathbf{a}\cdot\nabla(\nabla\cdot\mathbf{V}),$

(b) $\nabla\times(\mathbf{a}\cdot\nabla\,\mathbf{V}) = \mathbf{a}\cdot\nabla(\nabla\times\mathbf{V}) + \nabla\times\{(\nabla\times\mathbf{V})\times\mathbf{a}\}.$

104. The Gradient, Divergence and Curl Theorems.

In this chapter we have up to the present been concerned with differential operations on scalar-vector and vector-vector functions. We have now to consider certain integral operations on such functions. Let us interpret the variable vector $\mathbf{p}$ as the radius vector from the origin of a variable point P and let $U(\mathbf{p})$ and $\mathbf{V}(\mathbf{p})$ be scalar and vector functions of $\mathbf{p}$ which are continuous

for P in a certain region of space. Let S be a regular * surface lying wholly within this region of continuity and let us divide the surface S into a number n of sections, indicating by $\Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$ the areas of these portions. Let $\mathbf{p}_i$ be any value which $\mathbf{p}$ takes on for P on the i^{th} such section and let us form the sums,

$$\sum_{i=1} U(\mathbf{p}_i)\,\Delta\sigma_i, \qquad \sum_{i=1} \mathbf{V}(\mathbf{p}_i)\,\Delta\sigma_i.$$

We now proceed to the limit letting the number n of such sections become infinite and dividing S in such a way that the largest dimension of each section approaches zero. Under these circumstances it may be shown that the above sums will approach limits and that these limits will be independent of the particular manner of division of S and of the particular values of $\mathbf{p}_i$ chosen. We indicate these limits by,

$$(1)\quad \lim_{n\to\infty} \sum_{i=1}^{n} U(\mathbf{p}_i)\,\Delta\sigma_i = \iint U(\mathbf{p})\,d\sigma,$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{V}(\mathbf{p}_i)\,\Delta\sigma_i = \iint \mathbf{V}(\mathbf{p})\,d\sigma,$$

and call them the *integrals of* $U(\mathbf{p})$ *and* $\mathbf{V}(\mathbf{p})$ *over the surface* S.

It may happen that the function $U(\mathbf{p})$ or $\mathbf{V}(\mathbf{p})$ to be integrated is defined only for P on the surface S but this will in no way alter the definition nor the existence of the integral provided the functions are continuous on the surface including its boundary. For example, if we have given a scalar-point function $U(\mathbf{p})$ and a surface S we may form the function to be integrated by multiplying the unit vector $\mathbf{n}$ which is normal to S on a certain side at each of its points by the value of U at that point. The resulting vector-point function would then be defined only on the given surface, but could be integrated as above, giving us,

$$(2)\qquad \lim_{n\to\infty} \sum_{i=1}^{n} U(\mathbf{p}_i)\mathbf{n}(\mathbf{p}_i)\,\Delta\sigma_i = \iint_S U(\mathbf{p})\mathbf{n}\,d\sigma.$$

* We can not here enter upon a discussion of exactly what constitutes a regular curve, regular surface, regular region of space. From a practical point of view the word *regular* may here be thought of as used in its customary meaning since the ensuing theorems hold for all the surfaces and regions ordinarily encountered. An extensive treatment of this topic will be found in O. D. Kellogg, *Foundations of Potential Theory*, pp. 97–113.

In this case we may somewhat modify the manner of obtaining the integral without altering the result by representing each element of area $\Delta\sigma_i$ by the vector $\Delta\boldsymbol{\sigma}_i = \mathbf{n}(\mathbf{p}_i)\,\Delta\sigma_i$ normal to the surface at one of the points of the element and having a length equal to the area of the element. If we then form the sum $\sum_{i=1}^{n} U(\mathbf{p}_i)\Delta\boldsymbol{\sigma}_i$ and proceed to the limit as before, we obtain a result,

$$(3)\qquad \lim_{n\to\infty}\sum_{i=1}^{n} U(\mathbf{p}_i)\,\Delta\boldsymbol{\sigma}_i = \int\!\!\int U(\mathbf{p})\,d\boldsymbol{\sigma},$$

which by Duhamel's theorem is the same as that of equation (2) even though U and $\mathbf{n}$ are evaluated at different points of the elements. This device of representing each element of area by a normal vector is also convenient when we have given a vector-point function $\mathbf{V}(\mathbf{p})$ and wish to integrate over the surface the scalar or vector product of $\mathbf{V}$ with the unit normal vector to the surface. Here we have,

$$(4)\quad \lim_{n\to\infty}\sum_{i=1}^{n} \mathbf{V}(\mathbf{p}_i)\cdot\mathbf{n}(\mathbf{p}_i)\,\Delta\sigma_i = \lim_{n\to\infty}\sum_{i=1}^{n} \mathbf{V}(\mathbf{p}_i)\cdot\Delta\boldsymbol{\sigma}_i = \int\!\!\int \mathbf{V}(\mathbf{p})\cdot d\boldsymbol{\sigma},$$

$$(5)\quad \lim_{n\to\infty}\sum_{i=1} \mathbf{V}(\mathbf{p}_i)\times\mathbf{n}(\mathbf{p}_i)\,\Delta\sigma_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} \mathbf{V}(\mathbf{p}_i)\times\Delta\boldsymbol{\sigma}_i = \int\!\!\int_S \mathbf{V}(\mathbf{p})\times d\boldsymbol{\sigma}.$$

If $U(p)$ and $\mathbf{V}(\mathbf{p})$ are given scalar and vector point functions continuous in a certain region of space, let us consider a certain regular region T of space bounded by a regular surface S and lying wholly within this region of continuity. We divide the region T up into a number n of sections indicating by $\Delta\tau_1$, $\Delta\tau_2$, $\cdots$, $\Delta\tau_n$ the volumes of these portions and we form the sums,

$$\sum_{i=1}^{n} U(\mathbf{p}_i)\,\Delta\tau_i, \qquad \sum_{i=1}^{n} \mathbf{V}(\mathbf{p}_i)\,\Delta\tau_i,$$

where $\mathbf{p}_i$ is any value of $\mathbf{p}$ in the i^{th} section. We now proceed to the limit letting the number n of such sections become infinite

and dividing T in such a way that the largest dimension of each section approaches zero. Under these circumstances it may be shown that the above sums will approach limits and that these limits will be independent of the particular manner of division of T and of the particular values of $\mathbf{p}_i$ chosen. We indicate these limits by,

$$(6)\quad \lim_{n\to\infty} \sum_{i=1}^{n} U(\mathbf{p}_i)\,\Delta\tau_i = \iiint\limits_T U(\mathbf{p})\,d\tau,$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{V}(\mathbf{p}_i)\,\Delta\tau_i = \iiint\limits_T \mathbf{V}(\mathbf{p})\,d\tau,$$

and call them the *integrals of* $U(\mathbf{p})$ *and* $\mathbf{V}(\mathbf{p})$ *over the region* T.

We have defined the integrals of $U(\mathbf{p})$ and $\mathbf{V}(\mathbf{p})$ over a regular surface S and over a regular region of space T. There are three important closely related theorems connecting the integral of a function over a regular region T with the integral of a related function over the regular surface S bounding this region. They may be stated as follows:

If $U(\mathbf{p})$ together with its gradient ∇U and $\mathbf{V}(\mathbf{p})$ together with its divergence $\nabla\cdot\mathbf{V}$ and its curl $\nabla\times\mathbf{V}$ are continuous throughout a certain region of space and if T is any regular region of space bounded by a closed regular surface S both lying wholly within the above region of continuity, then we have,

The Gradient Theorem,

$$\iiint\limits_T \nabla U(\mathbf{p})\,d\tau = \iint\limits_S U(\mathbf{p})\mathbf{n}\,d\sigma,$$

The Divergence Theorem,

$$\iiint\limits_T \nabla\cdot\mathbf{V}(\mathbf{p})\,d\tau = \iint\limits_S \mathbf{n}\cdot\mathbf{V}(\mathbf{p})\,d\sigma,$$

The Curl Theorem,

$$\iiint\limits_T \Delta\times\mathbf{V}(\mathbf{p})\,d\tau = \iint\limits_S \mathbf{n}\times\mathbf{V}(\mathbf{p})\,d\sigma,$$

where $\mathbf{n}$ is the unit outward normal vector to the surface S.

We shall first prove the gradient theorem under the additional hypothesis that it is possible to find three noncoplanar unit vectors $\mathbf{u}$ such that every line parallel to $\mathbf{u}$ cuts the surface S in at most two points. We indicate by s the distance in the sense of $\mathbf{u}$ to any point P from a plane π perpendicular to $\mathbf{u}$. Then if U is the scalar-point function of the theorem, $\mathbf{u}\cdot(\nabla U)$ is the derivative $\frac{\partial U}{\partial s}$ of U in the direction of $\mathbf{u}$ and we have,

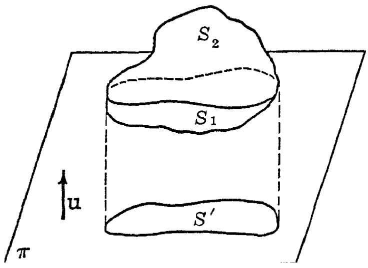

FIG. 87

$$\mathbf{u}\cdot\iiint_T \nabla U\, d\tau = \iiint_T \mathbf{u}\cdot\nabla U\, d\tau = \iiint_T \frac{\partial U}{\partial s}\, d\tau.$$

If S' is the projection of S on the plane π then the boundary of S' is the projection on π of a curve on S which divides S into two nappes which we shall call S_1 and S_2, the nappe S_1 corresponding to the smaller values of s. Now by the fundamental theorem of the integral calculus the above triple integral can be evaluated by two iterated integrals, first with respect to s from s_1 to s_2 and second over the plane area S'. Performing this integration with respect to s, we find,

$$\mathbf{u}\cdot\iiint_T \nabla U\, d\tau = \iint_{S'}\left\{\int_{s_1}^{s_2} \frac{\partial U}{\partial s}\, ds\right\} d\sigma' = \iint_{S'} (U_2 - U_1)\, d\sigma',$$

where U_1 and U_2 are the values of U at the point of S_1 and S_2 whose projection on π is the point considered in the integration over S'. But now if $\Delta\sigma_1$ and $\Delta\sigma_2$ are elements of the areas S_1 and S_2 both having projections $\Delta\sigma'$ on S' then,

$$\Delta\sigma' = -(\mathbf{u}\cdot\mathbf{n}_1 + \epsilon_1)\Delta\sigma_1 = (\mathbf{u}\cdot\mathbf{n}_2 + \epsilon_2)\Delta\sigma_2,$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit outward normals at some point of $\Delta\sigma_1$ and $\Delta\sigma_2$ and where ϵ_1 and ϵ_2 vanish uniformly with $\Delta\sigma_1$ and $\Delta\sigma_2$. Consequently by Duhamel's theorem the above integral may be

written,

$$\iint\limits_{S'} (U_2 - U_1)\, d\sigma' = \iint\limits_{S_1} \mathbf{u}\cdot\mathbf{n}_1\, U_1\, d\sigma_1$$
$$+ \iint\limits_{S_2} \mathbf{u}\cdot\mathbf{n}_2\, U_2\, d\sigma_2 = \iint\limits_{S} \mathbf{u}\cdot\mathbf{n}\, U\, d\sigma,$$

and we have finally,

$$\mathbf{u}\cdot \iiint\limits_{T} \nabla U\, d\tau = \mathbf{u}\cdot \iint\limits_{S} U\, \mathbf{n}\, d\sigma.$$

Since this holds for three noncoplanar values of $\mathbf{u}$ we find *The Gradient Theorem*,

$$(7) \qquad \iiint\limits_{T} \nabla U\, d\tau = \iint\limits_{S} U\, \mathbf{n}\, d\sigma$$

established for this case.

We may now readily remove the additional hypothesis concerning the existence of the unit vectors $\mathbf{u}$. Since the given region T and surface S are regular it is always possible to divide T by regular surfaces into sub-regions for which this hypothesis holds. For each of these regions we may write equation (7) and on adding these equations member for member find the theorem proven for the given region and surface. For the sum of the volume integrals obviously gives the integral over T, and although the surface integrals would then apply to the entire surface of all the sub-regions, yet the integration would cover all the dividing surfaces twice and the outward normals to the sub-regions on the two sides of this dividing surface would have opposite senses and thus cancel out these portions of the integral, leaving only the integral extended over the original surface S.

The gradient theorem being now established, the divergence and curl theorems follow at once. For if $\mathbf{V}(\mathbf{p})$ is the vector-point function of the theorems we may choose any three non-coplanar constant vectors, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and write $\mathbf{V}(\mathbf{p})$ in the form,

$$\mathbf{V}(\mathbf{p}) = U_1(\mathbf{p})\mathbf{a}_1 + U_2(\mathbf{p})\mathbf{a}_2 + U_3(\mathbf{p})\mathbf{a}_3.$$

Now by equation (7) we have, remembering equation § 101, (4),

$$\iiint_T \nabla\cdot(U_1\,\mathbf{a}_1)\,d\tau = \mathbf{a}_1\cdot\iiint_T \nabla U_1\,d\tau$$

$$= \mathbf{a}_1\cdot\iint_S U_1\,\mathbf{n}\,d\sigma = \iint_S \mathbf{n}\cdot(U_1\,\mathbf{a}_1)\,d\sigma$$

with similar equations for $U_2\,\mathbf{a}_2$ and $U_3\,\mathbf{a}_3$. On adding the three equations member for member we find *The Divergence Theorem*,

$$\text{(8)} \qquad \iiint_T \nabla\cdot\mathbf{V}\,d\tau = \iint_T \mathbf{n}\cdot\mathbf{V}\,d\sigma.$$

Likewise by equation (7) we have, remembering equation § 102, (4),

$$\iiint_T \nabla\times(U_1\,\mathbf{a}_1)\,d\tau = \left\{\iiint_T \nabla U_1\,d\tau\right\}\times\mathbf{a}_1$$

$$= \left\{\iint_S U_1\,\mathbf{n}\,d\sigma\right\}\times\mathbf{a}_1 = \iint_S \mathbf{n}\times U_1\,\mathbf{a}_1\,d\sigma$$

with similar equations for $U_2\,\mathbf{a}_2$ and $U_3\,\mathbf{a}_3$. On adding the three equations member for member we find *The Curl Theorem*,

$$\text{(9)} \qquad \iiint_T \nabla\times\mathbf{V}\,d\tau = \iint_S \mathbf{n}\times\mathbf{V}\,d\sigma.$$

All three theorems are thus established.

The divergence theorem was first proven by Gauss, 1813, and is often called by his name, although identities similar to this are to be found in the work of Lagrange as early as 1760–61. The systematic use of such identities in mathematical physics dates from Green, 1828. The theorems are easily remembered by observing that the integrands in the two members of each equation are alike except that ∇ in the volume integral is replaced by $\mathbf{n}$ in the surface integral.

EXERCISES

1. By application of the divergence theorem to the function $\mathbf{V} = \mathbf{p}$, show that the volume T of a regular region of space is given by

the formula,

$$T = \frac{1}{3}\iint\limits_S \mathbf{n}\cdot\mathbf{p}\, d\sigma,$$

where S is the surface bounding the region. Employ this formula to calculate the volume of a sphere, a cone, a pyramid.

2. Show that if S is a regular closed surface then,

$$\iint\limits_S d\boldsymbol{\sigma} = 0,$$

where $d\boldsymbol{\sigma}$ is the normal vector surface element as in equation (3).

3. Let the gradient ∇U, the divergence $\nabla\cdot\mathbf{V}$ and the curl $\nabla\times\mathbf{V}$ at a point be defined as,

$$\nabla U = \lim_{T\to 0}\frac{1}{T}\iint\limits_S U\,\mathbf{n}\, d\sigma,$$

$$\nabla\cdot\mathbf{V} = \lim_{T\to 0}\frac{1}{T}\iint\limits_S \mathbf{n}\cdot\mathbf{V}\, d\sigma,$$

$$\nabla\times\mathbf{V} = \lim_{T\to 0}\frac{1}{T}\iint\limits_S \mathbf{n}\times\mathbf{V}\, d\sigma,$$

where S is a regular surface surrounding the point considered and where the largest dimension of the enclosed region T approaches zero. With these definitions derive the coördinate formulas for the quantities defined.

4. Verify the gradient theorem for the functions $U = \mathbf{a}\cdot\mathbf{p}$ and $U = |\mathbf{p}|^n$ for the region bounded by the sphere $\mathbf{p}^2 = r^2$; for a cube with center at the origin and edges parallel or perpendicular to $\mathbf{a}$.

5. Verify the divergence and curl theorems for the functions $\mathbf{V} = |\mathbf{p}|^n\,\mathbf{p}$ and $\mathbf{V} = \mathbf{a}\times\mathbf{p}$ for the region bounded by the sphere $\mathbf{p}^2 = r^2$; for a cube with center at the origin and edges parallel or perpendicular to $\mathbf{a}$.

105. Stokes' and Green's Theorems.

In the previous article we defined the integral of a scalar-point function U and of a vector-point function $\mathbf{V}$ over a surface S and over a region of space T. The definitions of the integrals of such functions along a curve C are entirely analogous. We considered a special case of such a curve or line integral in § 72 in the definition of the work done by a given force upon a particle moving along the curve. If $U(\mathbf{p})$ and $\mathbf{V}(\mathbf{p})$ are scalar and vector-point

functions which are continuous for P in a certain region of space and if C is a curve lying wholly within this region of continuity, we divide the curve C up into a number n of sections indicating by $\Delta\gamma_1, \Delta\gamma_2, \Delta\gamma_3, \cdots, \Delta\gamma_n$ the lengths of these portions. We let $\mathbf{p}_i$ be any value which $\mathbf{p}$ takes on for P on the i^{th} such section and we form the sums,

$$\sum^n U(\mathbf{p}_i)\Delta\gamma_i, \qquad \sum_{i=1}^n \mathbf{V}(\mathbf{p}_i)\Delta\gamma_i.$$

We now proceed to the limit letting the number n of such sections become infinite and the length of every section approach zero. Under these circumstances it may be shown that the above sums approach limits and that these limits are independent of the particular manner of division of C into sections and of the particular values of $\mathbf{p}_i$ chosen. We indicate these limits by,

$$\lim_{n\to\infty} \sum_{i=1}^n U(\mathbf{p}_i)\Delta\gamma_i = \int_C U(\mathbf{p})d\gamma, \qquad \lim_{n\to\infty} \sum_{i=1}^n \mathbf{V}(\mathbf{p}_i)\Delta\gamma_i = \int_C \mathbf{V}(\mathbf{p})d\gamma$$

and call them the *integrals of* $U(\mathbf{p})$ *and* $\mathbf{V}(\mathbf{p})$ *along the curve* C.

No modification need be made in the above definitions if the function U or $\mathbf{V}$ to be integrated is defined only for points on the given curve C provided the condition of continuity is still satisfied. In particular this happens when the integrand involves the tangent to the given curve at the point considered. Thus if $\mathbf{t}(\mathbf{p})$ represent a unit tangent vector to C at P taken in a determined sense along the curve, we may form the integral,

$$\lim_{n\to\infty} \sum_{i=1}^n U(\mathbf{p}_i)\mathbf{t}(\mathbf{p}_i)\Delta\gamma_i = \int_C U\,\mathbf{t}\,d\gamma.$$

In this case we may somewhat modify the manner of obtaining the integral without altering the result by letting $\Delta\mathbf{p}_i$ represent the vector running in a determined sense from one end to the other of the i^{th} division of C. We then form,

$$\lim_{n\to\infty} \sum_{i=1}^n U(\mathbf{p}_i)\Delta\mathbf{p}_i = \int_C U\,d\mathbf{p},$$

which by Duhamel's theorem is the same as the preceding. This

device is also convenient when we wish to integrate the scalar or vector product of a given vector function $\mathbf{V}$ with the unit tangent vector to the curve. As before we find that,

$$\int_C \mathbf{V}\cdot\mathbf{t}\,d\gamma = \int_C \mathbf{V}\cdot d\mathbf{p}, \qquad \int_C \mathbf{V}\times\mathbf{t}\,d\gamma = \int_C \mathbf{V}\times d\mathbf{p}.$$

These integrals involving $\mathbf{t}$ or $d\mathbf{p}$ are evidently changed in sign if the sense of $\mathbf{t}$ or $d\mathbf{p}$ along the curve is reversed.

Analogous to the theorems of the preceding article we have the important *Stokes' Theorem* or *Circulation Theorem* connecting the integral of a vector-point function over a surface with the integral of a related function around the curve bounding that surface.

*Stokes' Theorem.** *If the vector-point function* $\mathbf{V}$ *together with its divergence and curl is continuous in a certain region of space and if S is a regular two sided surface lying wholly within this region and bounded by a regular closed curve C then,*

$$\int\int \mathbf{n}\cdot\nabla\times\mathbf{V}\,d\sigma = \int \mathbf{V}\cdot\mathbf{t}\,d\gamma,$$

where $\mathbf{n}$ *is the unit normal vector to S on one side and* $\mathbf{t}$ *is the unit tangent vector to C running around C in the positive sense of rotation relative to* $\mathbf{n}$.

To prove this theorem we shall first prove a simple lemma already presented as an exercise in § 102.

Lemma. If $\mathbf{a}$ *and* $\mathbf{b}$ *are any two constant vectors and if* $\mathbf{V}$ *is a vector point function possessing a curl and derivatives in the directions of* $\mathbf{a}$ *and* $\mathbf{b}$ *then,*

$$(1) \qquad \mathbf{a}\times\mathbf{b}\cdot(\nabla\times\mathbf{V}) = \mathbf{b}\cdot(\mathbf{a}\cdot\nabla\mathbf{V}) - \mathbf{a}\cdot(\mathbf{b}\cdot\nabla\mathbf{V}).$$

By definition of the curl, § 102 (1), we have,

$$(\nabla\times\mathbf{V}+\boldsymbol{\varepsilon})\cdot\Delta_1\mathbf{p}\times\Delta_2\mathbf{p} = \Delta_1\mathbf{V}\cdot\Delta_2\mathbf{p} - \Delta_2\mathbf{V}\cdot\Delta_1\mathbf{p}$$
$$\lim_{\substack{\Delta_1\mathbf{p}\to 0\\ \Delta_2\mathbf{p}\to 0}} \boldsymbol{\varepsilon} = 0$$

where $\Delta_1\mathbf{V}$ and $\Delta_2\mathbf{V}$ are the changes in $\mathbf{V}$ corresponding respec-

* Stokes' Theorem was proposed by him as a subject in Smith's Prize Examination Papers, 1854.

tively to the changes $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ in $\mathbf{p}$. In particular we may give $\Delta_1\mathbf{p}$ and $\Delta_2\mathbf{p}$ the direction and sense of $\mathbf{a}$ and $\mathbf{b}$ respectively, and write,

$$a\,\Delta_1\mathbf{p} = \alpha\,\mathbf{a}, \qquad b\,\Delta_2\mathbf{p} = \beta\,\mathbf{b}, \qquad \text{where} \qquad \begin{aligned} a &= |\mathbf{a}|, & b &= |\mathbf{b}|, \\ \alpha &= |\Delta_1\mathbf{p}|, & \beta &= |\Delta_2\mathbf{p}|, \end{aligned}$$

by which the previous equation takes the form,

$$\alpha\beta(\nabla\times\mathbf{V} + \boldsymbol{\varepsilon})\cdot\mathbf{a}\times\mathbf{b} = \beta a\,\Delta_1\mathbf{V}\cdot\mathbf{b} - \alpha b\,\Delta_2\mathbf{V}\cdot\mathbf{a},$$
$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}} \boldsymbol{\varepsilon} = 0.$$

Also by definition of the directional derivative of $\mathbf{V}$, § 100, (7), we have with the present notation,

$$a\,\Delta_1\mathbf{V} = \alpha\,\mathbf{a}\cdot\nabla\mathbf{V} + \alpha\,a\,\boldsymbol{\varepsilon}_1, \qquad \lim_{\alpha\to 0} \boldsymbol{\varepsilon}_1 = 0,$$

$$b\,\Delta_2\mathbf{V} = \beta\,\mathbf{b}\cdot\nabla\mathbf{V} + \beta\,b\,\boldsymbol{\varepsilon}_2, \qquad \lim_{\beta\to 0} \boldsymbol{\varepsilon}_2 = 0,$$

by which the above equation takes the form,

$$(\nabla\times\mathbf{V} + \boldsymbol{\varepsilon})\cdot\mathbf{a}\times\mathbf{b} = \mathbf{b}\cdot(\mathbf{a}\cdot\nabla\mathbf{V}) - \mathbf{a}\cdot(\mathbf{b}\cdot\nabla\mathbf{V}) + a\,\mathbf{b}\cdot\boldsymbol{\varepsilon}_1 - b\,\mathbf{a}\cdot\boldsymbol{\varepsilon}_2.$$

On proceeding to the limit allowing α and β to approach zero, we find equation (1) of the lemma established. This result also follows readily from the formula stated in § 102, Prob. 1.

We shall now prove Stokes' Theorem when the area S is any parallelogram, the method of proof closely following the mechanical interpretation of the curl of a vector-point function described in § 102. Let the area S be the parallelogram with successive vertices P_0ACB and let $\mathbf{a}$ and $\mathbf{b}$ be *unit* vectors having the direction and sense of P_0A and P_0B and let h and k be the lengths of these sides. Then the integral of $\mathbf{V}\cdot\mathbf{t}$ around the perimeter of the parallelogram in the sense P_0ACB will be,

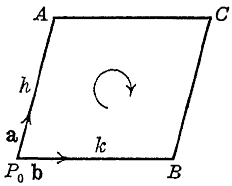

FIG. 88

$$\int_C \mathbf{V}\cdot\mathbf{t}\,d\gamma = \int_{P_0}^{A} \mathbf{V}\cdot\mathbf{a}\,d\gamma + \int_{A}^{C} \mathbf{V}\cdot\mathbf{b}\,d\gamma - \int_{C}^{B} \mathbf{V}\cdot\mathbf{a}\,d\gamma - \int_{B}^{P_0} \mathbf{V}\cdot\mathbf{b}\,d\gamma.$$

Taking the first and third terms of the second member we have,

$$\int_{P_0}^{A} \mathbf{V}\cdot\mathbf{a}\,d\gamma - \int_{C}^{B} \mathbf{V}\cdot\mathbf{a}\,d\gamma$$

$$= \int_0^h \{\mathbf{V}(\mathbf{p}_0 + \alpha\,\mathbf{a}) - \mathbf{V}(\mathbf{p}_0 + k\,\mathbf{b} + \alpha\,\mathbf{a})\}\cdot\mathbf{a}\,d\alpha$$

$$= -\int_0^h \left\{\int_0^k \mathbf{b}\cdot\nabla\mathbf{V}(\mathbf{p}_0 + \alpha\,\mathbf{a} + \beta\,\mathbf{b})\,d\beta\right\}\cdot\mathbf{a}\,d\alpha,$$

and similarly for the second and fourth terms we find,

$$\int_{A}^{C} \mathbf{V}\cdot\mathbf{b}\,d\gamma - \int_{B}^{P_0} \mathbf{V}\cdot\mathbf{b}\,d\gamma$$

$$= \int_0^k \{\mathbf{V}(\mathbf{p}_0 + h\,\mathbf{a} + \beta\,\mathbf{b}) - \mathbf{V}(\mathbf{p}_0 + \beta\,\mathbf{b})\}\cdot\mathbf{b}\,d\beta$$

$$= \int_0^k \left\{\int_0^h \mathbf{a}\cdot\nabla\mathbf{V}(\mathbf{p}_0 + \alpha\,\mathbf{a} + \beta\,\mathbf{b})d\alpha\right\}\cdot\mathbf{b}\,d\beta.$$

Now by the fundamental theorem of the integral calculus each of the iterated integrals obtained above may be evaluated as a double integral, and on adding them we find,

$$\int_C \mathbf{V}\cdot\mathbf{t}\,d\gamma = \int_0^k\int_0^h \{\mathbf{b}\cdot(\mathbf{a}\cdot\nabla\mathbf{V}) - \mathbf{a}\cdot(\mathbf{b}\cdot\nabla\mathbf{V})\}d\alpha\,d\beta,$$

or remembering equation (1) of the lemma,

$$\int_C \mathbf{V}\cdot\mathbf{t}\,d\gamma = \int_0^k\int_0^h \mathbf{a}\times\mathbf{b}\cdot(\nabla\times\mathbf{V})d\alpha\,d\beta.$$

If now the parallelogram S be divided into elementary parallelograms by lines parallel to the sides, the area $d\sigma$ of one of these elements of edges $d\alpha$ and $d\beta$ will be $|\mathbf{a}\times\mathbf{b}|\,d\alpha d\beta$ while the unit

normal vector **n** to the element in the sense of $\mathbf{a} \times \mathbf{b}$ will be $\mathbf{a} \times \mathbf{b}/|\mathbf{a} \times \mathbf{b}|$. Thus $\mathbf{a} \times \mathbf{b}\, d\alpha\, d\beta = \mathbf{n}\, d\sigma$ and we have,

$$\int \mathbf{V} \cdot \mathbf{t}\, d\gamma = \iint \mathbf{n} \cdot \nabla \times \mathbf{V}\, d\sigma \tag{2}$$

as was to be proven.

Stokes' Theorem being now established for any parallelogram within the region of continuity, we may readily extend it to any triangle within this region. Thus if ABC is such a triangle with PQR as the midpoints of the opposite sides, it appears upon examination of the figure that the integral of a function over the triangle ABC will be equal to the sum of its integrals over the parallelograms $ARPQ$, $BPQR$, $CQRP$ minus twice the integral over the triangle PQR. And likewise the integral of $\mathbf{V} \cdot \mathbf{t}$ extended in the chosen sense around the triangle ABC will equal the sum of its integrals around these parallelograms in this sense minus twice its integral around the triangle PQR. Remembering that Stokes' Theorem is known to hold for the parallelograms, it is evident that we have,

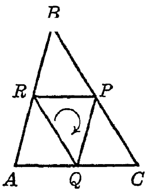

FIG. 89

$$\iint_{S_0} \mathbf{n} \cdot \nabla \times \mathbf{V}\, d\sigma - \int_{C_0} \mathbf{V} \cdot \mathbf{t}\, d\gamma$$
$$= -2\left\{ \iint_{S_1} \mathbf{n} \cdot \nabla \times \mathbf{V}\, d\sigma - \int_{C_1} \mathbf{V} \cdot \mathbf{t}\, d\gamma \right\},$$

where S_0 and C_0 are the original triangle and its boundary and S_1 and C_1 are the triangle PQR and its boundary. We may now apply to the triangle PQR the analysis just employed for ABC and continuing thus after n repetitions conclude that,

$$\iint_{S_0} n \cdot \nabla \times \mathbf{V}\, d\sigma - \int_{C_0} \mathbf{V} \cdot t\, d\gamma \tag{3}$$
$$= (-2)^n \left\{ \iint_{S_n} \mathbf{n} \cdot \nabla \times \mathbf{V}\, d\sigma - \int_{C_n} \mathbf{V} \cdot \mathbf{t}\, d\gamma \right\},$$

where S_n is a triangle having an area $1/4^n$ times that of the original

triangle and a perimeter $1/2^n$ times that of the original triangle. It appears at once that the term $(-2)^n \iint\limits_{S_n} \mathbf{n}\cdot\nabla\times\mathbf{V}\,d\sigma$ approaches zero as n becomes infinite since the integrand remains finite while 2^n times the area of integration approaches zero. It is likewise evident that the term $-(-2)^n \int\limits_{C_n} \mathbf{V}\cdot\mathbf{t}\,d\gamma$ approaches zero as n becomes infinite. For, letting $\mathbf{V}_0$ be the value of $\mathbf{V}$ at the point of intersection of the medians of ABC, we may set $\mathbf{V} = \mathbf{V}_0 + \Delta\mathbf{V}$ and write,

$$-(-2)^n \int\limits_{C_n} \mathbf{V}\cdot\mathbf{t}\,d\gamma = -(-2)^n \left\{ \mathbf{V}_0 \cdot \int\limits_{C_n} \mathbf{t}\,d\gamma + \int\limits_{C_n} \Delta\mathbf{V}\cdot\mathbf{t}\,d\gamma \right\}$$

$$= -(-2)^n \int \Delta\mathbf{V}\cdot\mathbf{t}\,d\gamma.$$

As **n** becomes infinite the integrand in the last member approaches zero due to the continuity of $\mathbf{V}$ while 2^n times the length of the path of integration remains constant, and the whole member consequently approaches zero. It thus appears that the left member of equation (3) is a constant independent of n which from the right member may be shown to be arbitrarily small. This constant must therefore be zero and Stokes' Theorem is proven for any triangle in the region of continuity.

If now S is any regular *two sided* surface lying within the region of continuity, then in forming the integrals of Stokes' Theorem, we may divide S into approximately triangular elements and divide the boundary C into elements which constitute the outer edges of the outside row of these elements of area. As the limiting processes involved in the definitions of the integrals proceed, the elements of area become more nearly triangular and Stokes' Theorem as proven for triangles may be applied to each element with the possible introduction of an error which will be an infinitesimal of higher order than the values of the integrals for the element. Since the surface S is two sided we may choose the normals **n** for the elements so that they all lie on the same side of S and so that

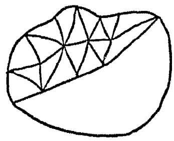

FIG. 90

consequently the integrals around the boundaries of the triangles are all taken in the same sense of rotation. Thus all the dividing lines between the elements will be covered by the integrations twice in opposite senses and these terms will thus disappear from the summation leaving only the terms arising from the elements of the boundary C. Since by Duhamel's theorem the infinitesimals of higher order in the terms have no effect on the limit of the sum, we find in this way that the application of Stokes' Theorem to each triangular element yields Stokes' Theorem for the given surface S. The theorem is thus proven in the general case.

Two remarks may be made concerning the argument in the last paragraph. We have assumed that the area of a region of a curved surface is a concept already defined and thus available for use in the definition of a surface integral. But as a matter of fact the area of a curved surface is itself a special case of a surface integral since it involves dividing the surface into elements and treating each element as if it were plane, followed by a summation and procedure to the limit. It would consequently be simpler in theory to define all surface integrals directly in terms of the elements as plane elements. If this be done in the above case, Stokes' Theorem for a curved surface follows at once *by definition* from Stokes' Theorem for triangles. It appeared in the last paragraph that an essential requirement of the surface S was that it be possible to distinguish two sides. A model of a surface for which this is not true and to which Stokes' Theorem is consequently not applicable is readily constructed. A rectangular strip of paper $ABCD$ has one end AD glued to the other end BC but with A falling on C and D on B. The resulting surface then has only one side and a single unbroken bounding curve. This model is known as Möbius' strip.

Fig. 91

In the preceding articles of this chapter and in the exercises at their ends we have numerous formulas giving the gradient, directional derivative, divergence, and curl of the various forms of vector and scalar products. These may be employed in connection with the Gradient, Divergence, Curl and Stokes' Theorems to yield a large number of useful formulas entirely analogous to the formula for *integration by parts* of the ordinary scalar-scalar

calculus. As an example of these we may derive what is probably the most important of them and known as,

*Green's First Identity.** If U and V are scalar-point functions continuous together with their gradients in a certain region of space and if furthermore ∇U has a continuous divergence in this region and if T is a regular region of space bounded by a closed regular surface S both lying wholly within the above region of continuity then,

$$(4) \quad \iiint V\,\nabla^2 U\,d\tau + \iiint \nabla U\cdot\nabla V\,d\tau = \iint V\,\mathbf{n}\cdot\nabla U\,d\sigma,$$

where $\mathbf{n}$ is the unit outward normal to the surface S.

To prove this we observe that by formula § 101, (10) we have

$$\nabla\cdot(V\,\nabla U) = V\,\nabla^2 U + \nabla V\cdot\nabla U,$$

and on applying the Divergence Theorem, § 104, (8) to the vector-point function $V\,\nabla U$ we find equation (4) at once.

If we further assume that ∇V as well as ∇U possesses a continuous divergence in the above region of continuity, we may interchange U and V in formula (4) and on subtracting the two equations find the important,

Green's Second Identity.

$$(5) \quad \iiint (U\,\nabla^2 V - V\,\nabla^2 U)d\tau = \iint (U\,\mathbf{n}\cdot\nabla \mathrm{V} - V\,\mathbf{n}\cdot\nabla U)d\sigma.$$

EXERCISES

1. Write out the Gradient, Divergence and Curl Theorems in Cartesian form letting $\mathbf{V} \equiv (P, Q, R)$ and $\mathbf{n} \equiv (\lambda, \mu, \nu)$. Write out the analogous theorems for a region S of the XY-plane bounded by a plane curve C, letting $\mathbf{V} \equiv (P, Q)$ and $\mathbf{n} \equiv (\lambda, \mu)$ where P and Q are functions of x and y. Show that all three of these theorems in the plane will follow if it be proven that,

$$\iint \frac{\partial U}{\partial x}\,d\sigma = \int \lambda U\,d\gamma.$$

Prove this formula by application of the fundamental theorem of the integral calculus.

* Green's identities were given in his celebrated paper: *An Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism*, 1828.

2. Write out Stokes' Theorem in Cartesian form for a region S of the XY-plane bounded by a plane curve C letting $\mathbf{V} \equiv (P, Q)$, $\mathbf{t}\, d\gamma \equiv (dx, dy)$ where P and Q are functions of x, y only. Show that this Stokes Theorem in the plane is a consequence of the formula,

$$\iint_S \frac{\partial U}{\partial x}\, d\sigma = \int_C \lambda U\, d\gamma$$

of Problem 1.

3. Prove that if the vector point function $\mathbf{V}$ together with its divergence and curl is continuous in a certain region of space and if a plane area S and its bounding regular closed plane curve C lie wholly within this region, then the Divergence Theorem,

$$\iint_S \nabla\cdot\mathbf{V}\, d\sigma = \int_C \mathbf{n}\cdot\mathbf{V}\, d\gamma,$$

holds over S and C provided that the derivative of $\mathbf{V}$ in the direction perpendicular to the plane is parallel to the plane at all points of S.

4. Show that Stokes' Theorem for a vector-point function $\mathbf{V}$ over a plane area S bounded by a regular closed plane curve C is identical with the plane Divergence Theorem over that same area and curve for the vector-point function $\mathbf{V}\times\mathbf{a}$ where $\mathbf{a}$ is the constant unit normal vector to S. Thus show that Stokes' Theorem for plane areas is a consequence of the theorem of Problem 3.

5. Prove that if U is a scalar-point function continuous together with its gradient in a certain region of space and if S is a regular two sided surface lying wholly within this region and bounded by a regular closed curve C then,

$$\iint_S \mathbf{n}\times\nabla U\, d\sigma = \int_C U\,\mathbf{t}\, d\gamma.$$

6. Show that a necessary and sufficient condition for the truth of Stokes' Theorem is the truth of the theorem of Problem 5.

7. Prove that, If U and $\mathbf{V}$ are scalar and vector point functions possessing continuous derivatives in all directions in a certain region of space and if C is a regular curve lying wholly within this region of space then,

$$\int_{P_1}^{P_2} \mathbf{t}\cdot\nabla U\, d\gamma = U(\mathbf{p}_2) - U(\mathbf{p}_1), \qquad \int_{P_1}^{P_2} \mathbf{t}\cdot\nabla\mathbf{V}\, d\gamma = \mathbf{V}(\mathbf{p}_2) - \mathbf{V}(\mathbf{p}_1).$$

8. Completely state and prove the theorems implied in the following equations:

$$(a) \quad \int_C P\,\mathbf{t}\cdot\nabla Q\,d\gamma + \int_C Q\,\mathbf{t}\cdot\nabla P\,d\gamma = 0 \qquad (C \text{ closed}),$$

$$(b) \quad \iint_S U\,\mathbf{n}\cdot\nabla\times\mathbf{V}\,d\sigma + \iint_S \mathbf{n}\cdot\nabla U\times\mathbf{V}\,d\sigma = \int_C U\,\mathbf{t}\cdot\mathbf{V}\,d\gamma,$$

$$(c) \quad \iint_S \mathbf{n}\cdot\nabla P\times\nabla Q\,d\sigma = \int_C P\,\mathbf{t}\cdot\nabla Q\,d\gamma = -\int_C Q\,\mathbf{t}\cdot\nabla P\,d\gamma,$$

$$(d) \quad \iiint_T U\,\nabla\cdot\mathbf{V}\,d\tau + \iiint_T \mathbf{V}\cdot\nabla U\,d\tau = \iint_S U\,\mathbf{n}\cdot\mathbf{V}\,d\sigma,$$

$$(e) \quad \iiint_T \nabla U\cdot\nabla\times\mathbf{V}\,d\tau = \iint_S \mathbf{n}\cdot\mathbf{V}\times\nabla U\,d\sigma,$$

$$(f) \quad \iiint_T U\,\nabla\times\mathbf{V}\,d\tau + \iiint_T \nabla U\times\mathbf{V}\,d\tau = \iint_S U\,\mathbf{n}\times\mathbf{V}\,d\sigma.$$

9. Prove that if a vector-point function $\mathbf{V}$ is normal to a given surface at each point of the surface, then the curl $\nabla\times\mathbf{V}$ is tangent to the surface at each point. Is the converse theorem true?

10. Show as a consequence of either the Divergence Theorem or Stokes' Theorem that,

$$\iint_S \mathbf{n}\cdot\nabla\times\mathbf{V}\,d\sigma = 0$$

if the surface S is closed.

106. Scalar and Vector Fields.

We have found that the gradient of a scalar-point function U and the divergence and curl of a vector-point function $\mathbf{V}$ give us considerable information concerning the behavior of these functions at each point of their region of definition. Instead of considering the behavior of U and $\mathbf{V}$ at individual points of the region we may investigate their behavior over the region taken as a whole. Considered in this way the region toegther with the associated values of U or $\mathbf{V}$ is known as a *scalar* or *vector field*, respectively.

We shall now consider the nature of a scalar field, recalling first some of the facts concerning it developed in § 100. Through each point of the region of definition of the scalar point function U we may pass a surface whose equation is of the form $U = (\text{const.})$. These surfaces along which U retains a constant value are known as *level surfaces*. Since the level surfaces never intersect each other they divide the region into layers or lamellas and for this reason we say that the distribution of a scalar field is *lamellar*. The gradient ∇U is at each point normal to the level surface through that point and thus if we visualize all the level surfaces of the field we have a picture indicating the surfaces along which U remains constant while the normals to these surfaces show the direction of the gradient, i.e. the direction in which U changes most rapidly. This picture may be made more informative by visualizing only the level surfaces for which U takes on an arithmetical sequence of values, U_0, $U_0 + a$, $U_0 + 2a$, $U_0 + 3a$, $\cdots$, in which a is any small constant. For let us consider the vector $\Delta\mathbf{p}$ running along the gradient from one of these surfaces to the next. By equation § 99, (1) we have,

$$(\nabla U + \boldsymbol{\varepsilon})\cdot\Delta\mathbf{p} = a, \qquad \lim_{\Delta\mathbf{p}\to 0} \boldsymbol{\varepsilon} = 0,$$

and since $\Delta\mathbf{p}$ is small and parallel to ∇U it appears that the distance $|\Delta\mathbf{p}|$ between two successive surfaces is approximately inversely proportional to the length of the gradient ∇U. Thus this particular family of level surfaces gives us a picture of the surfaces along which U remains constant and the normals to the surfaces indicate the direction of most rapid change in U while the distance between successive surfaces approximately varies inversely as the rate of change of U.

In the study of the vector field a concept of the greatest importance is that of the *vector lines*, also sometimes known as *lines of force* or *lines of flow*. A *vector line* is a curve in a vector field which is at each point tangent to the value of $\mathbf{V}$ at that point. If we think of the radius vector $\mathbf{p}$ of the point P on such a curve as a function of some scalar parameter t, then evidently this definition is equivalent to the differential equation,

$$(1) \qquad \mathbf{V}\times\mathbf{p}' = 0 \qquad \text{or briefly} \qquad \mathbf{V}\times d\mathbf{p} = 0.$$

In simple cases this differential equation may be solved directly in

vector form but in the more complicated cases it will be convenient to expand the equation in coördinates. Letting $\mathbf{V} \equiv (P, Q, R)$, $d\mathbf{p} \equiv (dx, dy, dz)$, equation (1) takes the form of a simultaneous system of the first order,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

It is shown in works on differential equations that if P, Q, R possess continuous derivatives with respect to x, y, z, then this differential system has two functionally independent solutions,

$$\alpha(x, y, z) = a, \qquad \beta(x, y, z) = b, \tag{3}$$

where a and b are arbitrary constants and these two equations treated as simultaneous determine the vector lines. We may pass a vector line through any given point of the field, for if its coördinates are substituted into equations (3) the values of a and b will be determined and with a and b so determined the two equations will determine a vector line through the given point. If any curve other than a vector line be drawn in the field, then through each point of this curve we may draw a vector line and the locus of these vector lines is called a *vector surface*. If the curve be closed the vector surface is called a *vector tube* or *solenoid*. The surfaces $\alpha(\mathbf{p}) = a$ and $\beta(\mathbf{p}) = b$ obtained above are themselves vector surfaces and the vector surface through an arbitrary curve $\mathbf{p} = \mathbf{f}(t)$ is obtained by eliminating t between the two equations,

$$\alpha(\mathbf{p}) = \alpha\{\mathbf{f}(t)\}, \qquad \beta(\mathbf{p}) = \beta\{\mathbf{f}(t)\}.$$

A vector surface is at each of its points tangent to the value of $\mathbf{V}$ at that point.

Of course there are in fact unlimitedly many vector lines in a vector field, but nevertheless, just as we found it convenient to visualize only a certain set of the level surfaces in a scalar field, so we find it convenient to speak of the *number of vector lines* passing in a certain sense through any given bounded two sided surface S in the vector field. By the number of vector lines through the surface S in the sense of the normal $\mathbf{n}$ we mean the value of the integral,

$$f = \iint_S \mathbf{n} \cdot \mathbf{V} \, d\sigma,$$

where $\mathbf{n}$ is the unit normal vector. This quantity f is also sometimes called the *flux* of $\mathbf{V}$ through S in the sense of $\mathbf{n}$. Evidently the number of vector lines crossing any portion of a vector surface is always zero.

There are two types of vector field which are of particular simplicity and importance and of frequent occurrence in the applications. They are,

The *lamellar field* which is defined as any field in which $\nabla \times \mathbf{V}$ is everywhere zero.

The *solenoidal field* which is defined as any field in which $\nabla \cdot \mathbf{V}$ is everywhere zero.

There are of course vector fields such as that of $\mathbf{V} = (\mathbf{a} \cdot \mathbf{p})\mathbf{p}$ which are neither lamellar nor solenoidal and there are others such as that of $\mathbf{V} = \mathbf{p}/|\mathbf{p}|^3$ which are both lamellar and solenoidal without being constant. We shall discuss the lamellar and solenoidal fields separately, *assuming the existence and continuity of the divergence and curl at all points of the field.*

In the case of the lamellar field we get at once a very important piece of information from Stokes' Theorem. Let C be any closed curve such that it is possible to find a two sided surface bounded by C and lying wholly within the field. This will clearly be the case if it is possible to shrink the closed curve C down to a point without having it anywhere leave the field. Then since the field is lamellar the first member of Stokes' Theorem,

$$\int\!\!\int \mathbf{n} \cdot \nabla \times \mathbf{V}\, d\sigma = \int \mathbf{V} \cdot \mathbf{t}\, d\gamma$$

will vanish for every two sided surface bounded by C and lying wholly within the field and the second member must consequently be zero. We define a region of space as *simply connected* if any closed curve drawn within it can be shrunk down to a point without leaving the region. We then have the theorem,

In a simply connected lamellar field we have

$$\int_C \mathbf{V} \cdot \mathbf{t}\, d\gamma = 0 \tag{4}$$

for every closed curve C.

From this it follows that if A and P are any two points of a simply connected lamellar field then the integral of $\mathbf{V}\cdot\mathbf{t}$ along a curve C from A to P depends only on the two end points A and P and not on the particular curve C along which the integration is carried out. For if C_1 and C_2 are any two curves of the field both running from A to P, then the curve C_1 from A to P and the curve C_2 from P to A form a closed curve to which we may apply theorem (4) and thus write,

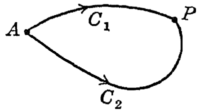

FIG. 92

$$\int_{C_1} \mathbf{V}\cdot\mathbf{t}\, d\gamma - \int_{C_2} \mathbf{V}\cdot\mathbf{t}\, d\gamma = 0,$$

in which now both integrations extend from A to P. On transposing the second term our statement is seen to be proven. If now the initial point A is fixed then the integral of $\mathbf{V}\cdot\mathbf{t}$ from A to P can only depend on the terminal point P and thus serves to determine a scalar-point function of P,

$$(5) \qquad U(\mathrm{p}) = \int^{P} \mathbf{V}\cdot\mathbf{t}\, d\gamma,$$

in which now the curve C joining A and P need no longer be specified. The scalar U thus defined as a function of P still involves the point A as a parameter, but we may readily see that any two such functions formed with different initial points A and B differ by a constant. For if we set

$$U_A(\mathrm{p}) = \int_A^P \mathbf{V}\cdot\mathbf{t}\, d\gamma, \qquad U_B(\mathrm{p}) = \int_B^P \mathbf{V}\cdot\mathbf{t}\, d\gamma,$$

then we have,

$$U_A(\mathrm{p}) - U_B(\mathrm{p}) = \int_A^B \mathbf{V}\cdot\mathbf{t}\, d\gamma = (\text{const.}).$$

Equation (5) defines a scalar-point function U in terms of the given vector-point function $\mathbf{V}$. We may conversely express $\mathbf{V}$ in terms of U. In fact we have the theorem,

If A and P are any two points of a simply connected lamellar vector field the integral,

$$U = \int_A^P \mathbf{V} \cdot \mathbf{t}\, d\gamma$$

is for fixed A a scalar-point function of P such that

$$\nabla U = \mathbf{V}.$$

We have seen that U is a scalar-point function of P and it only remains to show that the gradient of this U is $\mathbf{V}$. Let P_0 be any particular position of P and U_0 and $\mathbf{V}_0$ the corresponding values of U and $\mathbf{V}$. Now give to P the displacement $\Delta\mathbf{p}$ and let the corresponding change in U be ΔU. Let $\mathbf{a}$ be a unit vector having the direction and sense of $\Delta\mathbf{p}$ and let the length of $\Delta\mathbf{p}$ be Δs. Then $\Delta\mathbf{p} = \mathbf{a}\,\Delta s$ and we may write,

$$\Delta U = \int_{P_0}^{P} \mathbf{V} \cdot \mathbf{t}\, d\gamma = \mathbf{a} \cdot \int_0^{\Delta s} \mathbf{V}\, d\gamma,$$

where in the last integral we have chosen the straight line from P_0 to P as the path of integration. If we call $\mathbf{V} - \mathbf{V}_0 = \mathbf{n}$ we may write,

$$\int_0^{\Delta s} \mathbf{V}\, d\gamma = \int_0^{\Delta s} \mathbf{V}_0\, d\gamma + \int_0^{\Delta s} \mathbf{n}\, d\gamma = \mathbf{V}_0\, \Delta s + \int_0^{\Delta s} \mathbf{n}\, d\gamma.$$

If now $\mathbf{n}_1$ is the longest value assumed by $\mathbf{n}$ on the path considered we evidently may write,

$$\left| \int_0^{\Delta s} \mathbf{n}\, d\gamma \right| \leq \int_0^{\Delta s} |\mathbf{n}|\, d\gamma \leq \int_0^{\Delta s} |\mathbf{n}_1|\, d\gamma = |\mathbf{n}_1|\, \Delta s,$$

and consequently may set,

$$\int_0^{\Delta s} \mathbf{n}\, d\gamma = \boldsymbol{\varepsilon}\, \Delta s, \qquad \text{where} \qquad |\boldsymbol{\varepsilon}| \leq |\mathbf{n}_1|.$$

Due to the continuity of $\mathbf{V}$ we know that $\mathbf{n}_1$, and consequently $\boldsymbol{\varepsilon}$

must approach zero with Δs, so that we may write,

$$\Delta U = \mathbf{a}\cdot(\mathbf{V}_0\,\Delta s + \boldsymbol{\varepsilon}\,\Delta s) = \Delta\mathbf{p}\cdot(\mathbf{V}_0 + \boldsymbol{\varepsilon}) \qquad \text{where} \qquad \lim_{\Delta\mathbf{p}\to 0} \boldsymbol{\varepsilon} = 0.$$

On referring to the definition of gradient, § 99, (1) we see that $\mathbf{V}_0$ is the gradient of U at P_0 and our proposition is proven.

Since any lamellar vector function is thus the gradient of a scalar function the whole problem of the nature of a lamellar vector field reverts to the essentially simpler problem of the nature of a scalar field; a topic already discussed.

Since the gradient ∇U of a scalar-point function U is everywhere normal to the level surfaces of U it follows that for a lamellar vector field there always exists a family of surfaces filling the field and everywhere normal to the vector lines of the field, this family of surfaces being simply the level surfaces of the scalar U whose gradient is the given lamellar vector $\mathbf{V}$. Thus a *sufficient* condition that the vector lines satisfying the equation $\mathbf{V}\times d\mathbf{p} = 0$ have a family of normal surfaces is that the field be lamellar, i.e. that $\nabla\times\mathbf{V} = 0$. This condition is however not necessary and in fact we may prove that,

A necessary and sufficient condition that there exist a family of surfaces S filling the field and everywhere normal to the vector-point function $\mathbf{V}$ *is that* $\mathbf{V}$ *be everywhere perpendicular to its curl, i.e.,*

$$\mathbf{V}\cdot(\nabla\times\mathbf{V}) = 0.$$

We consider here two classes of surfaces. First we have the surfaces S described above. Evidently a necessary and sufficient condition that a surface be one of these surfaces is that the integral $\int_C \mathbf{V}\cdot\mathbf{t}\,d\gamma$ vanish for every curve C lying on the surface, whether C be closed or not. Next we consider the family of *curl surfaces* Σ which are the vector surfaces of the curl of $\mathbf{V}$, $\mathbf{W} = \nabla\times\mathbf{V}$. From Stokes' Theorem for such a surface,

$$\iint_\Sigma \mathbf{n}\cdot\mathbf{W}\,d\sigma = \int_\Gamma \mathbf{V}\cdot\mathbf{t}\,d\gamma,$$

it appears that a necessary and sufficient condition that a surface be a curl surface, i.e. be everywhere tangent to $\mathbf{W}$, is that the

integral $\int_\Gamma \mathbf{V}\cdot\mathbf{t}\,d\gamma$ vanish for every *closed* curve Γ lying on the surface. Thus every surface S is also a surface Σ. If now there exists a family of surfaces S filling the field then these surfaces are everywhere normal to $\mathbf{V}$ and tangent to $\mathbf{W}$ and it follows that $\mathbf{V}\cdot\mathbf{W} = 0$. The condition stated is thus necessary. To see that the condition is also sufficient we pass through any given point P_0 of the field any curve C_0 which is a vector line of any vector-point function which is everywhere perpendicular to $\mathbf{V}$. Then the curl surface Σ_0 of $\mathbf{V}$ which contains this curve C_0 is the desired surface S passing through P_0. For let A and B be any two points on Σ_0 and let A' and B' be the points where the vector lines of $\mathbf{W}$ through A and B intersect the curve C_0. Then the integral of $\mathbf{V}\cdot\mathbf{t}$ around the closed curve $AA'B'BA$ is zero because this closed curve lies on a curl surface. Likewise the integrals of $\mathbf{V}\cdot\mathbf{t}$ from A to A' and from B' to B are zero because these curves, being tangent to $\mathbf{W}$ must under the stated condition be everywhere normal to $\mathbf{V}$. Also the integral of $\mathbf{V}\cdot\mathbf{t}$ from A' to B' is zero by the manner of construction of the curve C_0. Consequently the integral of $\mathbf{V}\cdot\mathbf{t}$ between any two points A and B of the surface Σ_0 is zero and this surface must be a surface S. The theorem is thus proven.

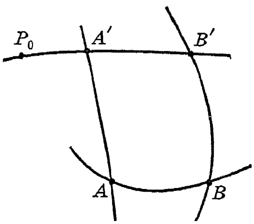

FIG. 93

The reader will observe that we have here proven in effect the familiar theorem of differential equations that a necessary and sufficient condition that the total differential equation,

$$P\,dx + Q\,dy + R\,dz = 0,$$

be integrable is that,

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

For if the family of surfaces S have its equation written in the form $U(xyz) = \lambda$ where λ is the parameter of the family, then this equation constitutes the solution of the differential equation, and conversely.

In the case of a solenoidal field we get an important piece of information from the divergence theorem. Let S be any closed surface such that $\mathbf{V}$ is solenoidal throughout its interior. Then the first member of the divergence theorem,

$$\iiint_T \nabla\cdot\mathbf{V}\,d\tau = \iint_S \mathbf{n}\cdot\mathbf{V}\,d\sigma,$$

must vanish and consequently the second also, and we conclude that,

In a solenoidal field we have,

$$\iint_S \mathbf{n}\cdot\mathbf{V}\,d\sigma = 0$$

for every closed surface S whose interior is entirely within the field. This theorem evidently has a certain analogy with the theorem that $\int_C \mathbf{V}\cdot\mathbf{t}\,d\gamma$ vanishes for every closed curve in a simply connected lamellar field.

If a vector tube or solenoid be drawn in a solenoidal field it follows from the above theorem that the flux through any cross section of the tube in a given sense is a constant independent of the form and position of the cross section. For if two cross sections S_1 and S_2 be drawn across the tube, then they together with the portion of the surface of the tube included between them form a closed surface to which we may apply the above theorem. But the surface of the tube is a vector surface and the flux through it is consequently zero, so that we have,

$$\iint_{S_1} \mathbf{n}\cdot\mathbf{V}\,d\sigma - \iint_{S_2} \mathbf{n}\cdot\mathbf{V}\,d\sigma = 0,$$

in which now the normal $\mathbf{n}$ has the same sense in the two cases. On transposing the second term our statement is seen to be proven.

The property just proven leads to the term *solenoidal field* applied to vector fields of this type. For if a surface other than a vector surface be drawn cutting across the field, we may draw curves on this surface dividing it up into portions such that the flux through each portion in a certain sense is some small positive

constant a. We then pass vector surfaces through each of these curves forming a packing of vector tubes filling the field. Since the flux through every cross section of every tube is equal to a, the area of the normal plane cross sections of the tubes varies approximately inversely as the length of $\mathbf{V}$ while the lines of contact of three or more tubes indicate the direction of $\mathbf{V}$. Evidently no tube can entirely close up, for since $\mathbf{V}$ remains finite this would require that the flux through the tube reduce to zero.

Analogous to the theorem that every lamellar vector function is the gradient of a scalar function, we have the theorem,

Every solenoidal vector function, $\mathbf{V}$, *is the curl of a vector function,* $\mathbf{W}$.

To see this we first recall (3) the two independent families

$$\alpha(\mathbf{p}) = a, \qquad \beta(\mathbf{p}) = b$$

of vector surfaces. Since $\mathbf{V}$ is at each point tangent to the surface of each family that passes through that point it follows that $\mathbf{V}$ is everywhere perpendicular to both $\nabla\alpha$ and $\nabla\beta$ and we may write,

$$\mathbf{V} = \varphi\, \nabla\alpha \times \nabla\beta,$$

where φ is a scalar point function. Since the field is solenoidal we have,

$$\begin{aligned} 0 = \nabla\cdot\mathbf{V} &= \varphi\, \nabla\cdot(\nabla\alpha \times \nabla\beta) + [\nabla\varphi\, \nabla\alpha\, \nabla\beta] \\ &= \varphi\, \nabla\cdot\{\nabla \times (\alpha\, \nabla\beta)\} + [\nabla\varphi\, \nabla\alpha\, \nabla\beta] \\ &= [\nabla\varphi\, \nabla\alpha\, \nabla\beta]. \end{aligned}$$

This last member being exactly the Jacobian $\dfrac{\partial(\varphi, \alpha, \beta)}{\partial(x, y, z)}$, it follows that φ is a function of α and β. If then γ is a scalar function of α and β such that,

$$\frac{\partial\gamma}{\partial\alpha} = \varphi(\alpha, \beta),$$

we have,

$$\nabla\gamma = \varphi\, \nabla\alpha + \frac{\partial\gamma}{\partial\beta}\nabla\beta$$

and,

$$\nabla\gamma \times \nabla\beta = \varphi\, \nabla\alpha \times \nabla\beta = \mathbf{V}.$$

Thus we have finally,

$$\mathbf{V} = \nabla\gamma \times \nabla\beta = \nabla \times (\gamma\, \nabla\beta) = \nabla \times \mathbf{W},$$

as we wished to prove.

The vector function $\mathbf{W}$ which has the given solenoidal vector $\mathbf{V}$ for its curl is however by no means unique. In fact the reader can readily establish that the most general solution $\mathbf{W}$ of the differential equation $\nabla \times \mathbf{W} = \mathbf{V}$, where $\mathbf{V}$ is a given solenoidal vector function, is of the form $\mathbf{W} = \mathbf{W}_1 + \nabla U$ where $\mathbf{W}_1$ is any particular solution of the equation and U is an arbitrary scalar function.

We have seen that a lamellar vector function can be written in the form,

$$\mathbf{V} = \nabla\alpha;$$

a vector function possessing a family of normal surfaces in the form,

$$\mathbf{V} = \varphi\,\nabla\alpha;$$

a solenoidal vector function in the form,

$$\mathbf{V} = \nabla\alpha \times \nabla\beta;$$

and any vector function in the form,

$$\mathbf{V} = \varphi\,\nabla\alpha \times \nabla\beta.$$

Thus the first type is said to be *simply scalar*, the next two *doubly scalar* and the last *triply scalar*.

EXERCISES

1. Prove that if the integral $\int_C \mathbf{V} \cdot \mathbf{t}\,d\gamma$ vanishes for every closed curve in a vector field, then the field is lamellar.
2. Prove that it follows from the theorem of Problem 1 that,
$$\nabla \times (\nabla U) = 0.$$
3. If P, Q, R are functions of x, y, z, show that the necessary and sufficient condition that there exist a function $U(x, y, z)$ such that,
$$\frac{\partial U}{\partial x} = P, \qquad \frac{\partial U}{\partial y} \quad Q, \qquad \frac{\partial U}{\partial z} = R,$$
is that,
$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$
4. Find the vector lines of the following vector fields without expanding in coordinates.

(*a*) $\mathbf{V} = \mathbf{a}$, (*d*) $\mathbf{V} = \mathbf{a} \times \mathbf{p} + \mathbf{b}$, (*g*) $\mathbf{V} = \mathbf{a} \times (\mathbf{p} \times \mathbf{b})$.
(*b*) $\mathbf{V} = \mathbf{p}$, (*e*) $\mathbf{V} = \mathbf{a} \times (\mathbf{p} \times \mathbf{a})$,
(*c*) $\mathbf{V} = \mathbf{a} \times \mathbf{p}$, (*f*) $\mathbf{V} = \mathbf{p} \times (\mathbf{a} \times \mathbf{p})$,

5. Determine which of the vector fields of Problem 4 possess normal surfaces and in case these surfaces exist, describe them.

6. Prove that if the integral $\displaystyle\iint_S \mathbf{n}\cdot\mathbf{V}\, d\sigma$ vanishes for every closed surface of a vector field, then the field is solenoidal.

7. Prove that it follows from the theorem of Problem 6 that,

$$\nabla\cdot(\nabla\times\mathbf{V}) = 0.$$

8. Show that any vector-point function $\mathbf{V}$ may be written in the form, $\mathbf{V} = \varphi\,\nabla\alpha + \nabla\beta$, where φ, α, β are scalar point functions. Show that if φ and α are functionally related then $\mathbf{V}$ is lamellar and that if either α and β or φ and β are functionally related then $\mathbf{V}$ is everywhere normal to a family of surfaces.

9. Show that $\mathbf{V} = \mathbf{a}\times\mathbf{p}/(\mathbf{a}\times\mathbf{p})^2$ is lamellar and find a scalar function U of which it is the gradient.

$$\textit{Ans.}\quad U = \frac{1}{|\mathbf{a}|}\arccos\frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{p})}{\sqrt{(\mathbf{a}\times\mathbf{b})^2(\mathbf{a}\times\mathbf{p})^2}}$$

10. Show that $\mathbf{V} = \dfrac{\mathbf{p}}{|\mathbf{p}|^3}$ is solenoidal and find a vector function $\mathbf{W}$ of which it is the curl.

$$\textit{Ans.}\quad \mathbf{W} = -\frac{\mathbf{a}\cdot\mathbf{p}}{|\mathbf{p}|}\,\frac{\mathbf{a}\times\mathbf{p}}{(\mathbf{a}\times\mathbf{p})^2}$$

11. If $\mathbf{V}$ is both lamellar and solenoidal show that,

$$\nabla(\mathbf{a}\cdot\mathbf{V}) + \nabla\times(\mathbf{a}\times\mathbf{V}) = 0,$$

where $\mathbf{a}$ is any constant vector.

12. Show that $\mathbf{V} = \dfrac{\mathbf{a}\times(\mathbf{p}\times\mathbf{a})}{(\mathbf{a}\times\mathbf{p})^2}$ is both lamellar and solenoidal and find a scalar function U of which it is the gradient and a vector function $\mathbf{W}$ of which it is the curl.

$$\textit{Ans.}\quad U = \frac{1}{2}\log(\mathbf{a}\times\mathbf{p})^2, \qquad \mathbf{W} = \frac{\mathbf{a}}{|\mathbf{a}|}\arccos\frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{p})}{\sqrt{(\mathbf{a}\times\mathbf{b})^2(\mathbf{a}\times\mathbf{p})^2}}$$

or

$$\mathbf{W} = \frac{\mathbf{a}\cdot\mathbf{p}}{(\mathbf{a}\times\mathbf{p})^2}\,(\mathbf{p}\times\mathbf{a})$$

CHAPTER XIV

POTENTIAL THEORY

107. Fields of Force.

In 1687 Sir Isaac Newton enunciated the *law of universal gravitation* which may be expressed as follows:

Every particle of matter attracts every other particle of matter with a force proportional to the product of their masses and inversely proportional to the square of their distance.

Thus two particles of masses m_1 and m_2 and separated by a distance r attract each other with a force of amount,

$$f = k^2 \frac{m_1 m_2}{r^2},$$

where k^2 is a constant proportionality factor known as the *constant of gravitation.* Experiment shows that this law holds to an extremely high degree of precision for all substances and for an immense range of distances. If the two particles are located at the two points P and Q having radius vectors $\mathbf{p}$ and $\mathbf{q}$ then the attraction at P due to Q is the vector,

$$\mathbf{f} = k^2 \frac{m_1 m_2}{r^3} (\mathbf{q} - \mathbf{p}), \qquad r = |\mathbf{q} - \mathbf{p}|.$$

By choosing as the unit of force the attraction of two unit particles at unit distance and assuming that the mass of P is unity, the above formula reduces to,

$$\mathbf{f} = \frac{m}{r^3} (\mathbf{q} - \mathbf{p}), \tag{1}$$

where m is the mass of the particle Q.

Coulomb established that the above law also holds for two point electric charges or two magnetic poles except that in this case the masses m_1 and m_2 must be regarded as positive or negative, the force being an attraction when their signs are unlike and a repulsion when their signs are like. The proportionality factor k^2 in these cases depends on the medium in which the particles lie.

If the unit particle P be attracted by n particles $Q_1, Q_2, \cdots, Q_n$ of masses $m_1, m_2, \cdots, m_n$ and radius vectors $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n$, then the force at P is, by the additive property of forces implied in Newton's second law of motion, given by,

$$(2) \qquad \mathbf{f} = \sum_{i=1}^{n} \frac{m_i}{r_i^3} (\mathbf{q}_i - \mathbf{p}), \qquad r_i = |\mathbf{q}_i - \mathbf{p}|.$$

If we now think of the unit particle P as variable and assuming any position in space except at the attracting particles Q_i, then this force $\mathbf{f}$ becomes a vector-point function of P and we may apply to this function the various concepts discussed in the preceding chapter. Thus the vector lines become *lines of force*, the vector tubes *tubes of force* and the vector flux through a surface the *flux of force* through the surface.

Our first observation concerning this field of force is,

The field of force due to the attraction of isolated particles is, except at the attracting particles, both lamellar and solenoidal.

Considering first the field due to a single attracting particle Q we have, $r^2 = (\mathbf{q} - \mathbf{p})^2 = \mathbf{p}^2 - 2\,\mathbf{p}\cdot\mathbf{q} + \mathbf{q}^2$ and so, $\mathbf{q}$ being constant, $\nabla(r^2) = -2(\mathbf{q} - \mathbf{p})$ and by formula § 99, (6),

$$\nabla\left(\frac{m}{r^3}\right) = \frac{3m}{r^5} (\mathbf{q} - \mathbf{p}).$$

Then by formula § 101, (10) we have,

$$(3) \quad \nabla\cdot\mathbf{f} = \nabla\cdot\left\{\frac{m}{r^3}(\mathbf{q} - \mathbf{p})\right\} = \frac{m}{r^3}\nabla\cdot(\mathbf{q} - \mathbf{p}) + \nabla\left(\frac{m}{r^3}\right)\cdot(\mathbf{q} - \mathbf{p})$$
$$= -\frac{3m}{r^3} + \frac{3m}{r^5}(\mathbf{q} - \mathbf{p})^2 = 0$$

and by formula § 102, (10),

$$(4) \quad \nabla\times\mathbf{f} = \nabla\times\left\{\frac{m}{r^3}(\mathbf{q} - \mathbf{p})\right\}$$
$$= \frac{m}{r^3}\nabla\times(\mathbf{q} - \mathbf{p}) + \nabla\left(\frac{m}{r^3}\right)\times(\mathbf{q} - \mathbf{p}) = 0.$$

The theorem being thus established for the field due to a single particle, follows by addition for the field due to any finite number of distinct particles.

Since $\frac{m}{r^3}(\mathbf{q} - \mathbf{p})$ is for constant $\mathbf{q}$ a lamellar function of $\mathbf{p}$ it

follows that it must be the gradient of some scalar point function U, and in fact the reader may readily verify that we have,

$$\text{(5)} \qquad \nabla U = \nabla\left(\frac{m}{r}\right) = \frac{m}{r^3}(\mathbf{q} - \mathbf{p}) = \mathbf{f}.$$

The scalar U whose gradient thus yields a given lamellar vector point function is known as the *scalar potential* of that vector. Similarly since $\frac{m}{r^3}(\mathbf{q} - \mathbf{p})$ is a solenoidal function of $\mathbf{p}$ it must be the curl of some vector function $\mathbf{W}$, and the reader may verify that if $\mathbf{a}$ is any non-zero constant vector we have,

$$\text{(6)} \quad \nabla\times\mathbf{W} = \nabla\times\left\{\frac{m\,\mathbf{a}\cdot(\mathbf{q} - \mathbf{p})}{r}\,\frac{\mathbf{a}\times(\mathbf{q} - \mathbf{p})}{\{\mathbf{a}\times(\mathbf{q} - \mathbf{p})\}^2}\right\}$$
$$= \frac{m}{r^3}(\mathbf{q} - \mathbf{p}) = \mathbf{f}.$$

A vector $\mathbf{W}$ whose curl thus yields a given solenoidal vector-point function is known as the *vector potential* of that vector.

Since the field of force due to any number of discrete particles is lamellar except at these particles we may apply to it the theorem that for lamellar fields the integral of the tangential component around a closed curve is zero, i.e.,

$$\text{(7)} \qquad \int_C \mathbf{f}\cdot\mathbf{t}\,d\gamma = 0,$$

it being of course assumed that the curve does not pass through any of the attracting particles. As pointed out in § 72 the integral of the tangential component of a force along any directed curve C is known as the *work* of the force along this path. Equation (7) thus states in effect,

The work of the force $\mathbf{f}$ *due to any number of discrete particles around any closed path not passing through any of these particles is zero.*

A field of force possessing this property is said to be *conservative*; a term exactly equivalent to the term *lamellar* applied to general vector fields.

Since the field of force is solenoidal except at the attracting particles we may apply the previously proven theorem for such fields and conclude that,

The integral of the normal component of the force over any closed surface S not enclosing any of the attracting particles is zero, i.e.,

$$\iint_S \mathbf{n}\cdot\mathbf{f}\,d\sigma = 0. \tag{8}$$

In other words the flux of force outward from any closed surface is zero if the surface does not enclose any of the attracting particles. It follows as before that any tube of force not containing any of the attracting particles has a constant flux of force across all its cross sections and consequently can never entirely close up. This is often expressed by saying that the tubes of force terminate only at the attracting particles.

The question now arises as to what the outward flux of force will be from a closed surface which does contain some of the attracting particles. Consider first a sphere S of radius r with an attracting particle of mass m at its center and no other particles within or on it. At each point of S the length of the force is m/r^2 and since its direction is opposite to that of the unit outward normal $\mathbf{n}$ to S at the point we see that $\mathbf{n}\cdot\mathbf{f}$ has the constant value $-m/r^2$. Consequently we have,

$$\iint_S \mathbf{n}\cdot\mathbf{f}\,d\sigma = -\frac{m}{r^2}\iint_S d\sigma = -\frac{m}{r^2}(4\pi r^2) = -4\pi m.$$

If now we consider any closed surface S enclosing a finite number n of discrete attracting particles P_i of masses m_i, we may surround each particle with a sphere S_i of sufficiently small radius δ_i so that it does not contain any other of the particles and lies wholly within S. Then the volume which is *outside* of these spheres and inside S contains none of the particles and by equation (8) we have for the flux through its surface of the force $\mathbf{f}$ due to all the particles,

$$\iint_S \mathbf{n}\cdot\mathbf{f}\,d\sigma + \iint_{S_1} \mathbf{n}\cdot\mathbf{f}\,d\sigma + \iint_{S_2} \mathbf{n}\cdot\mathbf{f}\,d\sigma + \cdots + \iint_{S_n} \mathbf{n}\cdot\mathbf{f}\,d\sigma = 0,$$

in which now the unit normal $\mathbf{n}$ being outward from the volume

considered will be directed toward the interior of each sphere S_i. Each of the spheres S_i contains but the one attracting particle P_i and so, as observed above, we have for each sphere when $\mathbf{n}$ is its *inward* normal,

$$\iint_{S_i} \mathbf{n}\cdot\mathbf{f}\, d\sigma = 4\pi m_i.$$

Consequently on substituting in the above equation we find,

$$(9) \qquad \iint_S \mathbf{n}\cdot\mathbf{f}\, d\sigma = -\, 4\pi m_0, \qquad m_0 = m_1 + m_2 + \cdots + m_n.$$

That is,

The outward flux of force through any closed surface not passing through any of the particles is $-\,4\pi$ *times the total mass of the enclosed particles.*

The attracting bodies actually encountered in nature are regarded by the physicist as sets of particles and consequently the above discussion might be directly applied to them except for the practical difficulty that the number of particles is ordinarily so great that it would be quite impossible to carry out the required summations. In such cases we find the attraction of the body on an exterior particle by the following procedure. We divide the space T occupied by the body into a large number of subdivisions, $\Delta\tau_1, \Delta\tau_2, \cdots, \Delta\tau_n$. We represent by $\rho_i\,\Delta\tau_i$ the mass of the particles contained in $\Delta\tau_i$, ρ_i being known as the *mean density* of that subdivision. Now it is *not* in general true that the attraction of the set of particles in $\Delta\tau_i$ will be equal to that of a single particle of mass $\rho_i\,\Delta\tau_i$ situated at some point in $\Delta\tau_i$. Nevertheless it may be shown * that the attraction on a given external particle of the particles in $\Delta\tau_i$ differs from that of a single particle of mass $\rho_i\,\Delta\tau_i$ located at any point of $\Delta\tau_i$ by a vector $\zeta_i\,\Delta\tau_i$, where every ζ_i can be made less in length than an arbitrarily chosen positive constant ϵ, independent of i, by taking the subdivisions $\Delta\tau_i$ sufficiently small in all their dimensions. This being the case, if we now assume that the mean density ρ_i of each subdivision is the value assumed by a certain continuous scalar point function ρ at a point in that subdivision, then it is shown in the integral calculus that formula

* *The Integral as the Limit of a Sum, and a Theorem of Duhamel's*, William F. Osgood, *Annals of Mathematics*, Sec. Ser., Vol. 4 (1902–3), p. 161.

(2) will become in the limit as the dimensions of the subdivisions approach zero,

$$(10)\quad \mathbf{f} = \iiint_T \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\tau, \qquad \mathbf{r} = \mathbf{q} - \mathbf{p}, \qquad r = |\mathbf{q} - \mathbf{p}|.$$

The scalar point function ρ is known as the *density*. It is not necessary that it vary continuously over the entire region T, as the above integral will exist and give the force $\mathbf{f}$ if the region T can be divided into a finite number of subdivisions in each of which ρ is continuous. We then say that ρ is *piecewise continuous.* It should be noted that the variable of integration in the above integral is $\mathbf{q}$ and that $\mathbf{f}$ is a function of $\mathbf{p}$.

Although such bodies can scarcely occur in nature, we may also consider the cases in which the attracting particles are distributed over a surface S or a curve C. The treatment follows the same lines as that employed above for volume distributions of the particles, and under the same hypotheses concerning the density we obtain the formulas,

$$(11)\quad \mathbf{f} = \iint_S \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\sigma, \qquad \mathbf{r} = \mathbf{q} - \mathbf{p}, \qquad r = |\mathbf{q} - \mathbf{p}|,$$

$$(12)\quad \mathbf{f} = \int_C \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\gamma$$

for the attraction of the set of particles of the surface S and of the curve C on the unit particle at P. In these cases ρ is known as the *surface density* and the *linear density*, respectively.

The fields of force given by equations (10), (11), (12) are, like that given by equation (2), both lamellar and solenoidal at all points outside of the attracting bodies. For since r never becomes zero for such external points the quantity $\rho(\mathbf{q})\mathbf{r}/r^3$ as a function of $\mathbf{p}$ possesses continuous derivatives in all directions and we may differentiate under the integral sign and find as in equations (3) and (4) that $\nabla \cdot \mathbf{f} = 0, \nabla \times \mathbf{f} = 0$ for $\mathbf{f}$ as given by equation (10), (11) or (12).

Since the force $\mathbf{f}$ due to these so-called *continuous distributions* of particles is lamellar outside of the attracting body it must in each case be the gradient of some scalar point function and in

fact we readily verify by differentiation under the integral sign that the scalar point functions,

$$(13)\qquad U(\mathbf{p}) = \iiint\limits_T \frac{\rho(\mathbf{q})}{r}\, d\tau,$$

$$(14)\qquad U(\mathbf{p}) \equiv \iint\limits_S \frac{\rho(\mathbf{q})}{r}\, d\sigma,$$

$$(15)\qquad U(\mathbf{p}) = \int \frac{\rho(\mathbf{q})}{r}\, d\gamma,$$

satisfy the relation $\nabla U = \mathbf{f}$ for the corresponding force fields. As before, U is known as the scalar potential for $\mathbf{f}$. Since these fields are solenoidal outside of the attracting bodies it follows from the divergence theorem that,

The flux of force through any closed surface not enclosing any attracting bodies is zero.

But in case an attracting body T lies wholly within some closed surface S we have for the outward flux of force through S,

$$\iint \mathbf{n}\cdot\mathbf{f}(\mathbf{p})\, d\sigma = \iint \mathbf{n}\cdot\left\{\iiint \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\, d\tau\right\} d\sigma.$$

Since r does not vanish the integrand is continuous and we may consequently change the order of integration obtaining,

$$\iint\limits_S \mathbf{n}\cdot\mathbf{f}(\mathbf{p})\, d\sigma = \iiint\limits_T \rho(\mathbf{q})\left\{\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, d\sigma\right\} d\tau.$$

But now the inner integral is simply the flux through S of the force due to a unit particle placed at Q and is consequently -4π, as previously observed. Thus we have,

$$(16)\qquad \iint\limits_S \mathbf{n}\cdot\mathbf{f}(\mathbf{p})\, d\sigma = -4\pi\iiint\limits_T \rho(\mathbf{q})\, d\tau = -4\pi m_0,$$

where m_0 is the total mass in T. Similar arguments hold for surface and line distributions and we conclude,

The outward flux through a closed surface S of the force due to a

continuous distribution wholly within S is -4π *times the total mass of the attracting body.*

The force fields (10), (11), (12) being solenoidal outside of the attracting body possess vector potentials which may readily be written out from that for a single particle as given in equation (6).

We have thus far considered the fields of force arising from the gravitational attraction of particles which are discrete or continuously distributed over certain volumes, surfaces, and curves. We now wish to consider the field of *magnetic force* due to discrete or continuously distributed *magnetic particles* or *doublets*. It is found that the magnetic properties of a bar magnet manifest themselves at two points near the ends of the bar known as the *poles* of the magnet. We designate one of these as the positive and the other as the negative pole and find that two like poles repel each other and two unlike poles attract each other with a force inversely proportional to the square of their distance and directly proportional to the product of the scalar quantities known as the *strengths* of the poles. The two poles of a magnet have numerically equal strengths and thus it is convenient to regard a bar magnet as two particles of masses $+m$ and $-m$ placed at the positive and negative poles respectively. We may then consider the field of force due to the action of these two particles on a single magnetic pole of strength $+1$. Let the particle of mass $-m$ be located at Q, the particle of mass $+m$ at Q' and the unit pole at P and call the vectors $PQ = \mathbf{r}$, $PQ' = \mathbf{r} + \Delta\mathbf{r}$. With a properly chosen unit of force we then have for the force on P due to the particles Q and Q',

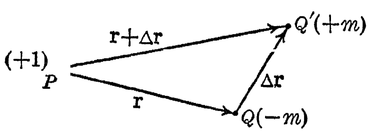

FIG. 94

$$\mathbf{f} = -m\frac{\mathbf{r}+\Delta\mathbf{r}}{|\mathbf{r}+\Delta\mathbf{r}|^3} + m\frac{\mathbf{r}}{|\mathbf{r}|^3} = -m\Delta\left(\frac{\mathbf{r}}{r^3}\right) \qquad r = |\mathbf{r}|,$$

where $\Delta(\mathbf{r}/r^3)$ designates the variation in $\mathbf{r}/r^3$ in passing from Q to Q'. If we now let $\mathbf{u}$ be a unit vector having the direction and sense of $\Delta\mathbf{r}$ and let Δs be the length of $\Delta\mathbf{r}$, then we may represent by $\mathbf{u}\cdot\nabla(\mathbf{r}/r^3)$ the derivative of $\mathbf{r}/r^3$ in the direction of $\Delta\mathbf{r}$ and so by equation § 100, (7) write,

$$\{\mathbf{u}\cdot\nabla(\mathbf{r}/r^3) + \boldsymbol{\varepsilon}\}\Delta s = \Delta(\mathbf{r}/r^3), \qquad \lim_{\Delta s\to 0} \boldsymbol{\varepsilon} = 0.$$

or

$$\mathbf{f} = -\, m\{\mathbf{u}\cdot\nabla(\mathbf{r}/r^3) + \boldsymbol{\varepsilon}\}\Delta s.$$

If now Q' approaches Q along the line joining them while at the same time m increases in such a way that the vector $m\,\Delta\mathbf{r}$ retains a constant value $\mathbf{m}$, we have in the limit,

$$\mathbf{f} = -\,\mathbf{m}\cdot\nabla_Q(\mathbf{r}/r^3), \tag{17}$$

where the subscript Q denotes that Q and not P has been regarded as the variable point. We say that this limiting force $\mathbf{f}$ is the force at P due to a *magnetic particle* or *doublet* of *moment* $\mathbf{m}$ at Q. Evidently the doublet is a mathematical abstraction suggested by a very short very strong bar magnet.

In the above limiting process the particle P was held fixed and $\mathbf{r}$ regarded as a vector-point function of Q. We may now consider the doublet at Q as determined by its moment $\mathbf{m}$ and the radius vector $\mathbf{q}$ of Q and regard $\mathbf{r}$ and $\mathbf{f}$ as functions of the radius vector $\mathbf{p}$ of P, where $\mathbf{r} = \mathbf{q} - \mathbf{p}$. We shall then have,

$$\mathbf{f} = \mathbf{m}\cdot\nabla(\mathbf{r}/r^3), \tag{18}$$

for evidently if $F(\mathbf{r})$ is any scalar or vector function of $\mathbf{r} = \mathbf{q} - \mathbf{p}$ the differential operators ∇, $\nabla\cdot$, $\nabla\times$ when applicable to F will yield results which will change only in sign when the independent variable is changed from $\mathbf{q}$ to $\mathbf{p}$ or vice-versa, the other of the two being held constant.

The force field due to a doublet is, like that due to a single particle, *both lamellar and solenoidal* except at the doublet, for we have by § 103, Prob. 8 and equations (3) and (4) of this article,

$$\nabla\cdot\mathbf{f} = \nabla\cdot\left(\mathbf{m}\cdot\nabla\frac{\mathbf{r}}{r^3}\right) = \mathbf{m}\cdot\nabla\left(\nabla\cdot\frac{\mathbf{r}}{r^3}\right) = 0, \tag{19}$$

$$\nabla\times\mathbf{f} = \nabla\times\left(\mathbf{m}\cdot\nabla\frac{\mathbf{r}}{r^3}\right) = \mathbf{m}\cdot\nabla\left(\nabla\times\frac{\mathbf{r}}{r^3}\right) = 0. \tag{20}$$

Being lamellar the field must possess a scalar potential which may be taken as $U = \mathbf{m}\cdot\nabla\left(\dfrac{1}{r}\right)$, for we have,

$$\nabla U = \nabla\left(\mathbf{m}\cdot\nabla\frac{1}{r}\right) = \mathbf{m}\cdot\nabla\left(\nabla\frac{1}{r}\right) = \mathbf{m}\cdot\nabla\frac{\mathbf{r}}{r^3} = \mathbf{f}. \tag{21}$$

Being solenoidal the field must possess a vector potential which

may be taken as $\mathbf{W} = -\mathbf{m} \times \nabla\left(\frac{1}{r}\right)$, for we have,

$$\nabla \times \mathbf{W} = -\nabla \times \left(\mathbf{m} \times \nabla \frac{1}{r}\right) \tag{22}$$

$$= \mathbf{m} \cdot \nabla\left(\nabla \frac{1}{r}\right) - \mathbf{m}\, \nabla \cdot \left(\nabla \frac{1}{r}\right)$$

$$= \mathbf{m} \cdot \nabla \frac{\mathbf{r}}{r^3} - \mathbf{m}\, \nabla \cdot \frac{\mathbf{r}}{r^3} = \mathbf{f}.$$

A continuous distribution of doublets over a two sided surface with the moments having everywhere the direction and sense of the unit normal vector directed to one side of the surface is known as a *double layer.* Evidently the double layer is a mathematical abstraction suggested by a thin sheet of iron or other magnetic substance magnetized in such a way that one side of the sheet is made up of positive poles and the other of equally strong negative poles. If we consider a large number of doublets Q_i placed on a continuously curved surface S with their moments $\mathbf{m}_i$ each having the direction and sense of the unit normal $\mathbf{n}_i$ to S at Q_i on a certain side, then we may set $\mathbf{m}_i = m_i \mathbf{n}_i$, and m_i being the length of the moment of the doublet is called the *scalar moment* or *strength* of the doublet. We now divide the surface S into a number of divisions $\Delta\sigma_i$ and represent the total strength of the doublets in $\Delta\sigma_i$ by $\mu_i \Delta\sigma_i$ where μ_i is called the *mean density* of the distribution in that subdivision. If we now assume that the mean density in each subdivision always equals the value assumed by a certain piecewise continuous scalar-point function μ at some point of that subdivision, then it may be shown by essentially the argument employed for continuous distributions of particles that in the limit as the subdivisions approach zero in all their dimensions the force at P due to the distribution of doublets is given by,

$$\mathbf{f} = -\iint_S \mu(\mathbf{q})\, \mathbf{n} \cdot \nabla_Q(\mathbf{r}/r^3)\, d\sigma, \qquad \mathbf{r} = \mathbf{q} - \mathbf{p}. \tag{23}$$

The scalar-point function μ is known as the *density* of the double layer.

Following the argument employed for continuous distributions of particles we may easily show that the force field of the double

layer is, except on the surface S, both lamellar and solenoidal and in fact possesses the scalar and vector potentials,

$$U = -\iint_S \mu(\mathbf{q})\, \mathbf{n}\cdot\nabla_Q \left(\frac{1}{r}\right) d\sigma, \tag{24}$$

$$\mathbf{W} = \iint_S \mu(\mathbf{q})\, \mathbf{n}\times\nabla_Q \left(\frac{1}{r}\right) d\sigma. \tag{25}$$

It may also be shown that if a closed surface encloses a finite number of doublets or encloses a double layer, then the flux of the resulting force through the surface is zero.

In dealing with magnetic and electrical forces it is not unusual to regard the force as the *negative* gradient of the scalar potential. If this be done the value of the scalar potential U for the doublet and the double layer will be the negative of that given above.

It may be shown (§ 108) that the force at any point due to an infinitely long slender rod of constant linear density is directed along the perpendicular from the point to the rod and varies in amount inversely as the distance from the rod. The properties of such a force field will be completely determined by a study of its behavior in any plane perpendicular to the rod. We find it convenient to think of the rod as replaced by a particle located at the point of intersection of the rod and the plane and producing the same force field in this plane. Such a particle would have a mass equal to twice the linear density of the rod and would attract unit particles in the plane with a force inversely proportional to their distance. Such particles are known as *logarithmic particles* and the study of their plane fields of force constitutes the two dimensional problem analogous to that of Newtonian force fields in space. See Problem 14.

EXERCISES

1. The amount of the Newtonian attraction of two particles is given in absolute units by the formula,

$$f = k^2 \frac{m_1 m_2}{r^2}.$$

If now new absolute units of length, mass, and time be introduced which are respectively $1/\lambda$, $1/\mu$, $1/\tau$ times as large as those em-

ployed above, show that the constant of gravitation k^2 becomes,

$$k'^2 = k^2 \frac{\lambda^3}{\mu\tau^2}.$$

Hence show that if in both cases the unit of mass be taken as that of a unit cube of water then,

$$k'^2 = \frac{k^2}{\tau^2}.$$

2. The constant of gravitation has the value $k^2 = 6.670 \times 10^{-8}$ in the absolute c.g.s. system. Show that $k^2 = 1.066 \times 10^{-9}$ in the absolute f.p.s. system.

3. Describe the lines of force and the equipotential surfaces of the field of force due to two particles of equal mass. Illustrate with a suitable figure.

4. Describe the lines of force and the equipotential surfaces of the field of force due to two particles having masses numerically equal but opposite in sign. Illustrate with a suitable figure.

5. Show that the potential $U = m/p$ at P due to a particle of mass m at O is the work done by the attraction of this particle in bringing a unit particle from an infinite distance to P. Extend this to fields of force due to continuous distributions of particles over a bounded volume, area or curve.

6. Express the flux through a surface S of the force due to a single attracting particle at O in terms of the area ω subtended at O by S on the unit sphere with O as center. This area ω is known as the *solid angle* subtended by S at O.

7. A field of force is said to be *central* if the force at any point P is always directed along the line through that point and a fixed point O called the *center of force* and if the amount and sense of the force depend only on the distance p of P from O. Show that the only field which is both central and solenoidal except at O is that due to the Newtonian force of a particle at O.

8. A field of force is said to be axial if the force at any point P is always directed along the perpendicular from P to a fixed axis L called the *axis of force* and if the amount and sense of the force depend only on the distance r of P from L. If the field is both axial and solenoidal except on L, find the law of force.

9. If S is a closed surface enclosing any finite number of discrete doublets, show that the flux through S of the force due to these doublets is zero.

10. If S is a closed surface enclosing another surface S' on which is a double layer, show that the flux through S of the force due to this double layer is zero.

11. Show that the scalar potential at a point P due to a double layer of density μ on a surface S not passing through P may be written in the form,

$$U \quad \iint_{S'} \mu' \, d\sigma,$$

where S' is that portion of a unit sphere with P as center which is subtended at P by S and where μ' takes on the same value at each point of Q' of S' that μ does at the point Q on S which is collinear with P and Q'.

12. By the aid of Problem 11 show that the potential at P due to a double layer of constant density μ has a sudden decrease of $4\pi\mu$ as P passes through the double layer in the direction and sense of the positive normal.

13. Show that the scalar potential of a double layer of constant density μ on a surface S has the value at any point P not on S of the negative of the flux through S in the sense of the layer, of the force due to a particle of mass μ at P.

14. Discuss the field of force *in a given plane* due to a particle of mass m in that plane attracting with a force inversely proportional to the *first* power of the distance r. If the force be regarded as remaining constant along every line perpendicular to the plane, show that the force field is both lamellar and solenoidal and that the scalar potential may be taken as $U = - m \log r$ and the vector potential as $\mathbf{W} = - m\theta \, \mathbf{u}$ where $\mathbf{u}$ is a unit vector perpendicular to the plane and θ is the angle which the line from the attracting particle to the point considered in the plane makes with any fixed line in the plane in the positive sense of rotation relative to $\mathbf{u}$. Discuss the flux of force through a closed curve either enclosing or not enclosing the particle. Express as integrals the force and the scalar potential due to continuous distributions of such *logarithmic particles* on curves and areas in the plane.

108. Force and Potential of Special Distributions.

We developed in the preceding article formulas for the force and scalar potential at a point P due to various distributions of attracting matter and doublets. There exists in the literature a very great number of applications of these formulas to special distributions of particular simplicity or interest. In most cases these applications are easily carried out by choosing a suitable coördinate system and determining the coördinates of the force and the value of the potential by the usual methods of the integral calculus. When so carried out the result may be of interest but

the procedure itself usually gives little information concerning the nature of the problem. Exercises of this sort are provided at the end of this article.

It occasionally happens that the force or potential due to a distribution may be directly determined without recourse to a coördinate system. We present some simple examples illustrating this method. Let us determine the force **f** at a point P due to a uniform distribution of attracting matter in a spherical shell bounded by two concentric spheres with centers at O and inner and outer radii a and b respectively. The point P may be either inside the inner sphere or outside the outer sphere and in each of these regions we know that the force **f** will be solenoidal. Due to the symmetry of the distribution we may evidently write,

$$\text{(1)} \qquad \mathbf{f} = -\,\varphi(p)\mathbf{p}, \qquad p = |\mathbf{p}|,$$

Then since **f** is solenoidal in the two regions considered we have,

$$0 = \nabla\cdot\mathbf{f} = -\,\varphi\,\nabla\cdot\mathbf{p} - \nabla\varphi\cdot\mathbf{p} = -\,3\varphi - p\,\frac{d\varphi}{dp}$$

or

$$\frac{3dp}{p} + \frac{d\varphi}{\varphi} = 0.$$

This differential equation integrates into $\varphi p^3 = c$ (const.) and we have,

$$\text{(2)} \qquad \mathbf{f} = -\,c\,\frac{\mathbf{p}}{p^3},$$

in which it only remains to determine the constant c for each region. From the symmetry of the distribution it is evident that **f** vanishes at the center O and this requires that c be zero. Thus for the interior region we have,

$$\text{(3)} \qquad \mathbf{f} = 0 \qquad \text{for} \qquad p < a.$$

From the law of the mean for integrals it is evident that the amount of the force at any exterior point will be less than if the entire mass of the shell were concentrated at the point of it which is nearest to P and greater than if the entire mass were concentrated at the point of its farthest from P. Thus we have,

$$\frac{m_0}{(p+b)^2} < \frac{c}{p^2} < \frac{m_0}{(p-b)^2},$$

where m_0 is the total mass of the shell. Hence,

$$m_0\left(\frac{p}{p+b}\right)^2 < c < m_0\left(\frac{p}{p-b}\right)^2,$$

and remembering that c is constant and that these inequalities hold for arbitrarily large values of p, it becomes apparent that $c = m_0$ and we have,

$$\text{(4)} \qquad \mathbf{f} = -\, m_0 \frac{\mathbf{p}}{p^3} \qquad \text{for} \qquad p > b.$$

The results (3) and (4) obviously still hold if the density ρ of the shell is merely a function $\rho(q)$ of the distance q from O without ρ necessarily being constant or even continuous, and the result (4) also holds for a solid sphere, this being the case $a = 0$. We may express this last fact by the theorem,

A sphere which is homogeneous in concentric layers attracts any exterior particle as if the entire mass of the sphere were concentrated at its center.

This theorem, of the greatest importance in astronomy, was originally proven by Sir Isaac Newton.

As a second illustration we may consider the attraction of a slender straight rod AB of constant linear density ρ at a point P which is at a distance c from the axis of the rod. If Q is any point of the rod we call $PQ = \mathbf{r}$ and by formula § 107, (10) we have,

$$\text{(5)} \qquad \mathbf{f} = \int_{AB} \frac{\rho\, \mathbf{r}}{r^3}\, d\gamma.$$

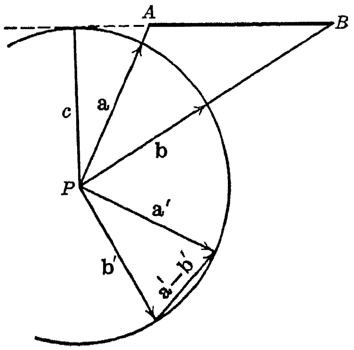

Fig. 95

We now indicate by $\mathbf{u}$ a vector having the direction and sense of $\mathbf{r}$ and the constant length c and by θ the angle which this vector makes in a given sense with the perpendicular from P upon the axis of the rod. We also indicate by $\mathbf{a}$, α and $\mathbf{b}$, β the values of $\mathbf{u}$, θ when Q is at A and B respectively. Then evidently,

$$\mathbf{r} = \mathbf{u} \sec\theta, \qquad r = c \sec\theta, \qquad d\gamma = r \sec\theta\, d\theta,$$

and f reduces to,

$$(6) \qquad \mathbf{f} = \int_\alpha^\beta \frac{\rho}{c^2}\,\mathbf{u}\,d\theta = \int_\alpha^\beta \rho\,\frac{\mathbf{u}}{c^3}\,c\,d\theta.$$

A little consideration of this last form yields the interesting conclusion that the force is exactly the same as that which would be produced at P by a piece of the rod curved along the arc of the circle of radius c and center P included between **a** and **b**. To evaluate the above integral we first observe that the derivative $\mathbf{u}'$ of **u** with respect to θ is the value of **u** for θ increased by $\frac{\pi}{2}$ and that consequently **u** is the derivative of the value of **u** for θ diminished by $\pi/2$, i.e. **u** is the derivative of $-\mathbf{u}'$. Thus we have,

$$(7) \qquad \mathbf{f} = \frac{\rho}{c^2}\int_\alpha^\beta \mathbf{u}\,d\theta = \frac{\rho}{c^2}\,(\mathbf{a}' - \mathbf{b}').$$

The force **f** is thus determined and evidently lies along the bisector of the angle between **a** and **b**. We may readily transform the above formula into the explicit form,

$$(8) \qquad \mathbf{f} = \frac{\rho}{c^2}\,\frac{(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}.$$

If the point P is held fixed while the wire is extended indefinitely in both directions formula (7) becomes,

$$(9) \qquad \mathbf{f} = \frac{2\rho}{c}\,\mathbf{i},$$

where **i** is the unit vector running from P perpendicularly toward the wire. Thus as P moves the amount of the force due to an infinitely long wire varies inversely as the distance c of P from the wire.

Since the scalar potential U of a Newtonian force field is a *scalar*-point function its determination for a given distribution of attracting matter is ordinarily less difficult than the direct determination of the force, which is a vector-point function. And of course the potential of the distribution having been found, the force follows at once as the gradient of the potential. The importance of the scalar potential of the force field lies, however, not

only in its simplicity but also in the fact that it has a physical significance as the negative of the potential energy of the mass distribution, or in other words, as the work done by the attracting forces in bringing the particles of the mass distribution to their actual position from a state of being scattered at infinite distances from each other. We can not, however, here go further into this aspect of the subject.

We conclude this section by the determination of the potential at any point P of a wire of constant linear density ρ having the form of a circle of radius a. Let c be the distance from P to the plane of the circle and d the distance from P to the line perpendicular to the plane of the circle at its center, O. Let Q be any point on the circle and θ the angle which the radius at Q makes with the radius at N, where N is the point of the circle nearest to P. Then evidently,

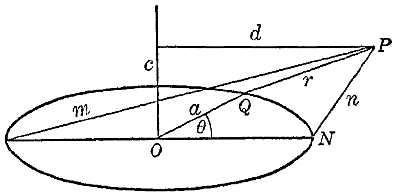

FIG. 96

$$\begin{aligned} PQ^2 &= r^2 \\ &= (d - a\cos\theta)^2 + (a\sin\theta)^2 + c^2 \\ &= a^2 + c^2 + d^2 - 2ad\cos\theta \end{aligned}$$

and the greatest and least values of r^2 are,

$$m^2 = a^2 + c^2 + d^2 + 2ad, \qquad n^2 = a^2 + c^2 + d^2 - 2ad,$$

so that we have,

$$r^2 = \frac{m^2 + n^2}{2} - \frac{m^2 - n^2}{2}\cos\theta = n^2\cos^2\theta/2 + m^2\sin^2\theta/2.$$

If we introduce an angle $\varphi = (\pi - \theta)/2$ we may write the potential at P due to the wire as,

$$(10) \qquad U = 2a\rho\int_0^{\pi}\frac{d\theta}{r} = 4a\rho\int_0^{\pi/2}\frac{d\varphi}{\sqrt{m^2\cos^2\varphi + n^2\sin^2\varphi}}.$$

The problem is thus solved except for the evaluation of this definite integral. This is an elliptic integral and can not be evaluated in closed form in terms of the elementary functions

but its value may be obtained in a variety of other ways. For example we may write,

$$(11)\quad U = \frac{4a\rho}{m} K(k), \qquad \text{where} \qquad K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

$$k^2 = 1 - \frac{n^2}{m^2},$$

and the quantity K here introduced is a well known tabulated function known as Legendre's complete elliptic integral of the first kind.*

The integral (10) may also be easily evaluated by a very ingenious device due to Gauss. We introduce a new variable of integration ψ defined by *Landen's transformation*,

$$(m + n) \sin (2\varphi - \psi) = (m - n) \sin \psi,$$

taking the determination of ψ which is zero when φ is zero, and after some trigonometric reduction find,

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{m^2 \cos^2 \varphi + n^2 \sin^2 \varphi}} = \frac{1}{2} \int_0^{\pi} \frac{d\psi}{\sqrt{\left(\frac{m+n}{2}\right)^2 \cos^2 \psi + mn \sin^2 \psi}}.$$

The last integrand evidently runs through the same set of values in the second quadrant that it does in the first so that we may drop the factor 1/2 and make the limits of integration 0 and $\pi/2$. If then we indicate by m_1 and n_1 respectively the arithmetic and geometric means of m and n,

$$m_1 = \frac{m + n}{2}, \qquad n_1 = \sqrt{mn},$$

we have,

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{m^2 \cos^2 \varphi + n^2 \sin^2 \varphi}} = \int_0^{\pi/2} \frac{d\psi}{\sqrt{m_1^2 \cos^2 \psi + n_1^2 \sin^2 \psi}}$$

or, in other words, the integral remains unchanged when m and n are replaced respectively by their arithmetic and geometric means. This substitution may be repeated as many times as desired, the successive arithmetic means $m_1, m_2, m_3, \cdots$, forming

* See page 128.

a diminishing sequence and the successive geometric means $n_1, n_2, n_3, \cdots$, forming an increasing sequence. It is not difficult to show that these two sequences converge to a common limit α, so that we have in the limit,

$$(12)\quad \int_0^{\pi/2} \frac{d\varphi}{\sqrt{m^2 \cos^2 \varphi + n^2 \sin^2 \varphi}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{\alpha^2 \cos^2 \varphi + \alpha^2 \sin^2 \varphi}} = \frac{\pi}{2\alpha}.$$

Thus we have finally,

$$(13)\quad U = \frac{2\pi a \rho}{\alpha} = \frac{M}{\alpha},$$

where M is the total mass of the circular wire. The value of α is readily computed to any desired degree of accuracy and then formula (13) gives the potential U to this same percentage of accuracy.

EXERCISES

Particles

1. If any three particles be placed at the vertices of an equilateral triangle, show that the attraction of any two of them upon the third is always directed through the centroid of the three particles.

2. Let three equal particles be placed at the vertices of an equilateral triangle and construct several of the equipotential curves in the plane of the triangle. Thus locate positions of equilibrium of an attracted particle other than the obvious one at the center of the triangle.

Curves

3. Find the amount of the attraction of a rod AB of constant linear density ρ and length l at a point P which is a units from A and b units from B. Find the point M of the rod towards which the attraction is directed. Evaluate for $\rho = 3$, $l = 4$, $a = 1$, $b = 4$.
Ans. $f = 4$, $AM = 4/5$

4. In the rod of Problem 3 show that the force becomes infinite as P approaches a point of the rod and that if P is on the line of the rod extended, the amount of the force is given by,
$$f = \pm \rho \left(\frac{1}{a} - \frac{1}{b}\right)$$

5. If a uniform rod is bent into the form of a triangle show that the force at the center of the inscribed circle is zero. Extend this result to any plane polygon which can be circumscribed about a circle.

6. Find the amount of the attraction of a circular wire of constant linear density ρ and radius a at a point P on the perpendicular to the plane of the circle at its center and c units from this center.

$$\textit{Ans.}\quad f = \frac{2\pi\rho ac}{(a^2 + c^2)^{3/2}}$$

7. Find the attraction at the center of a circle of radius a of a uniform wire lying along an arc of the circle of central angle 2α. Show that this attraction equals that of a particle placed at the mid-point of the arc and having the mass of a piece of the wire as long as the chord of the arc.

8. Three similar uniform rods of infinite length lie in the same plane. Show that a particle will be in equilibrium under the attraction of the rods if placed at the point of intersection of the medians of the triangle formed by them.

9. Find the potential of a rod AB of constant linear density ρ and length l at a point P which is a units from A and b units from B. Show that the equipotential surfaces are ellipsoids of revolution with A and B at the foci of the meridian sections. Show that as the rod becomes infinitely long the potential at P becomes infinite but that the *difference of potential* between two points P_0 and P approaches a definite limit depending only on the distances c_0 and c of P_0 and P from the rod.

$$\textit{Ans.}\quad U = \rho \log\left(\frac{a+b+l}{a+b-l}\right), \qquad U - U_0 = 2\rho \log\left(\frac{c_0}{c}\right)$$

10. Verify the results of Problem 3 by finding the gradient of U as determined in Problem 9.

11. Compute the potential of a circular wire of radius 5.5 and unit density at a point P which is 12 units from the plane of the circle and 10.5 units from the perpendicular to the plane of the circle at its center. Use two or more methods of evaluating the integral.

$$\textit{Ans.}\quad U = 2.11857$$

12. If m and n are two positive numbers, $m > n$, having an arithmetic mean m_1 and a geometric mean n_1 and if m_2 and n_2 are similarly the arithmetic and geometric means of m_1 and n_1, etc., show that the sequence $m, m_1, m_2, m_3, \cdots$ is decreasing and the sequence $n, n_1, n_2, n_3, \cdots$ is increasing and that they approach a common limit α.

Surfaces

13. Find the amount of the attraction of a circular disc of constant areal density ρ and radius a at a point P on the line perpendicular to the disc at its center and c units from it. What is the limit of this as the radius becomes infinite? What is the change in the attraction as P crosses the disc? *Ans.* $f = 2\pi\rho(1 - c/\sqrt{a^2 + c^2})$

14. Determine the potential of the disc of Problem 13 at the given point P and verify the result of Problem 13 by finding the force as the gradient of this potential. *Ans.* $U = 2\pi\rho(\sqrt{a^2 + c^2} - c)$

15. Find the amount of the attraction of a rectangular lamina of constant areal density ρ and dimensions $2a$, $2b$ at a point P on the perpendicular to the lamina at its center and c units from it. What is the limit of this as both dimensions of the lamina become infinite? What is the change in the attraction as P crosses the lamina?

$$\textit{Ans.}\quad f = 4\rho \text{ arc sin} \left(\frac{ab}{\sqrt{(a^2 + c^2)(b^2 + c^2)}} \right)$$

16. Prove *Playfair's Theorem* that the normal component of the attraction of a homogeneous plane lamina at any point P equals in amount the product $\rho\omega$ of the density ρ and the solid angle ω subtended at P by the lamina.

17. Prove that the attraction of any plane lamina on any point P is the same as that of the projection of this lamina from P on the spherical surface having the normal from P to the lamina as a diameter; the densities at corresponding points being equal.

18. Find the amount of the attraction of a spherical cap of constant areal density ρ at a point P of its axis on the concave side. Let a be the radius of the sphere and c be the distance from P to the center of the sphere, while r is the distance from P to the edge of the cap. Find the amount of the above attraction when P is on the convex side of the cap. Evaluate both results for the entire spherical shell. How much does the attraction change as P crosses the cap?

$$\textit{Ans.}\quad f = \frac{\pi\rho a}{c^2 r}(c + a - r)(c + r - a) \quad (c \text{ is positive or negative})$$

19. Determine the potential of the spherical cap of Problem 18 at the given point P and verify the results of Problem 18 by finding the force as the gradient of the potential. Evaluate the results for the entire spherical shell.

$$\textit{Ans.}\quad U = \frac{2\pi\rho a}{c}(r - |c - a|) \quad (c \text{ is positive or negative})$$

20. Find the amount of the attraction of a spherical cap of constant areal density ρ at the center of the sphere. Let 2α be the central angle of the cap. Show that the result is a special case of that of Problem 18. *Ans.* $f = \pi\rho \sin^2 \alpha$

Solids

21. Find the amount of the attraction of a spherical segment of constant density ρ and height h cut from a sphere of radius a at the point P at the vertex of the segment.

$$\textit{Ans.}\quad f = 2\pi\rho h\left(1 - \frac{1}{3}\sqrt{\frac{2h}{a}}\right)$$

22. Find the amount of the attraction of a right circular cylinder of constant density ρ and altitude h at a point P on the extension of the axis of the cylinder at distances a and b from the nearer and farther edges of the cylinder, respectively.

Ans. $f = 2\pi\rho(h + a - b)$

23. What must be the ratio of the radius of the base to the altitude of a homogeneous right circular cylinder of given volume in order that it exert the maximum attraction at the center of one of the bases?

Ans. $a/h = \frac{1}{8}(9 - \sqrt{17}) = .60961$

24. Find the amount of the attraction at the focus of a paraboloid of revolution of constant density ρ bounded by a plane perpendicular to the axis at a distance $b \leq a$ from the vertex; a being the distance from the vertex to the focus.

Ans. $f = 4\pi\rho\{b - a\log(1 + b/a)\}$

25. The space within a closed surface S is filled with matter of constant density ρ. Show that its potential at an external point P is given by,

$$U = \frac{\rho}{2}\iint\limits_S \frac{\mathbf{n}\cdot\mathbf{r}}{r}\,d\sigma,$$

where $\mathbf{r}$ is the vector from P to any point of S and $\mathbf{n}$ is the unit outward normal vector to S.

109. Force Fields in the Attracting Bodies.

In the first article of this chapter we developed several properties of the field of force due to various distributions of attracting matter and doublets, confining our attention to the nature of the field at points neither in nor on the attracting bodies. In this article we shall develop some properties of the force field and its scalar potential with special reference to their nature at points of the attracting bodies.

Volume Distributions

The force $\mathbf{f}$ and its scalar potential U at a point P due to a distribution of attracting matter over a volume T have been seen to be given by the integrals,

$$(1)\quad \mathbf{f} = \iiint\limits_T \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\tau, \quad U = \iiint\limits_T \frac{\rho(\mathbf{q})}{r}\,d\tau; \quad (\mathbf{r} = \mathbf{q} - \mathbf{p})$$

provided the point P is exterior to T and provided that the density ρ is piecewise continuous throughout T. If P is in or on T these integrals fail to exist and these formulas are meaningless because

the quantity r appearing in the denominator takes on the value zero within the region of integration. Both f and U are thus as yet undefined for points within or on T, but we shall not find it difficult to formulate a new definition of these quantities which is applicable to all positions of P, which includes the above as a special case, and which has been found to be consistent with the physical facts to a high degree of approximation. We construct a variable region T_0 having P as an interior point and δ as its largest dimension and we indicate by $T - T_0$ the region interior to T and exterior to T_0. The point P is then always an exterior point for the region $T - T_0$ and the integrals (1) are defined for this region. We now define f and U at any point P as the limits,

$$(2)\quad \mathbf{f} = \lim_{\delta \to 0} \iiint\limits_{T-T_0} \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\tau, \qquad U = \lim_{\delta \to 0} \iiint\limits_{T-T_0} \frac{\rho(\mathbf{q})}{r}\,d\tau.$$

The region T_0 is here allowed to vary in any way provided P is kept an interior point and provided the largest dimension δ approaches the limit zero. We continue to assume that ρ is piecewise continuous in T.

If P is outside T the region $T - T_0$ coincides with the region T as soon as δ becomes equal to or less than the least distance from P to T and the values of f and U as given by formulas (2) are equal to those given by formulas (1). It remains to prove that the limits in formulas (2) exist when P is in or on T. We use here a very general test for the existence of a limit due to Cauchy which in our case may be stated as follows.

The integrals,

$$\iiint\limits_{T-T_0} \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\tau, \qquad \iiint\limits_{T-T_0} \frac{\rho(\mathbf{q})}{r}\,d\tau,$$

will approach limits as δ approaches zero if corresponding to every arbitrarily chosen positive number ϵ there exists a positive number δ such that,

$$(3)\quad \left| \iiint\limits_{T-T_0} \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\tau - \iiint\limits_{T-T_0'} \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\tau \right| < \epsilon,$$

$$\left| \iiint\limits_{T-T_0} \frac{\rho(\mathbf{q})}{r}\,d\tau - \iiint\limits_{T-T_0'} \frac{\rho(\mathbf{q})}{r}\,d\tau \right| < \epsilon,$$

where T_0 and T'_0 are any two regular regions with P as an interior point and of maximum dimension not greater than δ.

To show that these conditions are satisfied we construct a sphere S with P as center and radius δ and another sphere S_0 with the same center P and radius $\delta_0 < \delta$ where δ_0 is not greater than the least distance from P to the regions $T - T_0$ and $T - T'_0$. Then the two regions $T - T_0$ and $T - T'_0$ differ only by regions lying wholly within the region $S - S_0$ and so conditions (3) will be satisfied if we can show that,

$$\iiint\limits_{S-S_0} \left|\frac{\rho}{r^3}\mathbf{r}\right| d\tau < \epsilon, \qquad \left|\iiint\limits_{S-S_0} \frac{\rho}{r}\, d\tau\right| < \epsilon$$

for δ sufficiently small and for every choice of δ_0. The density ρ being piecewise continuous, must have an upper bound k throughout T and we have,

$$\begin{aligned}
&\iiint\limits_{S-S_0} \left|\frac{\rho}{r^3}\mathbf{r}\right| d\tau \leq k \iiint\limits_{S-S_0} \frac{d\tau}{r^2} = k \int_{\delta_0}^{\delta} \frac{4\pi r^2\, dr}{r^2} \\
(4) \qquad &\qquad = 4\pi k(\delta - \delta_0) < 4\pi k\delta, \\
&\left|\iiint\limits_{S-S_0} \frac{\rho}{r}\, d\tau\right| \leq k \iiint\limits_{S-S_0} \frac{d\tau}{r} = k \int_{\delta_0}^{\delta} \frac{4\pi r^2\, dr}{r} \\
&\qquad = 2\pi k(\delta^2 - \delta_0^2) < 2\pi k\delta^2,
\end{aligned}$$

and it is obviously possible to choose δ so as to make these last members less than ϵ and thus satisfy conditions (3).

It being thus established that the limits (2) exist, we find it convenient to indicate their values by the same symbol employed in (1) for the case where P is exterior to the region T, but it must be borne in mind that when P is an interior or boundary point of T the integrals (1) are not definite integrals in the proper sense but are so called *improper integrals*. The fact that these limits exist is expressed by saying that these improper integrals are *convergent*.

By the aid of inequalities (4) we may now prove a lemma of value to us in the further discussion of the interior properties of the force and potential.

Lemma I. If T is any region having a volume equal to that of

a sphere of radius δ, then the integrals,

$$(5) \qquad \underset{T}{\iiint} \frac{\rho}{r^3}\mathbf{r}\,d\tau, \qquad \iiint \frac{\rho}{r}\,d\tau$$

will be less in absolute value than any given positive number ϵ for every choice of δ sufficiently small, and the choice of δ is independent of the shape and position of the region T and the position of the point P.

To establish this we construct a sphere S with the above radius δ and center at the point P, and let R be the region, if any, which T and S have in common. Then throughout $T - R$ we have $r \geqq \delta$ and throughout $S - R$ we have $r \leqq \delta$, but the regions $T - R$ and $S - R$ have the same volume $(S - R)$. Bearing in mind inequalities (4) we may now write,

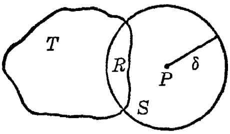

FIG. 97

$$\left| \underset{T}{\iiint} \frac{\rho}{r^3}\mathbf{r}\,d\tau \right| \leqq k \underset{T}{\iiint} \frac{d\tau}{r^2} \leqq k \underset{R}{\iiint} \frac{d\tau}{r^2} + \frac{k(S - R)}{\delta^2}$$

$$(6) \qquad \leqq k \underset{S}{\iiint} \frac{d\tau}{r^2} \leqq 4\pi k\delta,$$

$$\left| \underset{T}{\iiint} \frac{\rho}{r}\,d\tau \right| \leqq k \underset{T}{\iiint} \frac{d\tau}{r^2} \leqq k \underset{R}{\iiint} \frac{d\tau}{r^2} + \frac{k(S - R)}{\delta}$$

$$\leqq k \underset{S}{\iiint} \frac{d\tau}{r} \leqq 2\pi k\delta^2,$$

and it is obviously possible to choose δ so as to make these last members less than ϵ and thus establish the lemma.

With the aid of this lemma we now show that the functions (1) are *continuous* at each point P even if P is an interior or boundary point of T. We have here to establish that if P_0 and P are any two points with radius vectors $\mathbf{p}_0$ and $\mathbf{p}_0 + \Delta\mathbf{p}$ and if ϵ is any given positive number, then the inequalities,

$$(7) \quad |\mathbf{f}(\mathbf{p}_0 + \Delta\mathbf{p}) - f(\mathbf{p}_0)| < \epsilon, \qquad |U(\mathbf{p}_0 + \Delta\mathbf{p}) - U(\mathbf{p}_0)| < \epsilon,$$

hold whenever $|\Delta\mathbf{p}|$ is less than some suitably chosen number δ. To do this let S be a sphere of radius δ_1 and center P_0 and break

the functions $\mathbf{f}$ and U up into two functions each,

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2, \qquad U = U_1 + U_2,$$

where $\mathbf{f}_1$ and U_1 are given by the integrals (1) over $T - S$ and $\mathbf{f}_2$ and U_2 by these integrals over S. By the lemma we may choose δ_1 so small that the inequalities,

$$|\mathbf{f}_2| < \epsilon/3, \qquad |U_2| < \epsilon/3$$

hold for all positions of the point P. With δ_1 thus chosen the sphere S is fixed and as long as $|\Delta\mathbf{p}|$ remains less than δ_1, the functions $\mathbf{f}_1$ and U_1 are given by proper integrals of continuous functions over a fixed region $T - S$. They are therefore continuous functions and we may find a number δ_2 such that,

$$|\mathbf{f}_1(\mathbf{p}_0 + \Delta\mathbf{p}) - \mathbf{f}_1(\mathbf{p}_0)| < \epsilon/3, \qquad |U_1(\mathbf{p}_0 + \Delta\mathbf{p}) - U_1(\mathbf{p}_0)| < \epsilon/3,$$

whenever $|\Delta\mathbf{p}|$ is less than δ_2. If we now choose δ as the smaller of the two numbers δ_1 and δ_2, both of the last two sets of inequalities hold and inequalities (7) follow at once provided only that $|\Delta\mathbf{p}| < \delta$. The continuity of $\mathbf{f}$ and U is thus established.

It remains to be seen that the relation,

$$\nabla U = \mathbf{f}, \tag{8}$$

which we know holds for P an exterior point of T also holds when P is in or on T. On referring to the definition of the gradient ∇U of U, § 99 (1), we see that to establish that $\mathbf{f}(\mathbf{p}_0)$ is the gradient of U at P_0 we must show that there exists a vector $\boldsymbol{\varepsilon}$ satisfying the conditions,

$$\{\mathbf{f}(\mathbf{p}_0) + \boldsymbol{\varepsilon}\}\cdot\Delta\mathbf{p} = U(\mathbf{p}_0 + \Delta\mathbf{p}) - U(\mathbf{p}_0), \qquad \lim_{\Delta\mathbf{p}\to 0} \boldsymbol{\varepsilon} = 0.$$

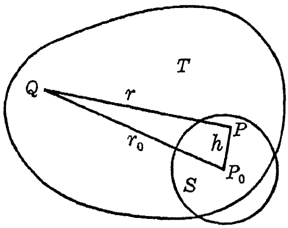

FIG. 98

We accordingly set up the quantity,

$$U(\mathbf{p}_0 + \Delta\mathbf{p}) - U(\mathbf{p}_0) - \mathbf{f}(\mathbf{p}_0)\cdot\Delta\mathbf{p} = \iiint_T \rho\left\{\frac{1}{r} - \frac{1}{r_0} - \frac{\mathbf{r}_0\cdot\Delta\mathbf{p}}{r_0^3}\right\} d\tau,$$

where the integral, being the sum of three convergent integrals, is of course convergent; and we observe that formula (8) will be established if we can show that this

integral can be written in the form,

$$\iiint_T \rho\left\{\frac{1}{r}-\frac{1}{r_0}-\frac{\mathbf{r}_0\cdot\Delta\mathbf{p}}{r_0^3}\right\}d\tau=|\Delta\mathbf{p}|\,\epsilon \qquad \text{where} \qquad \lim_{\Delta\mathbf{p}\to 0}\epsilon=0.$$

For if this can be done then the above vector $\boldsymbol{\varepsilon}$ may be chosen with the direction and sense of $\Delta\mathbf{p}$ and the length ϵ. Setting $\Delta\mathbf{p}=h\mathbf{u}$, where $\mathbf{u}$ is a unit vector and h is positive, we see that ϵ may be written as,

$$\epsilon=\iiint_T \frac{\rho}{h}\left\{\frac{1}{r}-\frac{1}{r_0}-\frac{\mathbf{r}\cdot\Delta\mathbf{p}}{r_0^3}\right\}d\tau$$

$$=\iiint_T \rho\left\{\frac{1}{r_0 r}\left(\frac{r_0-r}{h}\right)-\frac{\mathbf{r}_0\cdot\mathbf{u}}{r^3}\right\}d\tau.$$

To prove that this approaches zero with h we draw a sphere S with center P_0 and radius δ. Then for $h\leq\delta$ we have from the facts that,

$$\left|\frac{r_0-r}{h}\right|\leq 1, \qquad \left|\frac{\mathbf{r}_0\cdot\mathbf{u}}{r_0}\right|\leq 1, \qquad |\rho|\leq k, \qquad \frac{2}{r_0 r}\leq\frac{1}{r_0^2}+\frac{1}{r^2},$$

the result,

$$\left|\iiint \rho\left\{\frac{1}{r_0 r}\quad r_0-r \quad -\frac{\mathbf{r}_0\cdot\mathbf{u}}{r_0^3}\right\}d\tau\right.$$

$$\leq k\iiint_S\left(\frac{1}{r_0 r}+\frac{1}{r_0^2}\right)d\tau\leq\frac{k}{2}\iiint_S\left(\frac{1}{r^2}+\frac{3}{r_0^2}\right)d\tau\leq 8\pi k\delta,$$

where the last inequality follows from the first of inequalities (6). We may thus choose δ so small as to make the integral over S arbitrarily small in absolute value; and S being now fixed, the integral of the same quantity over $T-S$ may also obviously be made arbitrarily small in absolute value by choosing h sufficiently small, for this integral is for $h<\delta$ a proper integral of a continuous function and is zero for $h=0$. To see this last point we observe that by the cosine law $r^2=r_0^2+h^2-2h\,\mathbf{r}_0\cdot\mathbf{u}$ and hence that,

$$\frac{1}{r_0 r}\left(\frac{r_0-r}{h}\right)-\frac{\mathbf{r}_0\cdot\mathbf{u}}{r^3}=\frac{1}{r_0 r}\left(\frac{2\,\mathbf{r}_0\cdot\mathbf{u}-h}{r_0+r}\right)-\frac{\mathbf{r}_0\cdot\mathbf{u}}{r_0^3}.$$

For $h = 0$ we have $r = r_0$ and this quantity vanishes. The relation (7) is thus established. This of course carries with it the fact that the curl of $\mathbf{f}$ is zero wherever it exists and is continuous; i.e. $\mathbf{f}$ is everywhere lamellar.

To sum up the results obtained we may state the theorem,

The force $\mathbf{f}$ *and the potential* U *due to a distribution of piecewise continuous density over a regular region of space are given by the integrals* (1) *which are either proper integrals or convergent improper integrals. The functions* $\mathbf{f}$ *and* U *are everywhere continuous and satisfy the relation,* $\nabla U = \mathbf{f}$.

We established in § 107 the result,

$$\iint \mathbf{n}\cdot\mathbf{f}\, d\sigma = -\,4\pi m_0, \tag{9}$$

stating that the outward flux of force through a closed surface S due to a continuous distribution of matter wholly within S is $-\,4\pi$ times the total mass of the attracting bodies, while in case the attracting matter is wholly outside S the resulting flux is zero. We may now show that,

If the surface S *is any closed regular surface and* $\mathbf{f}$ *is the force due to attracting matter distributed with a piecewise continuous density over any volume* T, *then the above equation still holds provided we understand by* m_0 *the mass of this attracting matter lying within* S.

We divide the region T into three regions T', T_1, T_2 where T' is that portion of T lying within a distance δ of the surface S, T_1 is that portion of T lying outside of S and at a distance greater than δ from S, and T_2 is that portion of T lying inside of S and at a distance greater than δ from S. We indicate by $\mathbf{f}'$, $\mathbf{f}_1$, $\mathbf{f}_2$ the portions of $\mathbf{f}$ obtained by integrating over these respective regions. Then since T_1 is wholly exterior to S and T_2 is wholly interior, we have, as previously established,

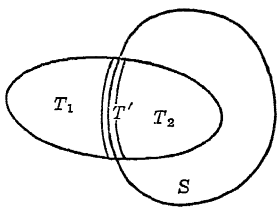

FIG. 99

$$\iint_S \mathbf{n}\cdot\mathbf{f}_1\, d\sigma = 0, \qquad \iint_S \mathbf{n}\cdot\mathbf{f}_2\, d\sigma = -\,4\pi \iiint_{T_2} \rho\, d\tau$$

and hence, since $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}'$, we have,

$$\iint_S \mathbf{n}\cdot\mathbf{f}\,d\sigma = 0 - 4\pi \iiint_{T_2} \rho\,d\tau + \iint_S \mathbf{n}\cdot\mathbf{f}'\,d\sigma.$$

On proceeding to the limit allowing δ to approach zero, the volume of T' will approach zero and $\mathbf{f}'$ approaches zero uniformly at all points of space, as proven in our lemma. The last integral in the above equation thus vanishes, while the region T_2 becomes the entire portion of T interior to S. We thus find equation (9) established, where m_0 is the total mass of matter distributed over the volume of T interior to S.

We saw in the first article of this chapter that the force field due to a volume distribution of attracting matter with piecewise continuous density is not only lamellar but solenoidal at all exterior points. Thus we have,

$$(10) \qquad \nabla\cdot\mathbf{f} = \nabla\cdot\nabla U = \nabla^2 U = 0.$$

The reader will recall (§ 103) that $\nabla^2 U$ is known as the Laplacian of U; that the differential equation $\nabla^2 U = 0$ is called Laplace's Equation and that any solution of Laplace's equation is said to be an harmonic function. Thus the potential due to any volume distribution with piecewise continuous density is harmonic at all exterior points. We are now to find that this property, unlike those previously discussed in this article, can not be extended unaltered to points within or on the attracting body. In fact if the density ρ of the distribution of matter over the volume T is merely continuous the resulting force $\mathbf{f}$ may not even possess a divergence at interior points. We may however show that,

If the density ρ of a distribution of attracting matter over a regular volume T possesses a continuous gradient at all interior points of T, then the resulting force $\mathbf{f}$ possesses continuous first derivatives in all directions at all interior points.

It should be noted that the theorem says nothing concerning the boundary points, and in fact $\mathbf{f}$ will not ordinarily possess derivatives there. To prove the theorem we observe that,

$$\mathbf{f} = -\iiint \rho\nabla_Q \frac{1}{r}\,d\tau = -\iiint \nabla_Q\left(\frac{\rho}{r}\right)d\tau$$

$$+ \iiint \frac{\nabla_Q \rho}{r}\,d\tau,$$

where the subscript Q on ∇ indicates that the gradient is taken with respect to the point of integration Q. Since the region T is regular we have by the gradient theorem,

$$\iiint \qquad \frac{\rho}{\ } \Big) \, d\tau = \iint \frac{\rho}{r} \mathbf{n} \, d\sigma,$$

where S is the regular surface bounding T and $\mathbf{n}$ is the unit outward normal to S. If we also resolve the gradient $\nabla_Q \rho$ in the directions of any three mutually perpendicular constant unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we have $\mathbf{f}$ written in the form,

$$\mathbf{f} = -\iint \frac{\rho}{r} \mathbf{n} \, d\sigma + \mathbf{i} \iiint \frac{\mathbf{i} \cdot \nabla_Q \rho}{r} \, d\tau$$

$$+ \mathbf{j} \iiint \frac{\mathbf{j} \cdot \nabla_Q \rho}{r} \, d\tau + \mathbf{k} \iiint \frac{\mathbf{k} \cdot \nabla_Q \rho}{r} \, d\tau.$$

The first integral is a proper integral for P at interior points of T and possesses continuous derivatives of all orders with respect to P, while the last three integrals are of the type yielding the potential of a volume distribution and have been seen to possess continuous first derivatives. Thus the force possesses a continuous divergence $\nabla \cdot \mathbf{f}$ at interior points of T and the potential U possesses a continuous Laplacian $\nabla^2 U$. Although this quantity $\nabla \cdot \mathbf{f} = \nabla^2 U$ can be obtained by differentiation under the integral sign in the above equation, it can not ordinarily be so obtained from the integrals (1).

Having seen that the divergence $\nabla \cdot \mathbf{f}$ is continuous at interior points of T it remains to determine its value there. Let T_1 be any regular region wholly interior to T and let S_1 be its regular boundary. Then by the divergence theorem and equation (9) we have,

$$\iiint_{T_1} \nabla \cdot \mathbf{f} \, d\tau = \iint_{S_1} \mathbf{n} \cdot \mathbf{f} \, d\sigma = -4\pi m_0 = -4\pi \iiint_{T_1} \rho \, d\tau,$$

where m_0 is the total mass of the attracting matter in T_1. Thus the equation,

$$\iiint_{T_1} (\nabla \cdot \mathbf{f} + 4\pi\rho) d\tau = 0,$$

holds for every such region T_1 interior to T, and due to the continuity of the integrand we have *Poisson's Equation*,

$$\nabla^2 U = \nabla \cdot \mathbf{f} = -4\pi\rho, \tag{11}$$

holding at all interior points. Poisson's Equation has here been proven under the hypothesis that the density ρ possesses a continuous gradient. Actually the equation may be established under slightly less stringent restrictions on the density, but the mere continuity of the density is not sufficient. Evidently Laplace's Equation (10) can be regarded as a special case of Poisson's Equation since the density ρ is identically zero at exterior points. If the force $\mathbf{f}$ or potential U is given at all points of T then equation (11) will determine the density of a distribution over T yielding this force or potential.

Surface Distributions

We shall now briefly discuss the force $\mathbf{f}$ and potential U due to a distribution of attracting matter of areal density ρ over a surface S. We have defined this force and potential by the integrals,

$$\mathbf{f} = \iint \frac{\rho(\mathbf{q})}{r^3}\,\mathbf{r}\,d\sigma, \qquad U = \iint \frac{\rho(\mathbf{q})}{r}\,d\sigma, \qquad \mathbf{r} = \mathbf{q} - \mathbf{p}, \tag{12}$$

and have observed that if ρ is piecewise continuous these integrals exist and satisfy the relations,

$$\nabla U = \mathbf{f}, \qquad \nabla^2 U = \nabla \cdot \mathbf{f} = 0,$$

at all points P not on S. For points P which are on S it is natural to attempt to proceed as in the case of volume distributions and extend these definitions by permitting the integrals (12) to become improper integrals. For the potential U this is done as follows. Let S be a regular surface and let P be any regular point of the surface. We construct a variable region S_0 of the surface having P as an interior point and δ as its maximum dimension and we indicate by $S - S_0$ the remainder of S. Then U is defined at P as the limit,

$$U = \lim_{\delta \to 0} \iint_{S-S_0} \frac{\rho(\mathbf{q})}{r}\,d\sigma \tag{13}$$

whenever it exists.

We may easily show that with this definition,

The potential U of a piecewise continuous distribution of attracting matter over a regular surface S exists and is continuous at all points of space.

The details of the proof will be left to the reader as the argument is substantially that employed for volume distributions, but the outline of the argument is as follows. We first establish,

Lemma II. If S is any plane region having an area equal to that of a circle of radius δ then the integral,

$$\int\int \frac{1}{r}\, d\sigma,$$

will be less in absolute value than any given positive number ϵ for every choice of δ sufficiently small, and the choice of δ is independent of the shape and position of the region S and the position of the point P.

We now surround the point P of the given surface S by a sphere of radius δ_1 so small that the normals to S at points within the sphere make an angle θ with the normal at P such that $\sec\theta < c$ where c is an arbitrary constant greater than 1. Then for regions S_0 of S within this sphere, having P as an interior point and of maximum dimension less than δ, we have,

$$\int\int \frac{\rho}{r}\, d\sigma < kc \int\int \frac{1}{r'}\, d\sigma',$$

where S_0', r', $d\sigma'$ are the projections of S_0, r, $d\sigma$ on the tangent plane at P. By the lemma the second member of this inequality can be made arbitrarily small by proper choice of $\delta < \delta_1$ and the existence of the limit (13) follows at once. The continuity of U as thus defined also follows as in the case of volume distributions, since the choice of δ is independent of the position of P. We agree to employ for U the symbol (12), already used for the proper integral when P is not on S, to indicate also the value of the improper integral when P is on S.

The analogous proof for the convergence of the improper integral for the force **f** fails. We therefore make no attempt to define **f** at points of the surface, but the behavior of **f** in the neighborhood of interior points of the surface is in part described by the following theorem.

If S is a regular surface possessing a continuously differentiable normal $\mathbf{n}$ *and covered with attracting matter having a continuously differentiable density ρ, then the force* $\mathbf{f}$ *at a point P approaches limits as P approaches any interior point Q_0 of S from each side of S. These two limits* $\mathbf{f}_+$ *and* $\mathbf{f}_-$ *satisfy the relation,*

$$\mathbf{f}_+ - \mathbf{f}_- = -\, 4\pi\rho_0\,\mathbf{n}_0,$$

where ρ_0 is the density at Q_0 and $\mathbf{n}_0$ *is the unit normal to S at Q_0 extended toward the side of* $\mathbf{f}_+$.

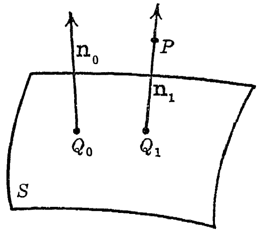

FIG. 100

We let P approach Q_0 from what we shall designate as the $+$ side of S. Then P will lie along the normal $\mathbf{n}_1$ to S at a point Q_1 of S and Q_1 will also approach Q_0. We now compare the force $\mathbf{f}$ at P and the vector $\mathbf{g}$ where,

$$\mathbf{f} = \iint \frac{\rho\,\mathbf{r}}{r^3}\,d\sigma, \qquad \mathbf{g} = \rho_1\,\mathbf{n}_1 \iint \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\,d\sigma,$$

ρ_1 being the density at Q_1. Examination of these expressions shows that they will differ very little if P is near Q_0 and Q is restricted to a small neighborhood about Q_0. In fact we shall establish that,

$$\mathbf{f} - \mathbf{g} = \mathbf{h} = \iint_S \left(\frac{\rho\,\mathbf{r}}{r^3} - \rho_1\,\mathbf{n}_1 \frac{\mathbf{n}\cdot\mathbf{r}}{r^3} \right) d\sigma,$$

is a continuous function of P at Q_0. But it is easily seen that $\mathbf{g}$ has at Q_0 exactly the properties described for $\mathbf{f}$ in the theorem. For the integral $\iint_S (\mathbf{n}\cdot\mathbf{r}/r^3)\,d\sigma$ is evidently the negative of the flux through S in the direction of $\mathbf{n}$ of the force due to a unit attracting particle at P. So if S is closed with $\mathbf{n}$ the outward normal and P passes from inside to outside at Q_0, this integral will suddenly decrease from 4π to zero, as previously noted, § 107, (8), (9). If S is not closed we may complete the closure about P by an additional surface and the integral over this additional surface, being a continuous function of P at Q_0, will not affect the existence

of these limits as P approaches Q_0 nor the discontinuity at Q_0. Thus in any case the integral $\iint_S (\mathbf{n}\cdot\mathbf{r}/r^3)d\sigma$ approaches limits as P approaches Q_0 and decreases by the amount 4π as P crosses S at Q_0 in the direction of $\mathbf{n}$. The statement in the theorem concerning $\mathbf{f}$ thus holds for $\mathbf{g}$.

To show that these statements also hold for $\mathbf{f}$ we have merely to prove that $\mathbf{h} = \mathbf{f} - \mathbf{g}$ is continuous at Q_0. Evidently $\mathbf{h}$ is the sum of the three functions,

$$\mathbf{h}' = -\iint_S \rho_1 \left\{ \nabla_Q \frac{1}{r} - \left(\mathbf{n}\cdot\nabla_Q \frac{1}{r}\right)\mathbf{n} \right\} d\sigma,$$

$$\mathbf{e}_1 = \iint_S (\rho_1 - \rho)\nabla_Q \frac{1}{r}\, d\sigma, \qquad \mathbf{e}_2 = \iint_S \rho_1(\mathbf{n}_1 - \mathbf{n})\mathbf{n}\cdot\nabla_Q \frac{1}{r}\, d\sigma.$$

To demonstrate the continuity of $\mathbf{h}'$ at Q_0 we first observe that by the application of Stokes' Theorem to the vector $\mathbf{a}\times\dfrac{\mathbf{n}}{r}$, where $\mathbf{a}$ is an arbitrary constant, we may easily show that,

$$\iint_S \left\{ \nabla_Q \frac{1}{r} - \left(\nabla_Q\cdot\frac{\mathbf{n}}{r}\right)\mathbf{n} \right\} d\sigma = \int_C \mathbf{t}\times\frac{\mathbf{n}}{r}\, d\gamma,$$

where C is the curve bounding S. Consequently we may write,

$$\mathbf{h}' = -\iint \rho_1 \left\{ \nabla_Q \frac{1}{r} - \left(\nabla_Q\cdot\frac{\mathbf{n}}{r}\right)\mathbf{n} + \frac{(\nabla_Q\cdot\mathbf{n})\mathbf{n}}{r} \right\} d\sigma$$

$$= \rho_1 \int \frac{\mathbf{n}}{r}\times\mathbf{t}\, d\gamma - \rho_1 \iint \frac{(\nabla_Q\cdot\mathbf{n})\mathbf{n}}{r}\, d\sigma.$$

The integral around C is continuous since $\mathbf{r}$ does not vanish on C, while the last integral may be broken up into components in three noncoplanar constant directions and the length of each component will then be the potential of a continuous surface distribution over S and consequently continuous, as previously observed. To demonstrate the continuity of $\mathbf{e}_1$ and $\mathbf{e}_2$ we draw a sphere of radius δ and center Q_0, calling the portion of S enclosed by it S_0. When δ has been chosen and P in its approach to Q_0

lies within this sphere, the portions of $\mathbf{e}_1$ and $\mathbf{e}_2$ due to integration over $S - S_0$ will be continuous since r will not vanish on $S - S_0$. As regards the portion of $\mathbf{e}_1$ and $\mathbf{e}_2$ due to integration over S_0, we observe that due to the continuous differentiability of ρ and $\mathbf{n}$ we may find positive constants c_1 and c_2 such that,

$$|\rho_1 - \rho| < c_1 r', \qquad |\mathbf{n}_1 - \mathbf{n}| < c_2 r',$$

where r' is the projection of r on the plane tangent to S at Q_1 and where these inequalities hold uniformly for all positions of P and Q within some distance δ_1 of Q_0. Also we may find a constant $c > 1$ such that $\sec\theta < c$ for all positions of P within some distance δ_2 of Q_0, where θ is, as before, the angle between the normals to S at Q_1 and at any point Q of S_0. Consequently we have, since $r' \leqq r$, $d\sigma \leqq c\, d\sigma'$,

$$\left|\iint\limits_{S_0} (\rho_1 - \rho) \frac{\mathbf{r}}{r^3}\, d\sigma\right| \leqq cc_1 \iint\limits_{S_0'} \frac{d\sigma'}{r'}$$

$$\left|\iint\limits_{S_0} \rho_1(\mathbf{n}_1 - \mathbf{n}) \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, d\sigma\right| \leqq kcc_2 \iint \frac{d\sigma'}{r'},$$

where S_0' is the projection of S_0 on the tangent plane at Q_1. By Lemma II, we may make each of these integrals over S_0' arbitrarily small by choosing δ less than some number δ_3. Thus finally for δ the least of the three numbers δ_1, δ_2, δ_3, we may make the portions of $\mathbf{e}_1$ and $\mathbf{e}_2$ due to integration over S_0 arbitrarily small, and the continuity of $\mathbf{e}_1$ and $\mathbf{e}_2$ follows by the argument presented in detail for the continuity of the force and potential due to a space distribution. The theorem just proven can be established under slightly less stringent restrictions on the density ρ, but the mere continuity of the density is not sufficient.

We have presented above the principal features of the force and potential fields due to volume and surface distributions of attracting matter. The fields due to a double layer distribution are somewhat less important and the arguments employed in their discussion are similar to those used above. For a full treatment of these topics in English the reader may consult,

O. D. Kellog, *Foundations of Potential Theory*,
W. D. MacMillan, *The Theory of the Potential.*

The development of the analogous properties in the plane of the force and potential due to distributions of logarithmic particles in this plane forms an instructive exercise for the reader.

EXERCISES

1. Show that the potential of a circular disc of constant areal density ρ and radius a at an interior point P of the disc at a distance p from the center is given by,

$$U = 4a\rho \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\sigma \qquad k = \rho/a.$$

Check with Problem 14 of § 108.

2. Find the amount of the attraction of a right circular cylinder of constant density ρ at an interior point of its axis at distances c_1 and c_2 from the centers of the bases and d_1 and d_2 from their respective edges. Check with Problem 22 of § 108 and show that f is continuous at the surface.

Ans. $f = 2\pi\rho\,|c_1 - c_2 + d_1 - d_2|$

3. Determine the force $\mathbf{f}$ and potential U at a point P which is $p \leq a$ units from the center O of a sphere of radius a and constant density ρ.

Ans. $f = \dfrac{4}{3}\pi\rho p, \qquad U = 2\pi\rho\left(a^2 - \dfrac{p^2}{3}\right)$

Check with the results found in § 108 and show that $\mathbf{f}$ and U are continuous at the surface.

4. Show that within a spherical cavity in a homogeneous sphere the force $\mathbf{f}$ is a constant vector equal to what the force would be at the center of the cavity if the sphere were solid.

5. Determine the amount of the force of the homogeneous oblate and prolate spheroids of constant density ρ and semiaxes a and b $(a > b)$ at an interior point P of the axis of revolution c units from the center. Check with Problem 3 of this set.

Ans. $f = \dfrac{4\pi\rho c}{e^3}\left(e - \sqrt{1 - e^2}\arcsin e\right), \qquad e = \text{eccentricity},$

$$f = \frac{4\pi\rho c(1 - e^2)}{e^3}\left(\log\sqrt{\frac{1+e}{1-e}} - e\right), \quad e = \text{eccentricity}$$

6. Determine the potential of the homogeneous prolate spheroid of density ρ and semiaxes a and b $(a > b)$ at the focus; at the pole.

Ans. $U = 2\pi\rho b^2, \qquad U = \dfrac{2\pi\rho b^2}{e^3}\left\{e - (1 - e^2)\log\sqrt{\dfrac{1+e}{1-e}}\right\},$

$e = \text{eccentricity}$

7. Find the amount f of the attraction at the focus P of a paraboloid of revolution of constant density ρ bounded by a plane perpendicular to the axis at a distance $b \geqq a$ from the vertex, a being the distance from the vertex to the focus. How far from the focus is the edge of the body if the force at the focus is zero?

$$\textit{Ans.}\quad f = 4\pi\rho a \left| \log\left(1 + \frac{b}{a}\right) - 1 \right|, \quad ea$$

8. Show that if two attracting surfaces are similar and similarly placed with respect to a point P and have the same density at corresponding points, then they have the same force at P.

9. By the aid of Problem 8 show that a semicircular area of constant density has an infinite force at the center of the straight edge.

10. A point P lies in the plane of a homogeneous rectangle at equal distances a from the extensions of two opposite edges and at distances c_1 and c_2 from the other two opposite edges. Show that the amount of the force at P due to the rectangle is,

$$f = \left| 2\rho \log \frac{(a + d_1)c_2}{(a + d_2)c_1} \right|,$$

where d_1 and d_2 are the distances from P to the corners of the rectangle. Assuming that the force at the center of the rectangle is zero, show that the above formula holds for interior as well as exterior points.

11. Discuss in a given plane the force and potential due to distributions in that plane of matter attracting inversely as the first power of the distance, the discussion of this logarithmic force and potential being concerned principally with points on and near the attracting masses.

12. Show that the logarithmic potential U due to a distribution of density ρ over a plane area S of matter attracting inversely as the first power of the distance satisfies in the plane *Poisson's Equation* for such matter,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -2\pi\rho.$$

110. Harmonic Functions.

We have seen that the scalar potentials of Newtonian forces satisfy Laplace's Equation $\nabla^2 U = 0$ at points where there are no attracting masses and we have agreed to call solutions of this equation harmonic functions. Harmonic functions also arise in connection with many other physical problems. To mention but a single instance we might remark that in the case of a steady flow of heat through a homogeneous isotropic medium it is found that the temperature within the medium is a harmonic function of

the point considered. It is of interest therefore to inquire what properties are common to all harmonic functions.

To be more specific we shall say that a scalar-point function U is harmonic at a point if its second derivatives exist and are continuous and satisfy Laplace's Equation $\nabla^2 U = 0$ throughout some neighborhood of that point. The function U is harmonic in an open region if it is harmonic at all points of that region. It is harmonic in a closed region if it is harmonic at all interior points and continuous on the boundary. In the case of an open region extending to infinity it is convenient to require also that $p|U|$ and $p^2|\nabla U|$ shall remain bounded for all sufficiently large values of p, where p is the distance from some fixed point O.

The indispensable tools in the study of harmonic functions are Green's identities proven in § 105. Green's first identity states that,

If U and V are scalar-point functions continuous together with their gradients ∇U, ∇V in a regular closed region T and if ∇U has a continuous divergence $\nabla^2 U$ in T then,

$$\text{(I)} \qquad \iiint\limits_T V\,\nabla^2 U\,d\tau + \iiint\limits_T \nabla U \cdot \nabla V\,d\tau = \iint\limits_S V\,\mathbf{n}\cdot\nabla U\,d\sigma,$$

where S is the boundary of T and $\mathbf{n}$ is the unit outward normal to S. This identity was proven in § 105 on the stronger hypothesis that both T and S lay *within* a region of continuity of ∇U, ∇V, $\nabla^2 U$ but the extension to the present statement is immediate if we understand that in defining ∇U and ∇V the increments $\Delta\mathbf{p}$ of the independent variable shall be confined to values which cause P to fall only at points of T and S. This permits us to employ the identity in case some of these quantities fail to exist outside of S or have other discontinuities there. The proof that (I) holds under these conditions then follows by applying the identity to a regular region T' within T whose boundary S' is everywhere within a distance δ of S. On proceeding to the limit allowing δ to approach zero, the present statement follows.

Our first theorem on harmonic functions follows at once from Green's first identity.

I. *If U is harmonic in a closed regular region T and possesses a continuous gradient on the boundary S of T, then the flux of the gradient of U through S is zero.*

For if in (I) we set $V \equiv 1$ and $\nabla^2 U \equiv 0$ we have,

$$\iint_S \mathbf{n}\cdot\nabla U \, d\sigma = 0 \tag{1}$$

and the theorem is proven.

Almost as simple is the proof of the uniqueness theorem.

II. *A function U harmonic in a closed regular region T and possessing a continuous gradient on the boundary S of T is uniquely determined in T by its values on S and is determined except for an additive constant by the values of its normal derivative on S.*

For let U_1 and U_2 be two functions harmonic in T and possessing continuous gradients on S. In Green's identity (I) we set $U = V = U_2 - U_1$ and find,

$$\iiint_T (\nabla U)^2 \, d\tau = \iint_S U \, \mathbf{n}\cdot\nabla U \, d\sigma.$$

If now U_1 and U_2 have the same values on S or if they have the same normal derivatives on S the second member of this equation vanishes. Since ∇U is real and continuous the first member can vanish also only if ∇U is zero throughout T. Thus U_1 and U_2 have the same gradient throughout T and they must differ, if at all, by a constant. But if U_1 and U_2 take on the same values on S this constant must be zero. The theorem is thus proven.

As a generalization of Theorem I, we have the following,

III. *If U and V are functions harmonic in a closed region T and possess continuous gradients on the boundary S of T then,*

$$\iint_S (U \, \mathbf{n}\cdot\nabla V - V \, \mathbf{n}\cdot\nabla U) d\sigma = 0. \tag{2}$$

This is an immediate consequence of *Green's second identity,*

$$\iiint_T (U \, \nabla^2 V - V \, \nabla^2 U) d\tau = \iint_S (U \, \mathbf{n}\cdot\nabla V - V \, \mathbf{n}\cdot\nabla U) d\sigma, \tag{II}$$

which follows from the first identity by interchange of U and V

and subtraction, provided we further assume that $\nabla^2 V$, as well as $\nabla^2 U$, exists and is continuous in T.

To prove our next theorem we shall need Green's third identity. If P is any exterior point to the closed region T we may choose V of Green's second identity (II) as $V = 1/r$ where $\mathbf{r} = PQ$ and where now Q is the variable of integration. Then $\nabla_Q V = -\mathbf{r}/r^3$, $\nabla_Q^2 V = 0$ and Green's second identity (II) becomes,

$$(3) \qquad \iiint_T \frac{\nabla_Q^2 U}{r}\, d\tau = \iint_S \left(\frac{\mathbf{n}\cdot\nabla_Q U}{r} - U\,\mathbf{n}\cdot\nabla_Q \frac{1}{r} \right) d\sigma.$$

If U is harmonic in T the first, and consequently the second, of these integrals will vanish. If however P is a point interior to T, we construct a sphere with center at P and radius δ so small that the sphere is wholly interior to T. Calling the interior of the sphere T_0 and its surface S_0, we apply equation (3) to the region $T - T_0$ obtaining,

$$(4) \qquad \iiint_{T-T_0} \frac{\nabla_Q^2 U}{r}\, d\tau = \iint_{S+S_0} \left(\frac{\mathbf{n}\cdot\nabla_Q U}{r} - U\,\mathbf{n}\cdot\nabla_Q \frac{1}{r} \right) d\sigma.$$

Since ∇U and $\nabla^2 U$ remain finite within T the integrals,

$$(5) \qquad \iiint_{T_0} \frac{\nabla_Q^2 U}{r}\, d\tau, \qquad \iint_{S_0} \frac{\mathbf{n}\cdot\nabla_Q U}{r}\, d\sigma,$$

vanish as δ approaches zero. Remembering that $\mathbf{n}$ is the *inward* normal for the sphere S_0 we have also,

$$(6) \quad \lim_{\delta\to 0} \iint_{S_0} U\,\mathbf{n}\cdot\nabla_Q \frac{1}{r}\, d\sigma = -\lim_{\delta\to 0} \iint_{S_0} U \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, d\sigma$$

$$= \lim_{\delta\to 0} \frac{1}{\delta^2} \iint_{S_0} U\, d\sigma = 4\pi U(P).$$

From equations (4), (5), (6) we have in the limit as δ approaches zero *Green's third identity.*

If U possesses a continuous Laplacian $\nabla^2 U$ within a closed regular region T and a continuous gradient ∇U on the surface S of T,

then the value of U at interior points P of T is given by,

$$\text{(III)}\quad U(P) = -\frac{1}{4\pi}\iiint\limits_T \frac{\nabla_Q^2 U}{r}\,d\tau + \frac{1}{4\pi}\iint\limits_S \frac{\mathbf{n}\cdot\nabla_Q U}{r}\,d\sigma - \frac{1}{4\pi}\iint\limits_S U\,\mathbf{n}\cdot\nabla_Q \frac{1}{r}\,d\sigma.$$

The striking feature of this equation is that it expresses any function U which merely satisfies certain differentiability conditions as the sum of three Newtonian potentials. The first is the potential of a volume distribution of density $-\nabla^2 U/4\pi$ in T, the second that of a surface distribution of areal density $\mathbf{n}\cdot\nabla U/4\pi$ on S, and the third that of a double layer of density $U/4\pi$ on S. As a special case of this we have the theorem,

IV. *If U is harmonic in a closed regular region T and possesses a continuous gradient ∇U on the surface S of T then U may be represented at all interior points of T as the sum of the potentials of a single and a double layer on S.*

It follows that a function which is harmonic in a region may be represented in regions interior to that region as the sum of Newtonian potentials in infinitely many ways. It also follows that,

V. *A function U which is harmonic at a point possesses derivatives of all orders at that point.*

For we may surround the point by a regular closed surface S at all points of which the function U is harmonic and then express U by the last two integrals of equation (III), and these *as functions of P* evidently possess derivatives of all orders at all interior points. We thus have the striking result that the mere fact that certain second derivatives of U exist, are continuous, and add up to zero at each point of a region insures the existence of *all* the derivatives of U at every interior point of that region.

A consequence of Green's third identity which is of peculiar simplicity and of frequent use is,

VI. *Gauss' Theorem of the Arithmetic Mean. If a function U is harmonic in a sphere, the value of U at the center of the sphere is the arithmetic mean of its values on the surface of the sphere.*

Let S be the sphere of the theorem with center P and radius a. Let S' be a concentric sphere of radius $a - \delta$ where $0 < \delta < a$.

Then Green's third identity (III) is applicable to U for S'. The first integral on the right in (III) vanishes because U is harmonic in S'; the second integral vanishes by equation (1) since r is constant for Q on S', and we have,

$$U(P) = -\frac{1}{4\pi}\iint_{S'} U\,\mathbf{n}\cdot\nabla_Q \frac{1}{r}\,d\sigma = \frac{1}{4\pi}\iint_{S'} \frac{U}{r^2}\,d\sigma$$

$$= \frac{1}{4\pi(a-\delta)^2}\iint U\,d\sigma.$$

On proceeding to the limit allowing δ to approach zero we find,

$$(7) \qquad U(P) = \frac{1}{4\pi a^2}\iint_S U\,d\sigma,$$

as stated in the theorem.

Important properties of harmonic functions may be derived from their expansions in series of certain types. Also these series often afford the most convenient means for computation of the values of the function. These expansions are based on preliminary expansions of the function $1/r$. Let P and Q be any two distinct points which are distinct from the origin O, and call $OP = \mathbf{p}$, $OQ = \mathbf{q}$. Then for $\mathbf{r} = PQ$ we have,

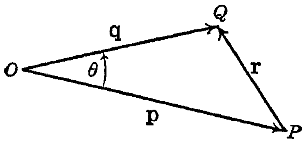

Fig. 101

$$\frac{1}{r} = \frac{1}{\sqrt{\mathbf{q}^2 - 2\,\mathbf{p}\cdot\mathbf{q} + \mathbf{p}^2}} = \frac{1}{q}(1 - 2\mu u + \mu^2)^{-1/2},$$

where

$$\mu = p/q, \qquad u = \cos\theta = \mathbf{p}\cdot\mathbf{q}/pq.$$

By the binomial theorem we have for any number x such that $|x| < 1$,

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots,$$

and consequently for $|2\mu u - \mu^2| < 1$ we may write,

$$(8) \quad \frac{q}{r} = 1 + \frac{1}{2}(2\mu u - \mu^2) + \frac{1\cdot 3}{2\cdot 4}(2\mu u - \mu^2)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(2\mu u - \mu^2)^3 + \cdots.$$

For $\mu < \sqrt{2} - 1$ we have,

$$(9)\quad |2\mu u - \mu^2| \leq 2\mu|u| + \mu^2 \leq 2\mu + \mu^2 \\ = \mu(2 + \mu) < (\sqrt{2} - 1)(\sqrt{2} + 1) = 1,$$

and consequently series (8) converges absolutely and uniformly in both its variables for $\mu < \sqrt{2} - 1$. Not only is this true, but the inequalities (9) show that for $\mu < \sqrt{2} - 1$ series (8) would still converge absolutely if the binomial powers were expanded and the parentheses around the resulting groups of terms removed. This fact permits us to expand the binomial powers, remove the parentheses and regroup the terms in any way. In particular we may collect the terms involving the same power of μ and write for $\mu < \sqrt{2} - 1$,

$$(10)\quad \frac{q}{r} = P_0(u) + P_1(u)\mu + P_2(u)\mu^2 + P_3(u)\mu^3 + \cdots,$$

where,

$$P_0(u) = 1, \qquad P_1(u) = u, \qquad P_2(u) = \frac{1}{2}(3u^2 - 1), \\ P_3(u) = \frac{1}{2}(5u^3 - 3u), \cdots.$$

These polynomials $P_n(u)$ are known as *Legendre polynomials* and are often used in the expansion of functions in series. Among their many important properties we shall only mention that they satisfy the recursion relation,

$$(n + 1)P_{n+1}(u) = (2n + 1)P_n(u)u - nP_{n-1}(u),$$

by which they may be successively computed.

If we let P and Q have the Cartesian coördinates, $P \equiv (x, y, z)$, $Q \equiv (\xi, \eta, \zeta)$, then,

$$(11)\quad \mu u = \frac{1}{q^2}(\xi x + \eta y + \zeta z), \qquad \mu^2 = \frac{1}{q^2}(x^2 + y^2 + z^2),$$

and these values may be substituted in series (8). Now inequalities (9) still hold if we replace u by $u_1 = (|\xi|\,|x| + |\eta|\,|y| + |\zeta|\,|z|)/pq$ and hence for $\mu < \sqrt{2} - 1$ series (8) will still converge absolutely if μu and μ^2 be replaced by their values (11), the powers expanded and the parentheses removed. We may therefore collect the terms of like degree in x, y, z and after dividing

through by q, write for $\mu < \sqrt{2} - 1$,

$$(12) \qquad \frac{1}{r} = H_0(x, y, z) + H_1(x, y, z) + H_2(x, y, z) + \cdots.$$

The functions $H_n(x, y, z)$ are then homogeneous polynomials of degree n in x, y, z, the coefficients in the polynomials being functions of ξ, η, ζ. These polynomials are themselves easily shown to be harmonic functions of P and are known as *spherical harmonics*. Furthermore the series (12) is term by term exactly series (10) simply divided through by q. For since u is homogeneous of degree zero in x, y, z, it follows that $P_n(u)\mu^n/q$ is homogeneous of degree n in x, y, z and consequently,

$$H_n(x, y, z) = P_n(u)\mu^n/q = p^n P_n(u)/q^{n+1}.$$

If in series (10) we merely replace μu and μ^2 by their values (11), expand the powers and remove the parentheses, we have a series of the form,

$$(13) \qquad \frac{1}{r} = \sum_{i,j,k=0} A_{ijk} x^i y^j z^k,$$

where the A_{ijk} are functions of ξ, η, ζ and which series is absolutely convergent for $\mu < \sqrt{2} - 1$. Since the order of terms is immaterial in an absolutely convergent series no special order of these terms need be specified. The series (13) is said to be a *power series in x, y, z* and any function which can thus be expanded in a power series in x, y, z for all points of some neighborhood of the origin O is said to be *analytic at O*.

We may now discuss the expansion of Newtonian potentials in series. In fact series (8) is the expansion of the potential of a particle of mass q at Q in terms of Legendre polynomials, and series (12) and (13) are the expansions of the potential of a unit particle at Q in spherical harmonics and in a power series respectively. If we have a volume distribution of attracting matter with continuous density ρ in a closed regular region T or a single or double layer with continuous density over a regular surface S, then we may choose O as any point of free space and indicate by a the least distance from O to any point of T or S. Then for P any point within a sphere of center O and radius $b = (\sqrt{2} - 1)a$ and for Q any point of T or S, the series (10), (12), (13) are abso-

lutely and uniformly convergent in all the variables $x, y, z, \xi, \eta, \zeta$. This suffices to show that the value of $1/r$ may be taken from the series and substituted into the expressions,

$$U = \iiint\limits_T \frac{\rho}{r}\, d\tau, \qquad U = \iint\limits_S \frac{\rho}{r}\, d\sigma,$$

for the potential of the volume distribution and the single layer distribution and the results integrated term by term. The resulting series will converge absolutely to the value of U for P within the above sphere and give us from (10) or (12),

$$(14) \quad U(P) = H_0(x, y, z) + H_1(x, y, z) + H_2(x, y, z) + \cdots,$$

where $H_n(x, y, z)$ is a spherical harmonic of degree n in x, y, z; while from (13) we have the power series,

$$(15) \qquad U(P) = \sum_{i,j,k=0}^{\infty} B_{ijk} x^i y^j z^k.$$

Thus the potentials of a volume distribution and a single layer with continuous densities are analytic at any point of free space.

The double layer potential,

$$U = -\iint\limits_S \mu\, \mathbf{n} \cdot \nabla_Q \frac{1}{r}\, d\sigma,$$

requires a little further consideration as the mere uniform convergence of a series does not permit the term by term differentiation. But for a *power* series in any variables the term by term differentiation with respect to these variables is permitted at interior points of the region of convergence. Series (13) is a power series in x, y, z and series (10) and (12) may be made so by permissible expansions and removals of parentheses. Thus the value of $1/r$ may be taken from these series and substituted in the formula,

$$U = \iint\limits_S \mu\, \mathbf{n} \cdot \nabla_P \frac{1}{r}\, d\sigma,$$

and the indicated operations performed term by term yielding series of the form (14) and (15) for the double layer potential

interior to the sphere of convergence. Thus the double layer potential is, like the volume potential and the single layer potential, an analytic function at all points of free space.

Returning now to harmonic functions in general and recalling Theorem IV that an harmonic function may be expressed as the sum of a single and a double layer potential, we have at once,

VII. *If a function U is harmonic in a region it is analytic at all interior points of that region.*

This in itself constitutes a proof of Theorem V, for the coefficients B_{ijk} in the power series expansion (15) of any function U analytic at O, are well known to be uniquely determined and equal to,

$$B_{ijk} = \frac{1}{i!\,j!\,k!}\left[\frac{\partial^{i+j+k}}{\partial x^i\,\partial y^j\,\partial z^k}\,U\right]_O,$$

where the subscript O indicates that the derivatives are evaluated at O.

As a conclusion to this brief discussion of harmonic functions we shall say something concerning *boundary value problems.* We saw in Theorem II that there is not more than one function harmonic in a closed region T and taking on a given set of values on the boundary, and that, save for an additive constant, there is not more than one function harmonic in T having given normal derivatives on the boundary. But this leaves open the important question as to whether in each case there is *even one* such function. The finding of conditions under which there exists a function harmonic in a region and taking on given values on the boundary, and the determination of the function when it exists, is known as the *Dirichlet problem.* The corresponding questions when the values of the normal derivative instead of those of the function itself are given on the boundary is known as the *Neumann problem.* These are the first and most important *boundary value problems.* Examples have been given of closed surfaces for which these problems have no solutions. Thus Lebesque in 1913 showed that for a closed surface with a sufficiently sharp inward pointing spine there could be no solution of the Dirichlet problem. On the other hand very general conditions have been found under which the problems admit a solution.

We shall discuss in detail only the very simple but important case of the Dirichlet problem for the sphere. We have here the theorem,

VIII. *If $f(P)$ is a function defined and continuous on the surface S of a sphere then there exists a function $U(P)$ harmonic in the interior T of S and taking on the values of $f(P)$ on S.*

Let the center of the sphere be O and the radius be a and let P be any interior or boundary point except O and let Q be any interior or boundary point except P. Let P' be the inverse of P with respect to the sphere, i.e. let P' be the point on the extension of OP through P such that,

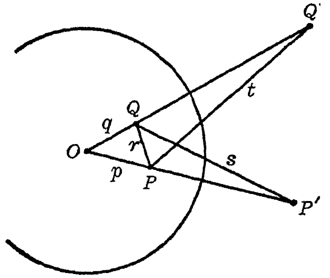

FIG. 102

$$pp' = a^2, \qquad p = OP, \qquad p' = OP'.$$

Under these conditions we define *Green's function* of Q for a given P as,

$$(17) \qquad G(Q, P) = \frac{1}{r} - \frac{a}{ps}, \qquad s = P'Q, \qquad r = PQ.$$

If we interchange P and Q and the restrictions upon them, we have similarly,

$$qq' \qquad\qquad q = OQ, \qquad q' = OQ',$$

and *Green's function* of P for a given Q is,

$$(18) \qquad G(P, Q) = \frac{1}{r} - \frac{a}{qt}, \qquad t = Q'P.$$

Evidently the triangles OPQ' and OQP' are similar and

$$\frac{p}{q} = \frac{q'}{p'} = \frac{t}{s},$$

and if P and Q are distinct from O and from each other, we have $ps = qt$ and consequently *Green's function is symmetric in P and Q,*

$$(19) \qquad G(Q, P) = G(P, Q).$$

We now define $G(Q, P)$ as a function of *any* two distinct interior or boundary points of S by formulas (17) and (18). At least one of these will always have a meaning when P and Q are distinct and they give the same value when both are available. The function is undefined if P and Q coincide.

We shall show that the formula,

$$U(P) = -\frac{1}{4\pi}\iint f(Q)\,\mathbf{n}\cdot\nabla_Q G(Q, P)\,d\sigma, \tag{20}$$

in which Q is the variable of integration, defines in the interior of the sphere the function whose existence is stated in the theorem. We have here to show that $U(P)$ is harmonic in the interior of the sphere and that as P approaches any point Q_0 of the surface of the sphere, $U(P)$ approaches the limit $f(Q_0)$. The proof of these facts depends on certain properties of Green's function which we shall now establish. All of these properties of course hold when P and Q are interchanged.

(*a*) *Green's function $G(Q, P)$ is zero if Q is on the surface S of the sphere.* For if Q is on S then $q = a$, $t = r$ and formula (18) gives $G(Q, P) = 0$.

(*b*) *For a given position of P Green's function is an harmonic function of Q except for Q at P.* For if P is not at O, by formula (17) $G(Q, P)$ is the sum of the potentials of a unit particle at P and a particle of mass $-a/p$ at P'. And if P is at O, by formula (18) $G(P, Q) = \dfrac{1}{r} - \dfrac{1}{a}$ which differs only by a constant from the potential of a unit particle at P.

(*c*) *If P and Q are both interior to the sphere S then $G(Q, P)$ is positive.* For, P and Q being given, we surround P by a small sphere S_0 excluding Q and wholly interior to S. Evidently if S_0 is small enough $G(Q, P)$ would be positive for Q on S_0 since the term $1/r$ becomes arbitrarily large for Q in the neighborhood of P. We have seen that $G(Q, P)$ is zero for Q on S and harmonic for Q in the region between S_0 and S. Consequently $G(Q, P)$ must be positive for Q anywhere in this region, since it follows from Gauss' Theorem of the Arithmetic Mean that a function harmonic in a closed region takes on its maximum and minimum values only on the boundary of the region unless the function is constant. See Problem 2.

(*d*) *For a given P the outward normal derivative* $\mathbf{n}\cdot\nabla_Q G(Q, P)$ *of $G(Q, P)$ with respect to Q on S has only negative and zero values.* This follows from the fact that for a given interior point P, $G(Q, P)$ is positive for Q within S and zero for Q on S; while if P is on S, $G(Q, P)$ is the constant zero.

(*e*) *For each interior position of P the integral over S of the normal derivative of G(Q, P) with respect to Q equals* -4π, *i.e.*

$$\iint \mathbf{n}\cdot\nabla_Q G(Q, P)d\sigma = -4\pi, \tag{21}$$

the variable of integration being Q. This follows at once by equations § 107, (8), (9) from the fact that $G(Q, P)$ is the sum of the potentials of a unit particle interior to S and an exterior particle, or is the potential of a unit interior particle plus a constant.

(*f*) *The function G(Q, P) possesses continuous derivatives of all orders with respect to both P and Q provided P and Q remain separated by some arbitrarily small fixed amount.* This follows at once by definitions (17) and (18).

These properties of Green's function having been established, we may now prove the stated properties of the function $U(P)$ of equation (20). If P is any point interior to the sphere S we have by properties (*b*) and (*f*),

$$\nabla_P^2 U(P) = -\frac{1}{4\pi}\iint_S f(Q)\, \nabla_P^2\, \{\mathbf{n}\cdot\nabla_Q G(Q, P)\}d\sigma$$

$$= -\frac{1}{4\pi}\iint_S f(Q)\, \mathbf{n}\cdot\nabla_Q\, \{\nabla_P^2 G(Q, P)\}d\sigma = 0.$$

For since P is an interior point and these derivatives are evaluated for Q on S, it follows that P and Q are separated as required by (*b*) and (*f*). *Thus U(P) is harmonic at all interior points of the sphere S.*

To see that as P approaches any point Q_0 of the surface of the sphere, $U(P)$ approaches $f(Q_0)$, we first multiply equation (21) through by $f(Q_0)$ obtaining in effect,

$$f(Q_0) = -\frac{1}{4\pi}\iint_S f(Q_0)\, \mathbf{n}\cdot\nabla_Q G(Q, P)d\sigma,$$

and then subtract from equation (20) finding,

$$U(P) - f(Q_0) = -\frac{1}{4\pi}\iint \{f(Q) - f(Q_0)\}\, \mathbf{n}\cdot\nabla_Q G(Q, P)d\sigma. \tag{22}$$

Now being given any arbitrarily small positive number ϵ, we construct a little sphere with center Q_0 and radius δ chosen small enough so that for Q within it we have due to the continuity of $f(Q)$,

$$|f(Q) - f(Q_0)| < \frac{\epsilon}{2}.$$

Then if S_0 is the portion of S lying within this little sphere, we have for all positions of P interior to S by means of properties (d) and (e)

$$(23) \quad \left| -\frac{1}{4\pi} \iint_{S_0} \{f(Q) - f(Q_0)\}\, \mathbf{n}\cdot\nabla_Q G(Q, P) \right|$$
$$< \frac{1}{4\pi}\frac{\epsilon}{2} \iint_{S_0} |\mathbf{n}\cdot\nabla_Q G(Q, P)|\, d\sigma \leq \frac{\epsilon}{2}.$$

On the other hand by property (a) $G(Q, Q_0)$ is zero for all positions of Q except Q_0; and so except at Q_0, $\nabla_Q G(Q, Q_0)$ is zero also. Thus when P in approaching Q_0 remains interior to the little sphere of radius δ the gradient $\nabla_Q G(Q, P)$ will by property (f) approach zero uniformly for all positions of Q outside this little sphere. It will thus be possible to take P so close to Q_0 that,

$$(24) \qquad -\frac{1}{4\pi} \iint_{S-S_0} \{f(Q) - f(Q_0)\}\, \mathbf{n}\cdot\nabla_Q G(Q, P) d\sigma < \frac{\epsilon}{2},$$

the integral being extended over that portion of S lying outside the little sphere. On adding inequalities (23) and (24) we find,

$$|U(P) - f(Q_0)| < \epsilon,$$

showing that $U(P)$ approaches $f(Q_0)$ as P approaches Q_0. The function $U(P)$ defined for points P interior to S by formula (20) may thus be continuously extended by defining it as equal to $f(P)$ on S and as so defined $U(P)$ has been seen to enjoy all the properties required of the solution of Dirichlet's problem for the sphere.

EXERCISES

1. Show that if a function U is harmonic in a sphere, the value of U at the center of the sphere is the arithmetic mean of its values throughout the interior of the sphere.

2. Show that if a function U is harmonic in a closed region T bounded by a closed surface S and if A and B, $A < B$, are the least and greatest values of U on S, then every value of U assumed within S satisfies the relation,
$$A < U < B.$$

3. Show that if a function U is harmonic at every point of space then it either takes on every value or is a constant.

4. Show that if a function U is harmonic in a closed connected region T and is constant in a region T_0 interior to T, then U is constant in T.

5. Show that if a function U is harmonic at every point of space and approaches a limit at infinity, then U is constant.

6. Show that if a function U is harmonic in a closed regular region T and has a continuous gradient on the boundary S of T and if U is constant on part of S and its normal derivative is zero on the rest of S, then U is constant in T.

7. Prove the converse of Gauss' Theorem of the Arithmetic Mean; i.e., If U is a function continuous in a closed region T and takes on at the center of every sphere in T the arithmetic mean of its values on that sphere, then U is harmonic in T.

8. If $U(\mathbf{p})$ is harmonic at a point show that $\mathbf{a}\cdot\nabla U$ and $\mathbf{p}\cdot\nabla U$ are also harmonic there. ($\mathbf{a}$ const.)

9. If $U(\mathbf{p})$ is harmonic at a point show that $\nabla^2(\mathbf{p}^2\, U)$ is also harmonic there.

10. If $U(\mathbf{p})$ is harmonic at a point, $\mathbf{p} \neq 0$, show that
$$\frac{1}{p}\, U\left(\frac{a^2\,\mathbf{p}}{p^2}\right) \qquad (a^2 \text{ const}).$$
is also harmonic there. (Lord Kelvin, 1847.)

11. Introduce the polar coördinates,
$$\mathbf{p} \equiv (p \sin\theta \cos\varphi,\ p\sin\theta\sin\varphi,\ p\cos\theta),$$
and show that,
$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial p^2} + \frac{1}{p^2}\frac{\partial^2 U}{\partial\theta^2} + \frac{1}{p^2\sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} + \frac{2}{p}\frac{\partial U}{\partial p} + \frac{\cot\theta}{p^2}\frac{\partial U}{\partial\theta}.$$
(Laplace, 1782.)

12. A regular closed surface S is an equipotential surface for a volume distribution lying within S. Show that at points P exterior to S the potential U of this volume distribution is the same as that of a surface distribution on S of areal density $-\,\mathbf{n}\cdot\nabla U/4\pi$. (Green's Equivalent Layer.)

13. If a function U is harmonic at the origin O and is expanded in a series,
$$U(\mathbf{p}) = U_0(\mathbf{p}) + U_1(\mathbf{p}) + U_2(\mathbf{p}) + \cdots,$$

where each $U_n(\mathbf{p})$ is a homogeneous polynomial in x, y, z, show that each $U_n(\mathbf{p})$ is harmonic at O.

14. Show that if a function U is harmonic at a point P then all its derivatives are also harmonic at P.

15. Write out explicitly in terms of x, y, z, ξ, η, ζ the first three spherical harmonics H_0, H_1, H_2 of series (12).

16. If $H_n(\mathbf{p})$ is a spherical harmonic of degree n, i.e. a harmonic function homogeneous of degree n in x, y, z, show that $H_n(\mathbf{p})/p^{2n+1}$ is also harmonic ($p \neq 0$).

17. Show that formula (20) may be written in the form,

$$U(\mathbf{p}) = \frac{a^2 - p^2}{4\pi a} \iint_S \frac{f(Q)}{r^3}\, d\sigma.$$

In this express $G(Q, P)$ as $G(p, q, \theta)$ where $\theta = \angle\, POQ$ and employ the fact that on S, $\mathbf{n} \cdot \nabla_Q G(Q, P) = \dfrac{\partial}{\partial q} G(p, q, \theta)$.

(Poisson's Integral, 1820.)

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